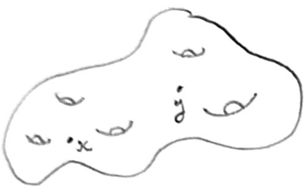
Two lectures ago: description of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is not really well-defined because it depends on a choice of $\overline{\mathbb{Q}}$. Thus $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is a "group up to conjugacy". I didn't explain this well, but the following analogy, which I learnt from K. Buzzard, is useful.

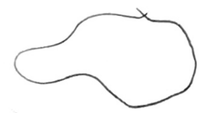
path connected space X



$\pi_1(X, x)$ fundamental group.

Isomorphism
$\pi_1(X, x) \xrightarrow{\sim} \pi_1(Y, y)$
depends on non-canonical choice of path $x \rightsquigarrow y$.

Grothendieck's picture:



$\mathbb{Q} \longleftrightarrow$ étale "site" $\mathrm{Spec}\,\mathbb{Q}$ (something like a space).

choice of $\overline{\mathbb{Q}}$ $\longleftrightarrow$ "base point" of $\mathrm{Spec}\,\mathbb{Q}$.

$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) = \pi_1^{et}(\mathrm{Spec}\,\mathbb{Q}, \overline{\mathbb{Q}})$.

VERBAL: This was

Punchline: $\pi_1(X, x)$ is not canonical but $\mathrm{Rep}\ \pi_1(X, x)$ is!

$\mathrm{Rep}\ \pi_1(X, x) \xrightarrow{\sim} \mathrm{Loc}(X) \leftarrow$ independent of choice of base point.

Last time we stated: Let K be a local field. W_K the Weil group.

Local Langlands conjecture for Gl_n;

$$\{n\text{-dim cts. reps of } W_K\} \xleftrightarrow{\sim} \{\text{irreducible representations of } \mathrm{Gl}_n(K)\}.$$

and we saw that it is true for $n=1$ by local class field theory.

Actually, this statement is still not quite correct, the correct statement is

$$\left\{\begin{array}{c} n\text{-dim } \underline{\text{Weil-Deligne}} \text{ reps} \\ \text{of } W_K \end{array}\right\} \xleftrightarrow{1:1} \left\{\begin{array}{c} \text{smooth admissible irreducible representations} \\ \text{of } \mathrm{Gl}_n(K) \end{array}\right\}$$

Rmk: irreducible $\Rightarrow$ admissible by a theorem of Jacquet, hence Deligne missing on LHS. ... missing on RHS.

No small subgroups argument:

This is a very useful lemma ... I'm not sure where it should go. So why not here!

A topological group G has <u>no small subgroups</u> if there exists a nbhd $\mathcal{U}$ of the identity s.t. any subgroup contained in $\mathcal{U}$ is trivial.

Groups w/ no small subgroups:

1. discrete groups, $U = \{id\}$ e.g. any finite group.

2. $\mathbb{R}$ 

or $\mathbb{C}$ 

3. any Lie group (use that $\exp : \mathrm{Lie}\, G \to G$ is local diffeo. and $\exp(mg) = \exp(g)^m$.)

<u>Lemma:</u> Suppose Γ is profinite and G has no small subgroups. Any continuous homomorphism $\varphi : \Gamma \to G$ has finite image.

<u>Proof:</u> Let $\mathcal{U} \subset G$ be as in def. of no small subgroups. Then $\varphi^{-1}(\mathcal{U}) \subset \Gamma$ is open. Hence there exists $N \subset \varphi^{-1}(\mathcal{U})$ normal, w/ G/N finite. Now $\varphi(N) \subset \mathcal{U}$ is a subgroup, hence is trivial. $\square$.

<u>Moral:</u> fractal like objects (p-adic groups, Galois groups) and Euclidean like objects (Lie groups) only intersect in finite groups.

E.g. Can't draw good pictures of $\mathbb{Z}_p$ in $\mathbb{C}$ comp. w/ addition/mult.

You could have guessed LLC for GL_2

Our goal here is to give a heuristic explanation for LLC.

Warning: This is imprecise, and much of what I say must be tweaked later. It is nonetheless extremely helpful (at least it was for me).

I learnt this way of looking at things from some beautiful lectures of Dipendra Prasad in Russia.

Start w/ finite reductive groups, e.g. $G = SL_2(\mathbb{F}_q)$.

