# LANGLANDS CORRESPONDENCE AND BEZRUKAVNIKOV'S EQUIVALENCE

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> "This regularity, at first blush so simple, is the germ of one of the major branches of the higher number theory and was the central theme of a development that began with Euler and Legendre in the eighteenth century, and continued down to our own time, with contributions by Gauss, Kummer, Hilbert, Takagi, and Artin. Current efforts to extend it will be the theme of this essay."

R. P. Langlands, introduction to [Lan90].

## Introduction

This course will be in two parts.

(1) I will attempt to give a picture of what the Langlands correspondence is about, from an arithmetical point of view. I will start of with some basic questions (e.g. counting points of varieties over finite fields) and show how they lead to interesting $L$-functions which should have an automorphic incarnation. Explaining this will involve rather a lot of algebraic number theory, which I will try to go over. (For students, Gus' course last semester will be very useful!) I will then pass to the local case. I will review the structure theory of local fields and their Galois groups and state the main theorems of local class field theory. I will then explain what the local Langlands correspondence should be, and we will see that it boils down to class field theory if the $GL_1$ case. I will sketch a beautiful heuristic argument for the local Langlands correspondence for $GL_2$ when $p$ is not 2, which can be turned into a proof which I will not go into (so-called Jacquet-Langlands correspondence).

(2) The second half of the course will focus on affine Hecke algebras and their categorifications. Roughly speaking this part of the course is about the local Langlands correspondence at the easiest "layers of difficulty", which is still extremely rich. I will explain the two simplest instances of the local Langlands correspondence for a general $G$, namely the case of unramified representations (where the Langlands correspondence boils down to the Satake isomorphism), and the case of tamely ramified representations (so-called Deligne-Langlands conjecture). The Deligne-Langlands conjecture was proved by Kazhdan and Lusztig (and later Ginzburg). I will try to outline their proof. I will then explain how categorifying Kazhdan and Lusztig's proof leads to a remarkable equivalence conceived by Bezrukavnikov in the late 90s but only recently written down. If time permits I will try to outline the main ideas of the proof of his theorem.

This is an ambitious course in terms of scope, and, unless you have significant background, will require work and reading outside of the lectures. I will certainly not prove everything, however I will try to make everything as explicit as I can for $GL_1$ and $GL_2$. There is some chance it will spill over into second semester, depending on the interest of the participants and the stamina of the lecturer! Below I have included a bibliography of the sources I have found useful whilst preparing this course. I will maintain an updated version of this document on the web-page of the course:

`http://www.maths.usyd.edu.au/u/geordie/LanglandsAndBezrukavnikov/`

## Questions / office hours

Please do *not* simply knock unannounced on my door! My e-mail address and office number is below. I am happy to answer questions about this course via e-mail, and to make appointments to discuss in detail. If you want to discuss the course you are also welcome to join our "working group lunch" on Tuesdays and Thursdays at 12:30. Topics related to this course will also be covered in our "informal Friday seminar" which takes place on Friday 3 - 5pm. For topics in this seminar see:

`https://sites.google.com/view/ifssydney/home`

## References


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