Ramification filtration revisited

$$\begin{tikzcd} L \arrow[r, phantom, "\supset"] \arrow[d, dash] & \mathcal{O}_L \arrow[r, phantom, "\supset"] \arrow[d, dash] & m_L \arrow[d, dash] \\ K \arrow[r, phantom, "\supset"] & \mathcal{O}_K \arrow[r, phantom, "\supset"] & m_K \end{tikzcd}$$

K local field $\supset \mathcal{O}_K \supset m_K = (\pi_K)$.

$$\begin{tikzcd} L \arrow[d, dash] \\ K \end{tikzcd}$$ finite extension: $I_{L/K} \hookrightarrow \mathrm{Gal}(L/K) \twoheadrightarrow \mathbb{Z}/(f)$.

Lemma: $\sigma \in I = I_{L/K}$ is determined by $\sigma(\pi_L)$.

⇝ ramification filtration $I_0 = I$

$I_j = \{\sigma \in I \mid \sigma(\pi_L)\pi_L^{-1} \in 1 + m_L^j\}$.

Key points: ① finite filtration: $\mathrm{Gal}(L/K) \supset I_0 \supset I_1 \supset \dots \supset \{1\}$.

② $I_0/I_1 \hookrightarrow k_L^\times : \sigma \mapsto \sigma(\pi_L)\pi_L^{-1} \in (\mathcal{O}_L/m_L)^\times = k_L^\times$.

Hence cyclic or oder prime to p.

③ $I_j/I_{j+1} \hookrightarrow (k_L, +) : \sigma \mapsto \sigma(\pi_L)\pi_L^{-1} \in (1+m_L^{j-1})/(1+m_L^j) \cong k_L$.

Hence a p-group, thus I_1 is Sylow p subgroup of I.

Picture:

$$\begin{array}{rcl}
\text{wild part}\ \Big\{ & \vdots & \\
& | & \\
& I_1 & \\
& | & \Big\}\ \text{cyclic of order prime to } p. \ (\text{tamely ramified}) \\
& I_0 & \\
& | & \Big\}\ \mathbb{Z}/f \quad (\text{unramified part}) \\
& \mathrm{Gal} &
\end{array}$$

These are significant constraints on local Galois groups, e.g. always solvable.

Example: ① $(p-1)^{st}$ roots of unity exist in $\mathbb{Q}_p$.

$$\begin{tikzcd} L = \mathbb{Q}_p(\overset{\pi_L}{\overset{\shortparallel}{\sqrt[p-1]{p}}}) \arrow[d, dash] \\ K = \mathbb{Q}_p \end{tikzcd}$$

then $\mathrm{Gal}(L/K) = \{\sigma : \pi_L \mapsto \zeta\pi_L \mid \zeta \in \mu_{p-1}\}$

$$\begin{tikzcd} \mathrm{Gal}(L/K) \arrow[d, phantom, "\shortparallel"] & \\ k_K^\times \arrow[r, hook] & k_L^\times. \end{tikzcd}$$

"tamely ramified extension".

② Examples of wild ramification are almost always **hard!**

Structure of absolute Galois groups of local fields

$$I \hookrightarrow \mathrm{Gal}(\bar{K}/K) \twoheadrightarrow \hat{\mathbb{Z}}$$

Would like to pass to the limit to get a ramification filtration on I. Problem is that if L' | L | K then ramification filt. on $I_{L'/K}$ are related to multiples of ramification filtration on $I_{L/K}$.

⇝ via a crazy procedure produce an "upper numbering" via $\mathbb{Q}_{\geq 0}$ which is compatible under field extensions.

$$\begin{tikzcd}
\vdots \\
I^{4/75} & & \text{wild part} \\
\vdots & \text{not canonical} \\
I^{>0} \arrow[d, dash] & \cong \prod_{\ell \neq p} \mathbb{Z}_\ell & \text{tame part} \\
I^{0} & = \hat{\mathbb{Z}} & \text{unramified part} \\
\mathrm{Gal} & \text{canonical}
\end{tikzcd}$$

Important points: ① first two steps only depend on residue char.