Up to conjugacy, G has two max tori $T_s = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, $a \in \mathbb{F}_q^\times$ "split". $\cong \mathbb{Z}/(q-1)\mathbb{Z}$

$T_a = \{ x \in \mathbb{F}_{q^2}^\times \subset GL_2(\mathbb{F}_q) \mid \mathrm{Norm}(x) = 1 \}$

"anisotropic". $\cong \mathbb{Z}/(q+1)$.

Roughly: irreducible reps of $SL_2(\mathbb{F}_q)$ $\overset{1:1}{\longleftrightarrow}$ $\left\{ \begin{matrix} \text{characters} \\ \text{of } T_s \end{matrix} \right\} \Big/ \chi \sim \chi^{-1}$ $\sqcup$ $\left\{ \begin{matrix} \text{characters} \\ \text{of } T_a \end{matrix} \right\} \Big/ \chi \sim \chi^{-1}$

"principal series" "discrete series".

count $\sim$ char. polys $\sim q$ $= \dfrac{q-1}{2} + \dfrac{q+1}{2}$.

Another example: $G = SL_2(\mathbb{R})$.

G has two max tori: $T_s = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, $a \in \mathbb{R}^\times$ "split"

$T_a = SO_2 = \begin{pmatrix} \cos & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ "anisotropic".

Roughly: irreducible admissible reps. of $SL_2(\mathbb{R})$ $\overset{1:1}{\longleftrightarrow}$ $\underset{\substack{\text{"principal series"}}}{\overset{\mathbb{C} \cup \mathbb{C} \,\cong}{\left\{ \begin{matrix} \text{characters} \\ \text{of } T_s \end{matrix} \right\}}}$ $\sqcup$ $\overset{\mathbb{Z} \,\cong}{\left\{ \begin{matrix} \text{characters} \\ \text{of } T_a \end{matrix} \right\}}$

$T_s \cong \mathbb{R}_{>0}^{\times} \times \{\pm 1\}$

"principal series" "discrete series"

Let us dream that something similar is true for $GL_2(K)$:

Conjugacy classes of max. tori in $GL_2(K)$ $\xleftrightarrow{\sim}$ Semi-simple K-algebras L s.t. $\dim_K L = 2$.

Two cases:

① "split" $\quad L \cong K \times K$

② "anisotropic" $\quad L$ is a degree two extension of K.

Applying our analogy from earlier:

$$\left\{ \begin{array}{c} \text{irreducible} \ldots \\ \text{reps of } GL_2(K) \end{array} \right\} \xleftrightarrow[1:1]{\text{Roughly}} \underbrace{\left\{ \begin{array}{c} \text{pairs of chs.} \\ \chi_1, \chi_2 : K^\times \to \mathbb{C}^\times \end{array} \right\}}_{\text{"principal series"}} \sqcup \left\{ \begin{array}{c} \text{characters} \\ \theta : L^\times \to \mathbb{C}^\times \text{ where} \\ L/K \text{ is degree } 2 \end{array} \right\}$$

By local class field theory, $W_K^{\times} \xrightarrow{\sim} K^\times$ and $W_L^{\times} \xrightarrow{\sim} L^\times$.

Moreover, $W_L \hookrightarrow W_K \to \mathbb{Z}/2\mathbb{Z}$.

$$\xleftrightarrow{\sim} \overset{①}{\left\{ \chi_1 \oplus \chi_2 : W_K \to GL_2(\mathbb{C}) \right\}} \cup \overset{②}{\left\{ \operatorname{ind}_{W_L}^{W_K} \theta : W_K \to GL_2(\mathbb{C}) \right\}}.$$

<u>Fact:</u> If $p \neq 2$, all cts Galois reps $W_K \to GL_2(\mathbb{C})$ are of types ① or ②.

This gives us LLC for GL_2 to the first degree of accuracy.

<u>Rmk:</u> When $p = 2$ the same matching works, but there are more objects on both sides.

Basic rep. theory of p-adic groups

K local field, $GL_n(K)$ is a topological group with

a nbhd basis of 1 given by $K_n = \{ g \in GL_n(\mathcal{O}_K) \mid g = \mathrm{id} \bmod \mathfrak{m}_K^n \}$.

i.e. $GL_n(\mathcal{O}_K)/K_n \cong GL_n(\mathcal{O}_K/\mathfrak{m}_K^n) \leftarrow$ finite group.

Rmk: $K_n \subset GL_n(\mathcal{O}_K)$ are normal.

