REPRESENTATION THEORY AND GEOMETRY

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http://www.maths.usyd.edu.au/u/geordie/Heilbronn.pdf

RECOLLECTIONS FROM LAST LECTURE

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Observation 1: If V has a positive-definite G-invariant geometric structure, then V is semi-simple.

If $U \subset V$ is a subrepresentation, then $V = U \oplus U^{\perp}$.

Observation 2: Any representation of G admits a positive-definite geometric structure.

Take a positive-definite geometric structure $\langle -, - \rangle$ on V. Then

$$\langle v, w \rangle_G := \frac{1}{|G|} \sum \langle gv, gw \rangle$$

defines a positive-definite and G-invariant geometric structure.

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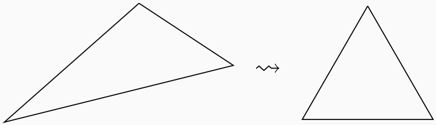
If V is simple and defined over the complex numbers, then Schur's lemma shows that the geometric structure is unique up to positive scalar.

This is an example of "unicity of geometric structure".



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In this lecture I will outline a bridge to geometry.

We will see that invariant forms appear naturally again.

THEMES OF THIS LECTURE

In this lecture I will outline a bridge to geometry.

We will see that invariant forms appear naturally again.

We will see a recurrence of the two themes:

"semi-simplicity via introduction of geometric structure" and "unicity of geometric structure"

THE FLAG VARIETY

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$$\mathrm{Flag}_n := \{0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n \mid \dim V_i = i\}.$$

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Example

$$\mathrm{Flag}_2 = \{0 \subset V_1 \subset \mathbb{C}^2\} = \text{lines in } \mathbb{C}^2 = \mathbb{P}^1(\mathbb{C}).$$

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Consider, the coordinate flag

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Its stabiliser in $SL_n(\mathbb{C})$ is

$$B = \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & * \end{pmatrix} \subset SL_n(\mathbb{C})$$

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Thus (“group theoretic description of flag variety”):

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B is a Borel subgroup, the flag variety makes sense for any reductive group.

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are the B-orbits on $\mathbb{P}^1\mathbb{C}$.

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Example

For $G = \mathrm{SL}_n(\mathbb{C})$ this is Gaußian elimination.

For $w \in W$, their closures

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For $n = 2$, $W = \{\mathrm{id}, s\}$ and

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This example is deceptive: Schubert varieties are usually singular, and have an extremely intricate structure.

Kazhdan-Lusztig conjecture (1979)

$$[\Delta_{x\cdot 0}] = \sum_{y \in W} P_{x,y}(1)[L_{y\cdot 0}]$$

(Recall that $P_{y,x} \in \mathbb{Z}[v]$ is a Kazhdan-Lusztig polynomial.)

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$$P_{y,w}(q) = \sum_i \dim IH^{2i}_{yB/B}(\mathrm{Schub}_w)q^i$$

Here $IH^{2i}_{yB/B}(\mathrm{Schub}_w)$ denotes the local intersection cohomology of a Schubert variety.

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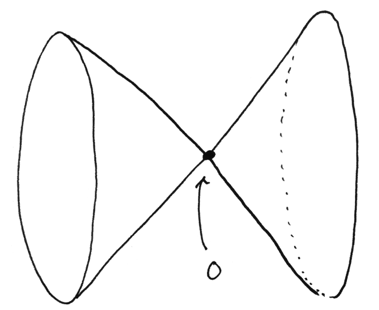
Example

Schubert varieties in Flag_2 are smooth $\Rightarrow (\Delta_\lambda : L_\mu) \in \{0, 1\}$ for $\mathfrak{sl}_2(\mathbb{C})$.

$$\{xy - zw = 0\} \subset \mathbb{C}^4$$



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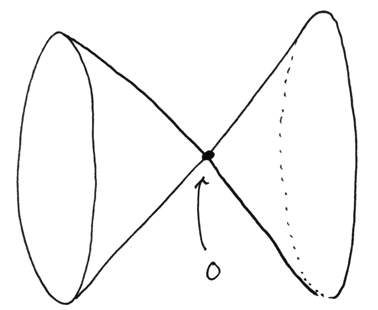