② Via class field theory, image of filtration on I in $W_K^{ab} \cong K^\times$ corresponds to filtration by $1 + \mathfrak{m}_K^j \subset \mathcal{O}_K^\times$

Fact that jumps are only at integers (as opposed to $\mathbb{Q}$) is the Hasse-Arf theorem.

More details on Weil-Deligne reps

We want to explain why a general Weil-Deligne rep. is "close" to a cts. rep of $\mathrm{Gal}(\overline{K}/K)$, and why $p=2$ is special in the local Langlands correspondence.

Proposition: Any indecomposable F-semi-simple Weil-Deligne rep. is isomorphic to $\mathrm{St}_n \otimes \rho$, for ρ an irreducible rep of W_K.

Here: e.g.
$$\mathrm{St}_4 = \left(\begin{pmatrix} |\cdot|^3 & & & \\ & |\cdot|^2 & & \\ & & |\cdot| & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \right).$$

This is left as a somewhat tricky exercise.
It becomes much easier if you know what the "weight filtration" of a nilpotent operator is.

Proposition:

ⓐ Let $\rho : W_K \longrightarrow GL(V)$ be an irreducible rep. Then there exists $\chi : W_K \to \mathbb{C}^\times$ cts such that $\rho \otimes \chi$ has finite image, and hence defines a representation $\mathrm{Gal}_K \xrightarrow{\rho\otimes\chi} GL(V)$.

ⓑ Suppose that $\rho : W_K \to GL(V)$ is irreducible and not induced from any proper subgroup. Then $\rho|_{I_{>0}}$ is irreducible. In particular, $\dim V$ is a power of p. (Any irreducible module over a p-group has dimension divisible by p.)

These two statements are reasonably easy consequences of:

Lemma: Suppose G has the form:

$$\Gamma \hookrightarrow G \twoheadrightarrow \mathbb{Z} \quad \text{w/ } \Gamma \text{ finite.}$$

Then any irreducible G-module is either irreducible over Γ or induced from a subgroup of the form $\Gamma \rtimes m\mathbb{Z}$.

The proof of this lemma is a worthwhile exercise!

Why is LLC for $p=2$ special? The above results imply that, for $n=2$, all irred. reps for $p \neq 2$ are induced. Do there exist non-induced reps for $p=2$? Yes!

$$\begin{tikzcd}
\mathrm{Gal}(\overline{K}/K) \arrow[r] & GL_2(\mathbb{C}) \arrow[d] & \\
& PGL_2(\mathbb{C}) \supset SO_3 & \text{maximal compact} \arrow[l]
\end{tikzcd}$$

finite subgroups of SO_3 are known:

- cyclic $\leftrightarrow$ reducible
- dihedral $\leftrightarrow$ induced
- Sym (tetrahedron) $= A_4$
- Sym (cube) $\overset{?}{=}$
- Sym (icosahedron) $= A_5$ (impossible, not solvable)

$$\mathbb{Z}/2 \times \mathbb{Z}/2 \hookrightarrow A_4 \twoheadrightarrow \mathbb{Z}/3\mathbb{Z}$$

$\uparrow$ this subgroup structure only possible for a local Galois group if $p = 2$!

A mystery to ponder during the break:

Let G denote a compact Lie group, e.g. a finite group.

Let $R(G)$ = representation ring of G.

What is a character $\theta : R(G) \to \mathbb{C}$?

There is a beautiful answer!

← Unramified representations:

One good way way to convince yourself that LLC is amazing is to see special cases already having striking consequences. Ex1: CFT. Ex2: Now!

LLC:

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{F-semi-simple} \\ \text{n-dim. Weil-Deligne} \\ \text{representations}\end{array}\right\}/\cong \arrow[r, leftrightarrow, "1:1", "\text{``canonical''}"'] \arrow[d, phantom, "\cup"] & \left\{\begin{array}{c}\text{irreducible} \\ \text{smooth admissible} \\ \text{reps of } GL_n(K)\end{array}\right\}/\cong \arrow[d, phantom, "\cup"] \\
\text{``unramified reps''} \left(\begin{array}{l}\text{i.e. } N=0 \text{ and } \rho : W_K \to GL_n(\mathbb{C}) \\ \text{factors over } W_K \twoheadrightarrow \mathbb{Z}\end{array}\right) \arrow[r, leftrightarrow, "1:1"] \arrow[d, phantom, "\cong"] & \text{``spherical reps''} \left(\begin{array}{c}\text{reps of } GL_n(K) \\ \text{w/ a } GL_n(\mathcal{O}_K) \\ \text{fixed vector}\end{array}\right) \arrow[d, leftrightarrow, "S"] \\
\{\text{semi-simple reps } \rho : \mathbb{Z} \to GL_n(\mathbb{C})\}/\cong \arrow[d, phantom, "\cong"] & \begin{array}{c}\text{irreducible reps of} \\ \text{``Hecke algebra''} \\ \mathcal{H}(GL_n(K), GL_n(\mathcal{O}_K))\end{array} \\
\{\text{semi-simple elts } x \in GL_n(\mathbb{C})\}/\text{conjugacy} \arrow[ur, leftrightarrow, "\text{``Satake isomorphism''}"'] &
\end{tikzcd}$$

The goal now is to explain this diagram. The left hand side should be pretty clear, the right hand side needs some work!

Remark: The LHS is independent of K and even p = residue characteristic.

Hecke algebras: Suppose G is a finite group.

Case 1: $N \trianglelefteq G$ normal:

Given V, $G \circlearrowright V^N$ (because $n \cdot gv = g \cdot \underbrace{g^{-1}ng}_{\in N} \cdot v = g \cdot v$)

and factors over G/N. Moreover:

$$\begin{tikzcd}
\mathrm{Rep}\, \mathbb{C}G \arrow[d, phantom, "\cup"] & \\
\langle \mathbb{C}[G/N] \rangle \arrow[r, "\sim", "{\mathrm{Hom}(\mathbb{C}[G/N], -)}"'] \arrow[d, phantom, "\shortparallel"] & \mathrm{Rep}\, \mathbb{C}[G/N] \\
\mathrm{ind}_N^G \mathbb{C} &
\end{tikzcd}$$

Case 2: $H \leq G$ not necessarily normal.

Given $V \in \mathrm{Rep}\, G$, what acts on V^H? The Hecke algebra!

$$\begin{tikzcd} V^H \arrow[r, "{[HgH]}"] \arrow[d] & V^H \\ V \arrow[r, "g\cdot"] & V \arrow[u, "\pi_H"'] \end{tikzcd}$$

$$\pi_H := \frac{1}{|H|} \sum_{h \in H} h : V \longrightarrow V^H$$

(projects to H-invariants)

(Note that g only depends on its (H, H) double coset.)

Alternative description: $\mathcal{H}(H, G) = {}^{H}\mathbb{C}[G]^{H}$ with multiplication

$$(f * f')(g) = \frac{1}{|H|} \sum_{g = hh'} f(h) f'(h)$$

$\uparrow$ factor makes $\mathbb{1}_H$ the unit.

Example: ① $N \leq G$ normal, $\mathcal{H}(N, G) = \mathbb{C}[G/N]$.

② $G = GL_n(\mathbb{F}_q) \supset B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, $\mathcal{H}(G, B) =$ "Hecke algebra of S_n at $q = |\mathbb{F}_q|$"

(almost independent of q).

Exercise (do it!) $\mathrm{End}(\mathbb{C}[G/H]) \xrightarrow{\sim} \mathcal{H}(H, G)$. Deduce that

$\mathbb{C}[G/H] = \mathrm{ind}_H^G \mathbb{C}$

$$\left\{ \begin{matrix} \text{irreps of } G \\ \text{w/ } H\text{-fixed} \\ \text{vector} \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{irred.} \\ \mathcal{H}(H,G)\text{-} \\ \text{modules} \end{matrix} \right\}$$

$\mathrm{Rep}\, \mathbb{C}G$

$\cup$

Hence: $\langle \mathbb{C}[G/H] \rangle \xrightarrow{\sim} \mathcal{H}(H, G)\text{-mod}$

Remark: There is a tendancy in the literature to concentrate on one H at a time, but one can consider all subgroups at a time, forming a "Hecke algebroid" ...