Exercise: K_0 is in fact a maximal subgroup.

$K_0 = GL_n(\mathcal{O}_K) \subset GL_n(K)$ is a <u>maximal compact subgroup.</u>

Eg: $GL_1(\mathbb{Q}_3) = \mathbb{Q}_3^\times = \mathbb{Z} \times \mathbb{Z}_3^\times = \mathbb{Z} \times \underset{\{\pm 1\}}{\underset{\cong}{\mu_2(\mathbb{Z}_3)}} \times \underset{\mathbb{Z}_3.}{\underset{\cong}{(1+3\mathbb{Z}_3)}}$.

$\cong \mathbb{Z} \times$  $\supset \{0\} \times$ 

maximal compact.

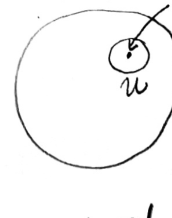
A space is <u>totally disconnected</u> if every point has a basis of nbhds which are compact and open.

Because each K_n is compact and open, $GL_n(K)$ is totally disconnected.

$\boxed{\text{We assume char } k = 0.}$

Let G be a totally disconnected group and V a vector space over a field k (possibly / probably!) ∞-dimensional. <u>We assume no topology on V.</u>

A representation $\pi : G \to GL(V)$ is



- <u>smooth</u> if $\forall v \in V$, the stabiliser of v is open.
- <u>admissible</u> if $\forall K \subset G$ open, V^K is finite dimensional.

Example 1: ① $\mathbb{k}$ is smooth, admissible.

$\mathbb{k}^\infty$ is smooth, not admissible.

② $GL_n(K) \curvearrowright K^n$ is not smooth, as $\mathrm{Stab}(\overset{0}{v})$ is not open.

In fact, if $\rho : GL_n(K) \to GL(V)$ is smooth, V f.d. $\Rightarrow \ker\rho = \bigcap \mathrm{Stab}(\text{basis})$ is open, normal

$\Rightarrow \ker\rho \supset SL_n(K) \Rightarrow \rho$ factors over det.

③ Consider: $\mathcal{F} = \{ \varphi : \mathbb{Z}_p \to \mathbb{C} \mid \varphi \text{ is locally constant} \}$.

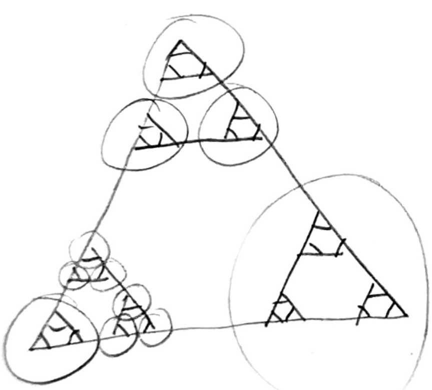
This is an ∞-dim rep of $\mathbb{Z}_p$ (the "smooth regular rep"),

p-adic analogue of $L^2(G)$, ...

Let us argue that $\mathcal{F}$ is smooth, admissible:

Fix $\varphi \in \mathcal{F}$, for any x we can find $U_x \ni x$ s.t. $f|_{U_x}$ constant.

ObdA, each U_x is of the form $x + p^{n_x}\mathbb{Z}_p$.



Compactness of $\mathbb{Z}_p \Rightarrow \exists$ finite subcover $U_{x_1} \cup \dots \cup U_{x_n}$.

Let $n = \max\{n_{x_i}\}$.

Then φ is fixed by $p^n\mathbb{Z}_p$. $\Rightarrow$ smooth.

A basis of open nbhds of 0 is given by $p^m\mathbb{Z}_p$.

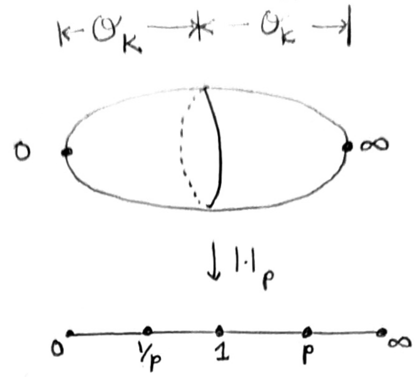
$$\mathcal{F}^{p^m\mathbb{Z}_p} = \left\{ \varphi \;\middle|\; \begin{array}{l}\varphi \text{ constant on} \\ p^m\mathbb{Z}_p \text{ cosets}\end{array} \right\} = \{ \varphi : \mathbb{Z}/p^m\mathbb{Z}_p \to \mathbb{C} \}$$

which is finite dim.