IH^i		0	1	2	3	…
local intersection at a smooth point	:	$\mathbb{Q}$	0	0	0	…
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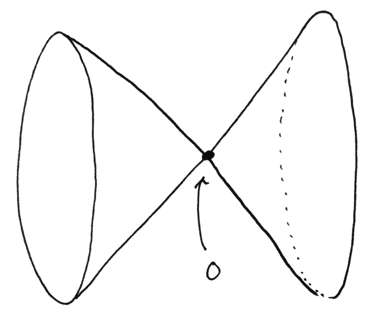

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Proof (1980) by Beilinson-Bernstein and Brylinski-Kashiwara:

$\mathfrak{g}$-modules (e.g. Verma modules)	$\rightsquigarrow$	D-modules on flag variety	$\rightsquigarrow$	perverse sheaves on flag variety	$\rightsquigarrow$	KL polynomials

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Beilinson-Bernstein (1990): similar ideas give Jantzen conjecture.

"The amazing feature of the proof is that it does not try to solve the problem but just keeps translating it in languages of different areas of mathematics (further and further away from the original problem) until it runs into Deligne's method of weight filtrations which is capable to solve it. So have a seat; it is going to be a long journey."

Joseph Bernstein, "Lectures on *D*-modules".

Consider $R = \mathbb{R}[x_1, x_2, \ldots, x_n]$ with its natural $W = S_n$-action.

Symmetric polynomials:

$$R^W = \mathbb{R}[\underbrace{x_1 + x_2 + \cdots + x_n}_{= e_1}, \underbrace{x_1x_1 + x_1x_3 + \cdots + x_{n-1}x_n}_{= e_2}, \ldots, \underbrace{x_1x_2 \ldots x_n}_{= e_n}]$$

Consider the coinvariant ring

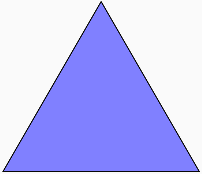
$$H := R/(e_1, e_2, \ldots, e_n).$$

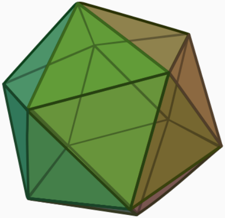
Borel isomorphism

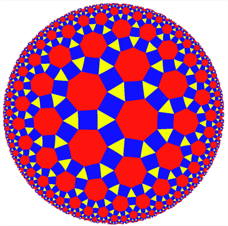
$$H^*(\mathrm{Flag}_n, \mathbb{R}) = H.$$

This gives a simple algebraic description of the cohomology ring.

SHADOWS OF HODGE THEORY







Weyl groups $\subset$ Real reflection groups $\subset$ Coxeter groups

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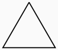
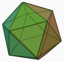
Example

$\mathrm{Sym}\Big($  $\Big) \circlearrowright \mathbb{R}^2$ or $\mathrm{Sym}\Big($  $\Big) \circlearrowright \mathbb{R}^3$.

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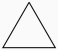
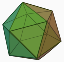
$\mathrm{Sym}\Big($  $\Big) \circlearrowright \mathbb{R}^2$ or $\mathrm{Sym}\Big($  $\Big) \circlearrowright \mathbb{R}^3$.

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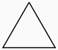
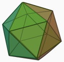
$\mathrm{Sym}\Big($  $\Big) \circlearrowright \mathbb{R}^2$ or $\mathrm{Sym}\Big($  $\Big) \circlearrowright \mathbb{R}^3$.

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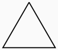
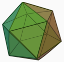
$\mathrm{Sym}\Big($  $\Big) \circlearrowright \mathbb{R}^2$ or $\mathrm{Sym}\Big($  $\Big) \circlearrowright \mathbb{R}^3$.

