

Ramification filtration revisited

K local field $\supset \mathcal{O}_K \supset \mathfrak{m}_K = (\pi_K)$.

$$\begin{array}{ccc} L \supset \mathcal{O}_L \supset \mathfrak{m}_L \\ | \quad | \quad | \\ K \supset \mathcal{O}_K \supset \mathfrak{m}_K \end{array}$$

L/K finite extension: $I_{L/K} \hookrightarrow \text{Gal}(L/K) \twoheadrightarrow \mathbb{Z}/(f)$.

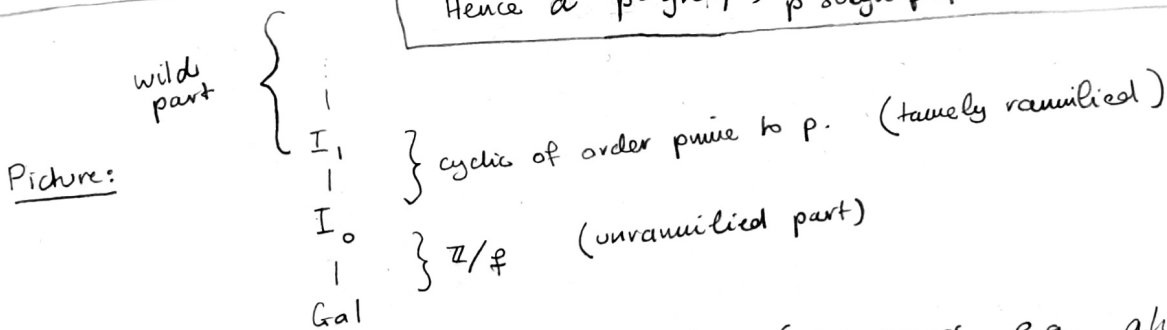
Lemma: $\sigma \in I = I_{L/K}$ is determined by $\sigma(\pi_L)$.

\Rightarrow ramification filtration $I_0 = I$
 $I_j = \{ \sigma \in I \mid \sigma(\pi_L) \pi_L^{-1} \in 1 + \mathfrak{m}_L^j \}$.

Key points: (1) finite filtration: $\text{Gal}(L/K) \supset I_0 \supset I_1 \supset \dots \supset \{1\}$.

(2) $I_0/I_1 \hookrightarrow k_L^\times : \sigma \mapsto \sigma(\pi_L) \pi_L^{-1} \in (\mathcal{O}_L/\mathfrak{m}_L)^\times = k_L^\times$.
 Hence cyclic of order prime to p .

(3) $I_j/I_{j+1} \hookrightarrow (k_L, +) : \sigma \mapsto \sigma(\pi_L) \pi_L^{-1} \in (1 + \mathfrak{m}_L^{j-1}) / (1 + \mathfrak{m}_L^j)$.
 Hence a p -group, thus I_1 is Sylow p -subgroup of I .



These are significant constraints on local Galois groups, e.g. always solvable.

Example: (1) $(p-1)^{\text{st}}$ roots of unity exist in \mathbb{Q}_p .

$$L = \mathbb{Q}_p(\sqrt[p-1]{\pi_L}) \\ | \\ K = \mathbb{Q}_p$$

then $\text{Gal}(L/K) = \{ \sigma : \pi_L \mapsto \delta \pi_L \mid \delta \in \mu_{p-1} \}$
 $k_K^\times \hookrightarrow k_L^\times$.

"tamely ramified extension"

(2) Examples of wild ramification are almost always hard!