$B = \{z \mid \|z\| \leq 1\}$

$\mathbb{P}^1\mathbb{C} = B \sqcup B^{-1}$

④ Final example (very important).



Consider: $\mathbb{P}^1(K) = K \cup \{\infty\}$

$\mathbb{P}^1(K)$ covered by compact sets $\Rightarrow$ compact

(i.e. $\mathcal{O}_K = \{\lambda \in \mathbb{P}^1 \mid |\lambda|_p \leq 1\}$
$\cup \{\lambda \mid |\lambda| \geq 1\}$.)

Now: same arguments as before show that

$I = \{\varphi : \mathbb{P}^1(K) \to \mathbb{C} \mid \varphi \text{ locally constant}\}$ is smooth and admissible.

Exercise: $I/_{\text{triv}}$ is irreducible. $I/_{\text{triv}}$ is the Steinberg module.

(caveat: not sure how hard this is with current techniques)

Back to the case of GL_n. (For connoisseurs, theory works for any totally disconnected group). If $K_0, K_1, \ldots$ denote our sequence of open nbhds of the identity we can consider the chain

$$V^{K_0} \subset V^{K_1} \subset V^{K_2} \subset \cdots$$

smooth $\Rightarrow$ every vector lives in $V^{\mathcal{U}}$ for some $\mathcal{U} < G$ open, hence in some V^{K_i}. filtration is exhaustive.

admissible $\Rightarrow$ each V^{K_i} is finite dimensional.

K_0 action on V^{K_i} factors over $K_0/K_i \cong GL_n(\mathcal{O}_K/m_K^i)$

$\uparrow$ finite group e.g. $GL_n(\mathbb{Z}/p^m\mathbb{Z})$.

Hence: $$V^{K_i} = \bigoplus_{\rho \in \widehat{K_0/K_i}} \underbrace{V^{K_i}(\rho)}_{\rho \text{ isotypic component.}}$$

Passing to the limit: $$V = \bigoplus_{\rho \in \widehat{K_0}} \underbrace{V(\rho)}_{\rho \text{ isotypic component}}$$

<u>Lemma:</u> V admissible $\iff$ each $V(\rho)$ is finite-dimensional.

<u>Proof:</u> If $V(\rho)$ is infinite-dimensional for some ρ then $\ker \rho$ contains a K_i for some i, then $V(\rho) \subset V^{K_i}$ is ∞-dimensional, hence V^K is smooth.

For the other direction, note that ρ can occur in at most one of the reps

V^{K_0} $\quad V^{K_1}/V^{K_0}$ $\quad V^{K_2}/V^{K_1}$ $\quad V^{K_3}/V^{K_2}$ $\quad \ldots$

indeed, if $\rho \in V^{K_{i+1}}/V^{K_i}$ then ρ has K_{i+1} invariants, but no K_i invariants. $\square$

Because each of these spaces is finite dimensional, the result follows.

<u>Rmk:</u> This is like the theory of K-finite vectors in the rep. theory of real Lie groups. A big difference is that here we have no idea about the rep theory of e.g. $GL_{10}(\mathbb{Z}/p^{60}\mathbb{Z})$, whereas we know the rep. theory of compact Lie groups rather well.

<u>Examples:</u> ① $\mathcal{F} = \{\varphi : \mathbb{Z}_p \to \mathbb{C} \mid \varphi \text{ loc. constant}\}$, $\mathcal{F}^{p^m\mathbb{Z}_p} = \{\varphi \mid \varphi \text{ constant on } p^m\mathbb{Z}_p\text{-orbits}\}$

$= \text{reg rep. of } \mathbb{Z}_p/p^m\mathbb{Z}_p$.

Hence $$\mathcal{F} = \bigoplus_{\substack{\chi : \mathbb{Z}_p \to \mathbb{C}^\times \\ \text{cts}}} \mathbb{C}\chi$$

② $I = \{\varphi : \mathbb{P}^1(K) \to \mathbb{C} \mid \text{loc. constant}\}$. $\quad \mathcal{F}^{K_n} = \{\varphi : \mathbb{P}^1(\mathcal{O}_K/m_K^n) \to \mathbb{C}\}$

$$\mathcal{F} = \varinjlim \mathbb{C}[\mathbb{P}^1(\mathcal{O}_K/m_K^n)].$$