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Remark

If W is the Weyl group of a complex semi-simple Lie algebra $\mathfrak{g}$, then H is isomorphic to the cohomology of the flag variety of $\mathfrak{g}$ (the “Borel isomorphism”).

$$H := R/(R^W_+)$$

$d :=$ number of reflecting hyperplanes in $\mathfrak{h}_{\mathbb{R}}$

"complex dimension of flag variety"

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There exists a unique (up to scalar) bilinear form

$$\langle -, - \rangle : H^{d-\bullet} \times H^{d+\bullet} \to \mathbb{R}$$

satisfying $\langle \gamma c, c' \rangle = \langle c, \gamma c' \rangle$ for all $\gamma, c, c' \in H$ (the invariant form).

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Remark

$\langle -, - \rangle$ is the analogue of the intersection form on cohomology.

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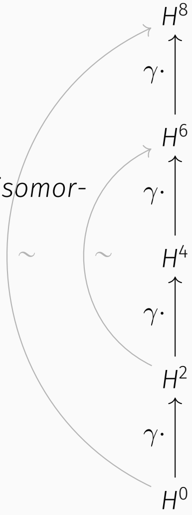
Theorem (toy model)

For all $\gamma \in K$ and $i \geqslant 0$:

(a) (Hard Lefschetz) Multiplication by γ^i induces an isomorphism

$$H^{d-i} \xrightarrow{\sim} H^{d+i}$$



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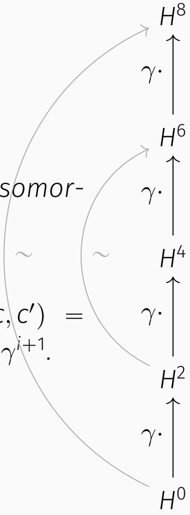

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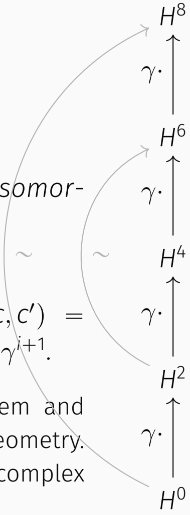
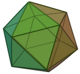
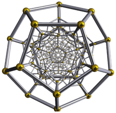


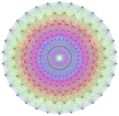
The theorem is identical to the hard Lefschetz theorem and Hodge-Riemann bilinear relations in complex algebraic geometry. In the Weyl group case the theorem can be deduced from complex algebraic geometry, but not in general.

H_3: symmetries of  :

1 3 5 7 9 11 12 12 12 12 11 9 7 5 3 1

H_4: symmetries of 120 cell  $\subset \mathbb{R}^4$:

1 4 9 16 25 36 49 64 81 100 121 144 168 192 216 240 264 288 312 336 359 380 399 416 431 444 455 464 471 476 478 476 471 464 455 444 431 416 399 380 359 336 312 288 264 240 216 192 168 144 121 100 81 64 49 36 25 16 9 4 1

E_8: 

1 8 35 112 294 672 1386 2640 4718 8000 12978 20272 30645 45016 64470 90264 123829 166768 220849 287992 370250 469784 588833 729680 894613 1085880 1305640 1555912 1838523 2155056 2506798 2894688 3319268 3780640 4278429 4811752 5379194 5978792 6608029 7263840 7942628 8640288 9352240 10073472 10798593 11521896 12237428 12939064 13620586 14275768 14898464 15482696 16022740 16513208 16949127 17326016 17639957 17887656 18066494 18174568 18210722 18174568 18066494 17887656 17639957 17326016 16949127 16513208 16022740 15482696 14898464 14275768 13620586 12939064 12237428 11521896 10798593 10073472 9352240 8640288 7942628 7263840 6608029 5978792 5379194 4811752 4278429 3780640 3319268 2894688 2506798 2155056 1838523 1555912 1305640 1085880 894613 729680 588833 469784 370250 287992 220849 166768 123829 90264 64470 45016 30645 20272 12978 8000 4718 2640 1386 672 294 112 35 8 1

BACK TO THE KAZHDAN-LUSZTIG CONJECTURE

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We provided an algebraic proof of (1) as a consequence of

Theorem (Elias-W.)

The hard Lefschetz and Hodge-Riemann relations hold for H_w.

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Diagrammatic algebra crucial to calculate and discover correct statements.