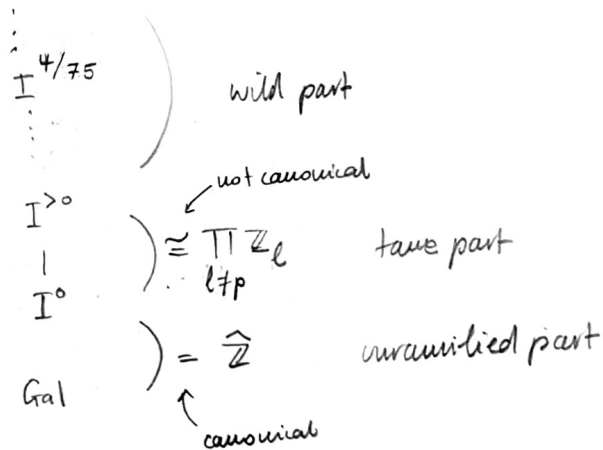
Structure of absolute Galois groups of local fields

$$I \hookrightarrow \text{Gal}(\bar{K}/K) \twoheadrightarrow \widehat{\mathbb{Z}}$$

Would like to pass to the limit to get a ramification filtration on I . Problem is that

if $\begin{matrix} L' \\ | \\ L \\ | \\ K \end{matrix}$ then ramification filt. on $I_{L'/K}$ are related to multiples of ramification filtration on $I_{L/K}$.

via a crazy procedure produce an "upper numbering" via $\mathbb{Q}_{\geq 0}$ which is compatible under field extensions.



Important points: ① first two steps only depend on residue char.

② Via class field theory, image of filtration on I in $W_K^{ab} \cong K^\times$ corresponds to filtration by $1 + m_K^j \subset \mathcal{O}_K^\times$

Fact that jumps are only at integers (as opposed to \mathbb{Q})

is the Hasse-Arf theorem.

More details on Weil-Deligne reps

We want to explain why a general Weil-Deligne rep. is "close" to a cts. rep of $\text{Gal}(\bar{K}/K)$, and why $p=2$ is special in the local Langlands correspondence.

Proposition: Any indecomposable F -semi-simple Weil-Deligne rep. is isomorphic to $\text{St}_n \otimes \rho$, for ρ an irreducible rep of W_K .

Here: e.g. $\text{St}_4 = \left(\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \right)$.

This is left as a somewhat tricky exercise.

It becomes much easier if you know what the "weight filtration" of a nilpotent operator is.

Proposition:

(a) Let $\rho: W_K \rightarrow \text{GL}(V)$ be an irreducible rep. Then there exists $\chi: W_K \rightarrow \mathbb{C}^\times$ cts such that $\rho \otimes \chi$ has finite image, and hence defines a representation $\text{Gal}_K \xrightarrow{\rho \otimes \chi} \text{GL}(V)$.

(b) Suppose that $\rho: W_K \rightarrow \text{GL}(V)$ is irreducible and not induced from any proper subgroup. Then $\rho|_{I \geq 0}$ is irreducible.

In particular, $\dim V$ is a power of p . (Any irreducible

module over a p -group has dimension divisible by p .)

These two statements are reasonably easy consequences of:

Lemma: Suppose G has the form:

$$\Gamma \hookrightarrow G \twoheadrightarrow \mathbb{Z} \quad \text{w/ } \Gamma \text{ finite.}$$

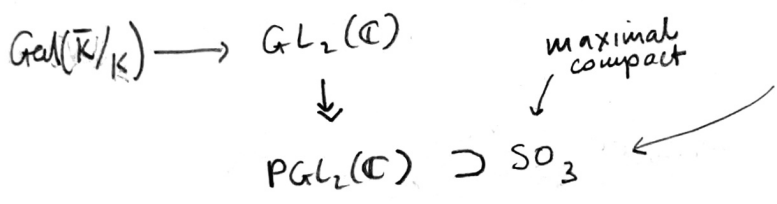
Then any irreducible G -module is either irreducible over Γ or induced from a subgroup of the form $\Gamma \rtimes m\mathbb{Z}$.

The proof of this lemma is a worthwhile exercise!

Why is LLC for $p=2$ special?

The above results imply that,

for $n=2$, all irred. reps for $p \neq 2$ are induced. Do there exist non-induced reps for $p=2$? Yes!