The Hecke algebra(s) of a p-adic group

Let $G = GL_n(K)$

$$\mathcal{H}^{BIG} = \{ \varphi : G \to \mathbb{C} \mid \varphi \text{ locally constant and compactly supported} \}$$

nice exercise!

$$= \bigcup_i \{ \varphi : G \to \mathbb{C} \mid \varphi \text{ constant on } K_i \text{ double cosets, } \neq 0 \text{ on finitely many} \}$$

Algebra under convolution: $(f * f')(g) = \int_G f(h) f'(h^{-1}g)\, d\mu$

μ Haar measure on G

E.g.

$$(\mathbb{1}_{K_i} * \mathbb{1}_{K_i})(g) = \int_G \mathbb{1}_{K_i}(h)\, \mathbb{1}_{K_i}(h^{-1}g)\, d\mu$$

$$= \int_{K_i} \mathbb{1}_{K_i}(h^{-1}g)\, d\mu = \begin{cases} 0 & \text{if } g \notin K_i \\ \int_{K_i} d\mu = \mu(K_i) & \text{if } g \in K_i \end{cases}$$

Hence $\mathbb{1}_{K_i}$'s are "quasi-idempotents": $\left(\frac{1}{\mu(K_i)} \mathbb{1}_{K_i}\right)^{*2} = \frac{1}{\mu(K_i)} \mathbb{1}_{K_i}$.

Because any smooth admissible rep has $V^{K_i} \neq 0$ for some K_i, $\mathcal{H}^{BIG}$ can be used to understand all smooth admissible reps. On the other hand, it is very complicated.

Assume $\mu(K_0) = 1$, then $\mathcal{H}_{sph} = \mathcal{H}(G(\mathcal{O}_K), G(K))$ is the <u>spherical Hecke algebra</u>.

Cartan decomposition: $GL_n(K) = \bigsqcup_{\substack{\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \\ \lambda_i \in \mathbb{Z}}} GL_n(\mathcal{O}_K) \begin{pmatrix} \pi_K^{\lambda_1} & & \\ & \ddots & \\ & & \pi_K^{\lambda_n} \end{pmatrix} GL_n(\mathcal{O}_K).$

Hence $\mathcal{H}_{sph} = \bigoplus_{\substack{\underline{\lambda} \\ \underline{\lambda} = (\lambda_1 \geq \ldots \geq \lambda_n)}} \mathbb{C} \mathbb{1}_{\underline{\lambda}}$

Theorem: ① $\mathcal{H}_{sph}$ is commutative.

② There exists a canonical isomorphism

$$\operatorname{Rep} {}^L GL_n(\mathbb{C}) \xrightarrow{\sim} \mathcal{H}_{sph}.$$

(Satake isomorphism)

Rmk: ${}^L GL_n(\mathbb{C}) \cong GL_n(\mathbb{C})$, but theorem is true for general reductive groups and there dual group is important.

This theorem can be used to establish unramified LLC:

$$\begin{tikzcd}
\text{spherical reps} \arrow[d, leftrightarrow, "\text{def}"] & & & \\
\left\{\begin{array}{c}\text{irreducible smooth} \\ \text{admissible reps of} \\ GL_n(K) \text{ w/ } GL_n(\mathcal{O}_K)\text{-} \\ \text{fixed vector}\end{array}\right\} \arrow[r, "1:1", "\text{Hecke alg. magic}"'] & \left\{\begin{array}{c}\text{irreducible} \\ \mathcal{H}_{sph}\text{-modules}\end{array}\right\} \arrow[r, leftrightarrow, "1:1", "\mathcal{H}_{sph}\text{ commutative}"'] & \left\{\begin{array}{c}\text{characters} \\ \chi : \mathcal{H}_{sph} \to \mathbb{C}\end{array}\right\}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, leftrightarrow, "1:1", "\text{Satake isomorphism}"'] & \left\{\begin{array}{c}\text{characters} \\ \chi : R({}^L GL_n) \to \mathbb{C}\end{array}\right\} \arrow[r, leftrightarrow, "1:1"] & \left\{\begin{array}{c}\text{Semi-simple elts} \\ \text{in } {}^L GL_n(\mathbb{C})\end{array}\right\} / \text{conj.} \arrow[r, leftrightarrow, "\text{def.}"'] & \begin{array}{c}\text{unramified} \\ n\text{-dim} \\ \text{Weil-Deligne reps}\end{array}
\end{tikzcd}$$