Basic lemma

Suppose that $\langle -, - \rangle_t$ is a family of geometric structures, for $t \in (a, b) \subset \mathbb{R}$. Then all $\langle -, - \rangle_t$ have the same signature.

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This implies:

Deformations of Lefschetz operators

Suppose that L_t is a family of Lefschetz operators, for $t \in (a,b)$.

If one L_t satisfies the Hodge-Riemann relations, then they all do.

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Corollary (Elias-W. 2013)

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A mystery for the 21^{st} century?

Similar structures arise in the theory of non-rational polytopes (due to McMullen, Braden-Lunts, Karu, ...) and in recent work of Apridisato-Huh-Katz on matroids. Why?

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Theorem (W. 2016)

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Again invariant forms, their deformations and the Hodge-Riemann relations play a crucial role.

MODULAR REPRESENTATIONS

We now turn to modular representations. That is, representations over fields of characteristic $p > 0$.

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There are fascinating connections to number theory and to algebraic geometry.

Recall the following example from the first lecture:

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JORDAN-HÖLDER THEOREM

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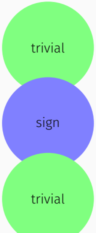
We write ("Grothendieck group")

$$[\mathbb{F}_3^3] = [L] + [H/L] + [\mathbb{F}_3^3/H] = 2[\mathrm{trivial}] + [\mathrm{sign}].$$



These are examples of decomposition numbers.

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In fact, the study of simple representations of arbitrary algebraic groups reduces to the case of reductive algebraic groups.

There are analogies between infinite-dimensional representations of Lie algebras, and algebraic representations of algebraic groups. The analogue of the Kazhdan-Lusztig conjecture in this setting is the Lusztig conjecture (1980).

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Invariant forms (now defined over the integers) still play a decisive role in the theory.

Lusztig conjecture (1980)

$$[\hat{\Delta}_A] = \sum_B q_{A,B}(1)[\hat{L}_B]$$

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(d) False for primes growing exponentially in the rank (W. 2014, following He-W. 2013), e.g. false for $p = 470\,858\,183$ for SL_{100}.

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${}^p q_{A,B}$ are computable via diagrammatic algebra + computer.

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We obtain in this way a bijection:

$$\left\{ \begin{array}{c} \text{simple representations} \\ \text{of the symmetric group } S_n \end{array} \right\}_{/\cong} \xrightarrow{\sim} \{\text{partitions } \lambda \text{ of } n\}.$$

Because the Specht modules V_λ are represented by integral matrices, we can reduce them modulo p to obtain modular representations $\mathbb{F}_p \otimes_{\mathbb{Z}} V_\lambda$.

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If λ has two rows then the answer is given by a fractal tree.

DECOMPOSITION NUMBERS FOR TWO ROW PARTITIONS

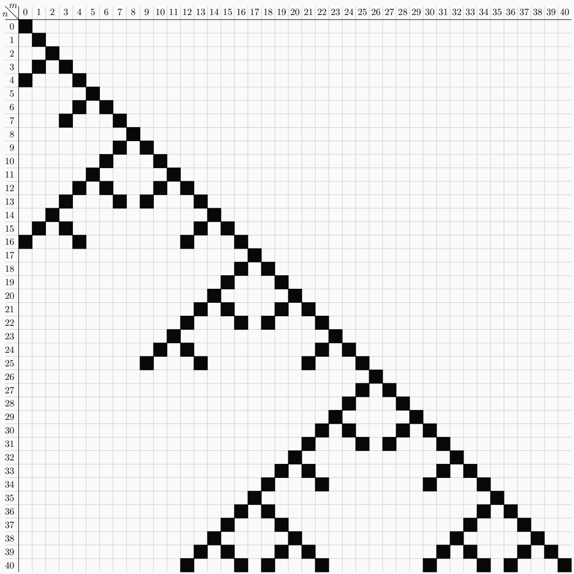

FIGURE 1. The multiplicities of $\Delta(m)$ in $T(n)$ for $p = 3$.

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The conjecture predicts that these numbers are given by a "discrete dynamical system"...

Billiards and tilting characters:
`https://www.youtube.com/watch?v=Ru0Zys1Vvq4`