- finite subgroups of SO_3 are known:
- cyclic \leftrightarrow reducible
 - dihedral \leftrightarrow induced
 - Sym (tetrahedron) = A_4
 - Sym (cube) = ?
 - Sym (icosahedron) = A_5 (impossible not solvable)

$$\mathbb{Z}/2 \times \mathbb{Z}/2 \hookrightarrow A_4 \twoheadrightarrow \mathbb{Z}/3\mathbb{Z}$$

↑ this subgroup structure only possible for a local Galois group if $p=2$!

A mystery to ponder during the break:

Let G denote a compact Lie group, e.g. a finite group.

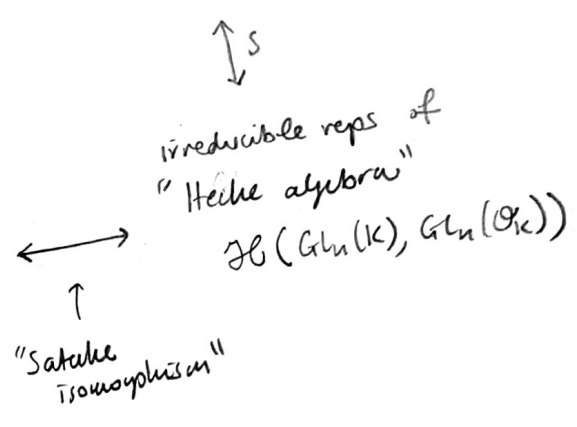
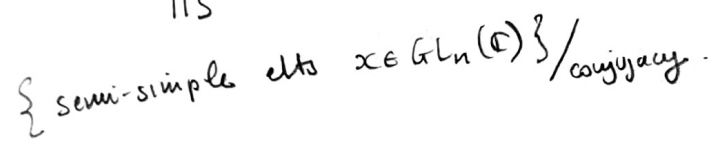
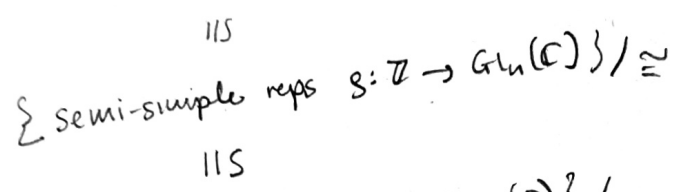
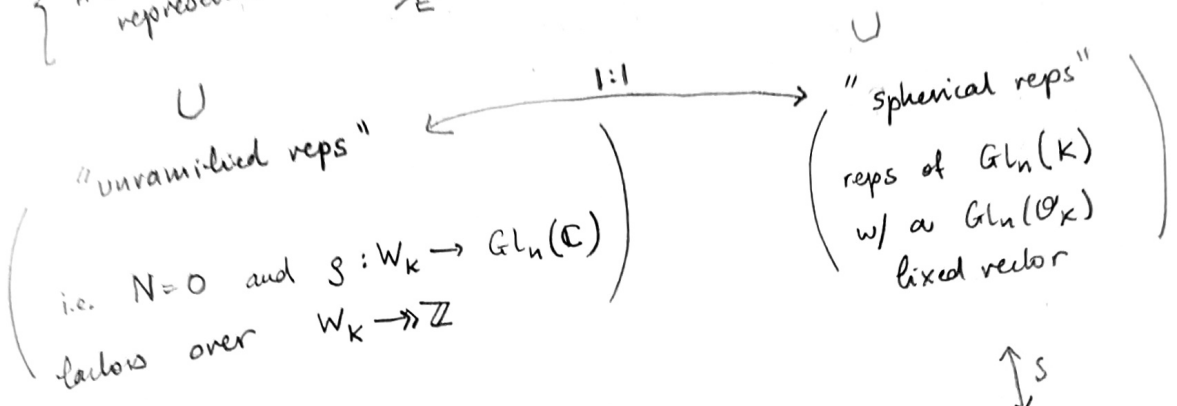
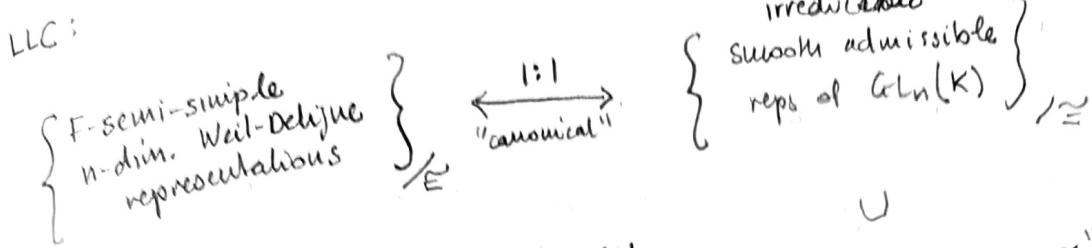
Let $R(G)$ = representation ring of G .

What is a character $\Theta: R(G) \rightarrow \mathbb{C}$?

There is a beautiful answer!

Unramified representations:

One good way way to convince yourself that LLC is amazing is to see special cases already having striking consequences. Ex1: CFT. Ex2: Now!



The goal now is to explain this diagram. The left hand side should be pretty clear, the right hand side needs some work!

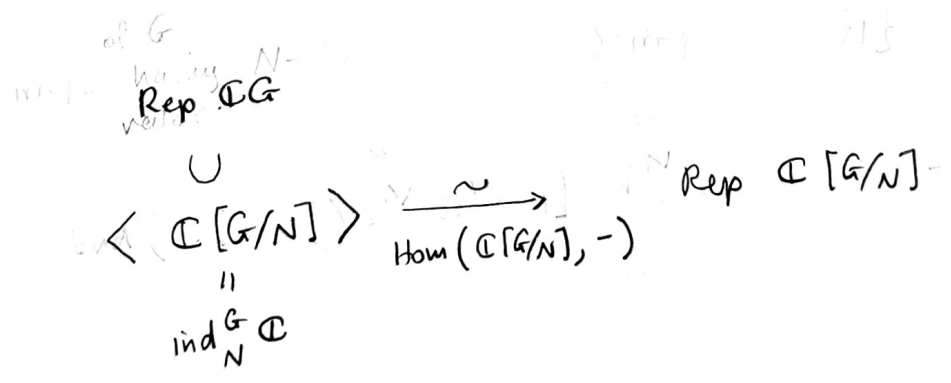
Remark: The LHS is independent of K and even $p \neq$ residue characteristic.

Hecke algebras: Suppose G is a finite group.

Case 1: $N \leq G$ normal:

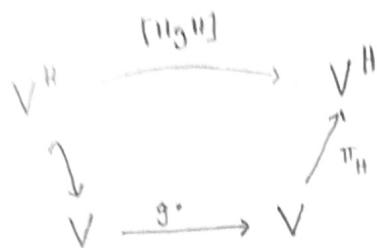
Given V , $G \subset V^N$ (because $n \cdot gv = g \cdot \overset{N}{g^{-1}ng} \cdot v = g \cdot v$)

and factors over G/N . Moreover:



Case 2: $H \leq G$ not necessarily normal.

Given $V \in \text{Rep } G$, what acts on V^H ? The Hecke algebra!



$$\pi_H := \frac{1}{|H|} \sum_{h \in H} h : V \rightarrow V^H$$

(projects to H -invariants)

(Note that g only depends on its (H, H) double coset.)

Alternative description: $\mathcal{H}(H, G) = {}^H \mathbb{C}[G]^H$ with multiplication

$$(f * f')(g) = \frac{1}{|H|} \sum_{g=hh'} f(h) f'(h')$$

↑ factor makes $\mathbb{1}_H$ the unit.

Example: ① $N \leq G$ normal, $\mathcal{H}(N, G) = \mathbb{C}[G/N]$.

② $G = GL_n(\mathbb{F}_q) \supset B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, $\mathcal{H}(G, B)$ = "Hecke algebra of S_n at $q = |\mathbb{F}_q|$ ".

(almost independent of q).

Exercise (do it!) $\text{End}(\mathbb{C}[G/H]) \cong \mathcal{H}(H, G)$. Deduce that

$$\text{Rep } \mathbb{C}G \quad \text{ind}_H^G \mathbb{C}$$

$$\left\{ \begin{array}{l} \text{irreps of } G \\ \text{w/ } H\text{-fixed} \\ \text{vector} \end{array} \right\} \cong \left\{ \begin{array}{l} \text{irred.} \\ \mathcal{H}(H, G)\text{-} \\ \text{modules} \end{array} \right\}$$

Hence: $\langle \mathbb{C}[G/H] \rangle \cong \mathcal{H}(H, G)\text{-mod}$

Remark: There is a tendency in the literature to concentrate on one H at a time, but one can consider all subgroups at a time, forming a "Hecke algebra" ...

The Hecke algebra(s) of a p-adic group

Let $G = \text{GL}_n(K)$

$$\mathcal{H}^{\text{BIG}} = \left\{ \varphi : G \rightarrow \mathbb{C} \mid \begin{array}{l} \varphi \text{ locally constant and} \\ \varphi \text{ compactly supported} \end{array} \right\}$$

Nice exercise!

$$\Downarrow \bigcup_i \left\{ \varphi : G \rightarrow \mathbb{C} \mid \begin{array}{l} \varphi \text{ constant on } K_i \text{ double cosets,} \\ \neq 0 \text{ on finitely many} \end{array} \right\}$$

Algebra under convolution:
 μ Haar measure on G

$$(\varphi * \varphi')(g) = \int_G \varphi(h) \varphi'(h^{-1}g) d\mu$$

E.g.:

$$(\mathbb{1}_{K_i} * \mathbb{1}_{K_i})(g) = \int_G \mathbb{1}_{K_i}(h) \mathbb{1}_{K_i}(h^{-1}g) d\mu$$

$$= \int_{K_i} \mathbb{1}_{K_i}(h^{-1}g) d\mu = \begin{cases} 0 & \text{if } g \notin K_i \\ \int_{K_i} d\mu = \mu(K_i) & \text{if } g \in K_i \end{cases}$$

Hence $\mathbb{1}_{K_i}$'s are "quasi-idempotents": $\left(\frac{1}{\mu(K_i)} \mathbb{1}_{K_i} \right)^{*2} = \frac{1}{\mu(K_i)} \mathbb{1}_{K_i}$.

Because any smooth admissible rep has $v^{K_i} \neq 0$ for some K_i ,

\mathcal{H}^{BIG} can be used to understand all smooth admissible reps.
On the other hand, it is very complicated.

Assume $\mu(K_0) = 1$, then $\mathcal{H}_{\text{sph}} = \mathcal{H}(G(\mathcal{O}_K), G(K))$

is the spherical Hecke algebra.

Cartan decomposition:

$$GL_n(K) = \prod_{\substack{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \\ \lambda_i \in \mathbb{Z}}} GL_n(\mathcal{O}_K) \begin{pmatrix} \pi_K^{\lambda_1} & & \\ & \dots & \\ & & \pi_K^{\lambda_n} \end{pmatrix} GL_n(\mathcal{O}_K).$$

Hence $\mathcal{H}_{\text{sph}} = \bigoplus_{\substack{\lambda \\ \lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)}} \mathbb{C} \mathbb{1}_{\lambda}$

Theorem:

① \mathcal{H}_{sph} is commutative.

② There exists a canonical isomorphism

$$\text{Rep } {}^L GL_n(\mathbb{C}) \xrightarrow{\sim} \mathcal{H}_{\text{sph}}. \quad (\text{Satake isomorphism})$$

Remark: ${}^L GL_n(\mathbb{C}) \cong GL_n(\mathbb{C})$, but theorem is true for general reductive groups and here dual group is important.

This theorem can be used to establish unramified LLC:

