LECTURES ON THE GEOMETRY AND MODULAR REPRESENTATION THEORY OF ALGEBRAIC GROUPS

JOSHUA CIAPPARA AND GEORDIE WILLIAMSON

ABSTRACT. These notes provide a concise introduction to the representation theory of reductive algebraic groups in positive characteristic, with an emphasis on Lusztig's character formula and geometric representation theory. They are based on the first author's notes from a lecture series delivered by the second author at the Simons Centre for Geometry and Physics in August 2019. We intend them to complement more detailed treatments.

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INTRODUCTION

0.1. **Group actions.** In mathematics, group actions abound; their study is rewarding but challenging. To make problems more tractable, an important approach is to *linearise* actions and focus on the *representations* that arise. Historically, the passage from groups to representations was a non-obvious step, arising first in the works of Dedekind, Frobenius, and Schur at the turn of the last century.[1] Nowadays it pervades modern mathematics (e.g. the Langlands program) and theoretical physics (e.g. quantum mechanics and the standard model).

In the universe of all possible representations of a group, the ones we encounter by linearising are typically well behaved in context-dependent ways; we say these representations "occur in nature":

(1) Any representation of a finite group occurs inside a representation obtained by linearising an action on a finite set; thus all representations of finite groups "occur in nature". Over the complex numbers Maschke's theorem, Schur's lemma, and character theory provide powerful tools for understanding the entire category of representations.
(2) Lie group actions on smooth manifolds M induce representations on $L^2(M, \mathbb{C})$ and, more generally, on the sections and cohomology spaces of equivariant vector bundles on M. Here it is the *unitary representations* which are most prominent. The study of continuous representations of Lie groups is the natural setting for the powerful Plancherel theorems and abstract harmonic analysis.
(3) The natural permutation of polynomial roots by a Galois group Γ produces interesting representations after linearising (so-called *Artin representations*). More generally, Galois group actions on étale and other arithmetic cohomology theories produce continuous representations (so-called *Galois representations*) which are fundamental to modern number theory.

These notes concern algebraic representations of algebraic groups. In algebraic geometry, the actions that occur in nature are the *algebraic actions*; linearising leads to *algebraic representations*.[2] For example, an algebraic group G acting on a variety then acts algebraically on its regular functions. More generally, G acts algebraically on the sections and cohomology groups of equivariant vector bundles.

Among all algebraic groups, our main focus will be on the representation theory of *reductive algebraic groups*. These are the analogues in algebraic geometry of compact Lie groups. Indeed, over an algebraically closed field of characteristic zero, the representation theory of reductive algebraic groups closely parallels the theory of continuous finite-dimensional representations of compact Lie groups: the categories involved are semi-simple, simple modules are classified by highest weight, and characters are given by Weyl's famous formula.

Over fields of characteristic p the classification of simple modules is still by highest weight, but a deeper study of the categories of representations yields several surprises. First among these is the Frobenius endomorphism, which is a totally new phenomenon in characteristic p, and implies immediately that the categories of representations must behave differently to their characteristic zero cousins.

[1] For a fascinating account of this history, we recommend [Cur99].

[2] One often finds the term *rational representations* in the literature. We try to avoid this terminology here, as we find it often leads to confusion.

0.2. **Simple characters.** A basic question underlying these notes is the determination of the characters of simple modules. Understanding their characters is a powerful first step towards understanding their structure. Equally or perhaps more importantly, the pursuit of character formulas has motivated and been parallel to rich veins of mathematical development.

A beautiful instance of this was in the conjecture and proof of a character formula for simple highest weight modules over a complex semi-simple Lie algebra $\mathfrak{g}$. The resolution of the Kazhdan–Lusztig conjecture by Brylinski–Kashiwara [BK81] and Beilinson–Bernstein [BB81] in 1981 hinged on a deep statement relating D-modules on the flag variety of G to representations of $\mathfrak{g}$. The geometric methods introduced in [BB81] were one of the starting points of what is now known as *geometric representation theory*, and the localisation theorem remains a tool of fundamental importance and utility in this area.

The analogous question over algebraically closed fields of positive characteristic has resisted solution for a longer period and demanded the adoption of totally different approaches. From the time it was posited [Lus80] until very recently, the state of the art has been Lusztig's conjectural character formula for simple G-modules. Our main goal in these lectures will be to state and then examine this conjecture, particularly in terms of its connections to perverse sheaves and geometry. We conclude with a brief discussion of how the conjecture was found to be correct for large p, but also how the expected bounds were too optimistic. Moving along a fast route towards fundamental open questions in modular representation theory, we will encounter many of the objects, results, and ideas which underpin this discipline.

0.3. **Outline of contents.**

Lecture I: We introduce algebraic groups and their representations, as well as the Frobenius morphisms which give the characteristic p story its flavour.

Lecture II: We narrow the lens to reductive groups G and their root data, before making connections between the representation theory of G and the geometry of the flag variety G/B (for B a Borel subgroup).

Lecture III: We explore two analogous character formula conjectures: one for semi-simple Lie algebras in characteristic 0, due to Kazhdan–Lusztig, and one for reductive groups in characteristic p, due to Lusztig.

Lecture IV: We state Lusztig's conjecture more explicitly, before explaining its relation to perverse sheaves on the affine Grassmannian via the Finkelberg–Mirković conjecture.

Lecture V: We discuss the phenomenon of torsion explosion and its bearing on estimates for the characteristics p for which Lusztig's conjecture is valid. To finish, we give an illustrative example in an easy case, as well as indications of how the theory of intersection forms can be applied to torsion computations in general.

0.4. **Notation.** Throughout these notes, we fix an algebraically closed field k of characteristic $p \geqslant 0$; our typical focus will be $p > 0$. Unadorned tensor products are taken over k. Unless otherwise noted, modules are *left* modules.

0.5. **Acknowledgements.** We would like to thank all who took part in the summer school for interesting discussions and an inspiring week. We are grateful to the Simons Centre for hosting these lectures and for their hospitality.

Lecture I

1. Algebraic groups

We start by introducing algebraic groups and their duality with commutative Hopf algebras, using the functor of points formalism. We follow Jantzen [Jan03]. Readers desiring to pursue this material is greater depth will certainly require further details on both the algebraic and geometric sides; for this we recommend [Har77] and [Wat79] in addition to [Jan03].

Definition 1.1. A *k-functor* $\mathcal{X}$ is any (covariant) functor from the category of (commutative, unital) k-algebras to the category of sets:

$$\mathcal{X} : k\text{-Alg} \to \text{Set}.$$

Such k-functors form a category k-Fun with natural transformations as morphisms.

When first learning algebraic geometry, we think of a k-variety X as the subset of affine space k^n defined by the vanishing of an ideal $I \subseteq k[x_1, \ldots, x_n]$. Grothendieck taught us to widen this conception of a variety by considering the vanishing of I over any base k-algebra A:

$$X(A) = \{a \in A^n : f(a) = 0 \text{ for all } f \in I\}.$$

In other words, $X(A)$ is the solutions in A of the equations defining X. The association $A \mapsto X(A)$ extends to a k-functor $\mathcal{X} : k\text{-Alg} \to \text{Set}$ as follows: if $\varphi : A \to B$ is a k-algebra homomorphism, then the identity

$$f(\varphi(a)) = \varphi(f(a)), \quad f \in I,$$

shows there is an induced mapping $X(A) \to X(B)$. In this way, k-varieties provide the most important examples of k-functors.

The bijection $X(A) \cong \text{Hom}_{k\text{-Alg}}(k[X], A)$ gives a coordinate-free (though perhaps less intuitive) construction of $\mathcal{X}$ from X, and it underlies the next definition.

Definition 1.2. Let R be a k-algebra.

- The *spectrum* $\text{Sp}_k(R)$ is the representable k-functor $\text{Hom}(R, -)$.
- The category of *affine k-schemes* is the full subcategory of the category of k-functors given by spectra of k-algebras R.

In this way, we have a contravariant functor $\text{Sp}_k : k\text{-Alg} \to k\text{-Fun}$, since an algebra homomorphism $\varphi : A \to B$ induces a natural transformation

$$\text{Sp}_k(B) \to \text{Sp}_k(A)$$

via pre-composition with φ. The anti-equivalence of k-Alg with the category of affine k-varieties now shows we have embedded the latter category inside of k-Fun. We henceforth drop the distinction in notation between X and $\mathcal{X}$.

Definition 1.3. Let $\mathbb{A}^n_k = \text{Sp}_k(k[x_1, \ldots, x_n])$ be affine n-space over k.

- If X is a k-functor, then define

$$k[X] = \text{Hom}_{k\text{-Fun}}(X, \mathbb{A}^1_k),$$

the *regular functions* on X. It is a k-lagebra under pointwise addition and multiplication.
- Say the affine k-scheme X is *algebraic* if $k[X]$ is of finite type over k, and *reduced* if it contains no non-zero nilpotents.

This generalises the algebra of global functions on a k-variety.

We should now define *k-schemes* to be k-functors which are locally affine k-schemes in an appropriate sense. Since we will work directly with relatively few non-affine schemes, we omit the precise technical developments here and refer the reader to [Jan03, Chapter 1].

Exercise 1.4. Consider the k-functor F defined by the rule

$$F(A) = \{a \in A^{\mathbb{N}} : a_i = 0 \text{ for all but finitely many } i \in \mathbb{N}\}.$$

Show that F is not an affine k-scheme.

Definition 1.5. A *k-group functor* is a functor k-Alg $\to$ Grp. A *k-group scheme* (resp. *algebraic k-group*) is a k-group functor whose composite with the forgetful functor Grp $\to$ Set is an affine k-scheme (resp. algebraic affine k-scheme).

Equivalently, k-group schemes are group objects in the category of affine k-schemes. From this viewpoint, it is straightforward to see that they correspond to commutative Hopf algebras in k-Alg under the aforementioned anti-equivalence:

$$\{k\text{-group schemes}\} \cong \{\text{commutative Hopf algebras}\}^{\text{op}}, \quad G \mapsto k[G].$$

In some situations it is more convenient to specify an algebraic group by its Hopf algebra. For more discussion of this, see [Jan03, Section I.2.3–2.4].

Examples 1.6.

(1) The *additive group* $G = \mathbb{G}_a$ is defined on k-algebras by

$$G(A) = (A, +).$$

We have $k[G] = k[z]$, a polynomial ring in one variable, with

$$\Delta(z) = 1 \otimes z + z \otimes 1, \quad \varepsilon(z) = 0, \quad S(z) = -z$$

as comultiplication, counit, and antipode.

(2) The *multiplicative group* $G = \mathbb{G}_m$ is defined by

$$G(A) = A^{\times}.$$

Here $k[G] = k[z, z^{-1}]$ and

$$\Delta(z) = z \otimes z, \quad \varepsilon(z) = 1, \quad S(z) = z^{-1}.$$

A *torus* over k is any k-group isomorphic to an n-fold product $\mathbb{G}_m^n$.

(3) The *m-th roots of unity* μ_m are a k-subgroup scheme of $\mathbb{G}_m$ defined by

$$\mu_m(A) = \{a \in A^{\times} : a^m = 1\}.$$

We have $k[\mu_m] = k[z]/(z^m - 1)$.

(4) Let M be a k-vector space and define GL_M by

$$\mathrm{GL}_M(A) = \mathrm{End}_A(M \otimes A)^{\times}.$$

This is an affine k-scheme if and only if $M \cong k^m$ is finite dimensional, in which case it is an algebraic group with

$$k[GL_M] = k[GL_n] = k[z_{ij}]_{1 \leqslant i,j \leqslant n}[(\det(z_{ij})^{-1}],$$

where we write GL_n for GL_{k^n}. (Indeed, if $\{m_i\}_{i \in I}$ is a basis of M, then there are regular coordinate functions $X_{ij} \in k[GL_n]$ for $i, j \in I$ whose non-vanishing sets would give an open cover of $GL_M(k)$ if it were the spectrum of some ring; by quasi-compactness, this forces I to be finite.) Notice that $\mathrm{GL}_1 = \mathbb{G}_m$.

(5) The upper triangular matrices with diagonal entries 1 form a k-subgroup scheme $U_n \subseteq \mathrm{GL}_n$.

2. Representations

Definition 2.1. A *representation* of G is a homomorphism of k-group functors

$$G \to \mathrm{GL}_V,$$

where V is some k-vector space.

Suppose G is reduced and $V \cong k^n$ is finite dimensional. A representation of G on V is equivalent to a group homomorphism

$$G(k) \to \mathrm{GL}_n(k), \quad g \mapsto (z_{ij}(g)),$$

where the matrix coefficients are regular functions $z_{ij} \in k[G]$. This is an intuitive way to picture representations. The category of finite-dimensional representations will be denoted $\mathrm{Rep}(G)$.

Definition 2.2. Let V be a k-vector space. The *k-group functor V_a associated to V* is given by $V_a(A) = (V \otimes A, +)$.

This generalises the additive group $\mathbb{G}_a$, which we recover for $V = k$.

Definition 2.3. Let G be an algebraic k-group and V a k-vector space. A *(left) G-module* structure on V is an action of G on the k-functor V_a, i.e. a natural transformation

$$G \times V_a \to V_a,$$

such that $G(A)$ acts by A-linear endomorphisms on $V \otimes A$ for each k-algebra A.

The classical dictionary between representations and modules is reflected in the following proposition:

Proposition 2.4. There are natural equivalences of abelian tensor categories:

$$\{\text{representations of } G\} \cong \{\text{left } G\text{-modules}\} \cong \{\text{right } k[G]\text{-comodules}\}.$$

Exercise 2.5. Prove Prop. 2.4.

Results on representations can sometimes be obtained more expediently via comodules; an example follows.

Proposition 2.6. If G is an algebraic group and V is a G-module, then V is locally finite: any finite-dimensional subspace of V is contained in a finite-dimensional G-stable subspace of V.

Proof. View V as a right $k[G]$-comodule with action map $a : V \to V \otimes k[G]$, and suppose that for some fixed v we have

$$a(v) = \sum_{i=1}^{r} v_i \otimes f_i \tag{2.1}$$

with respect to a fixed choice of basis $\{v_i\}$ of V. Since $(a \otimes 1) \circ a = (1 \otimes \Delta) \circ a$, we can apply $a \otimes 1$ to the right-hand side of (2.1) and expand it in two different ways inside $V \otimes k[G] \otimes k[G]$:

$$\sum_i a(v_i) \otimes f_i = \sum_i \left(\sum_k v_k \otimes f_k^i \right) \otimes f_i = \sum_i v_i \otimes \Delta(f_i).$$

If ε denotes evaluation at 1 (the counit of $k[G]$), we have $\varepsilon_g = \varepsilon \circ \rho_{-g}$ for all $g \in G$, evaluation at g. Apply $1 \otimes \varepsilon_g \otimes 1$ to the previous equation and simplify:

$$\sum_i (g \cdot v_i) \otimes f_i = \sum_i v_i \otimes \eta_i,$$

for $\eta_i = ((\varepsilon_g \otimes 1) \circ \Delta)(f_i) \in k[G]$. This shows $W = \oplus_{f_i \neq 0} k v_i$ is a finite-dimensional G-stable subspace of V containing v, since $\eta_i \neq 0$ implies $f_i \neq 0$.

Corollary 2.7. Simple representations of V are finite dimensional.

Examples 2.8.
(1) For any algebraic group G and any vector space V, we have the *trivial representation* V_{triv}, via the trivial group homomorphism $G \to GL_V$.
(2) The prototypical representation is the *regular representation* $k[G]$, obtained by viewing $k[G]$ as a comodule over itself. The comodule action map

$$a : V \to V \otimes k[G]$$

of a comodule V can be interpreted as an embedding

$$V \hookrightarrow V_{\text{triv}} \otimes k[G].$$

This shows that any represention embeds within a direct sum of regular representations, and also that any irreducible representation is a submodule of $k[G]$.
(3) There is a decomposition of $k[\mathbb{G}_m]$ into one-dimensional $\mathbb{G}_m$-stable subspaces,

$$k[\mathbb{G}_m] = \oplus_m k z^m,$$

with $a \cdot z^m = a^{-m} z^m$. It turns out that these pieces are precisely the simple $\mathbb{G}_m$-modules, and that any representation of $\mathbb{G}_m$ is semi-simple.
(4) In the regular representation of $G = \mathbb{G}_a$ on $k[z]$, there is an increasing filtration by indecomposable submodules

$$V_i = \{f \in k[z] : \deg \text{f} \leqslant i\}.$$

Indeed, $\lambda \cdot z = z + \lambda$, for $\lambda \in \mathbb{G}_a$.

Suppose now that $p > 0$. Then $k \oplus k z^p$ is a G-stable subspace we would not see in characteristic zero. To understand why it arises is one of the goals of the next section.

Exercise 2.9. Let $T = \mathbb{G}_m^r$ be the split torus over k. Show that there is a canonical equivalence of categories

$$\text{Rep}(T) \cong \{X\text{-graded } k\text{-modules}\},$$

where $X = \text{Hom}(T, \mathbb{G}_m)$ is the *character lattice* of T. (Hint: this becomes very transparent in the language of comodules.)

3. Frobenius kernels

In this section we assume $p > 0$. Let A be an $\mathbb{F}_p$-algebra. The p-th power map

$$\sigma_A : A \to A, \quad x \mapsto x^p,$$

is a morphism of $\mathbb{F}_p$-algebras, and hence induces an *absolute Frobenius* morphism of affine $\mathbb{F}_p$-schemes $\mathrm{Sp}_{\mathbb{F}_p}\sigma_A : \mathrm{Sp}_{\mathbb{F}_p}(A) \to \mathrm{Sp}_{\mathbb{F}_p}(A)$. Any affine k-scheme X now admits a pullback square as follows:

$$\begin{array}{ccc} X^{(1)} & \xrightarrow{\mathrm{Fr}_a} & X \\ \downarrow & & \downarrow \\ \mathrm{Sp}_{\mathbb{F}_p}(k) & \xrightarrow{\mathrm{Sp}_{\mathbb{F}_p}\sigma_k} & \mathrm{Sp}_{\mathbb{F}_p}(k) \end{array}$$

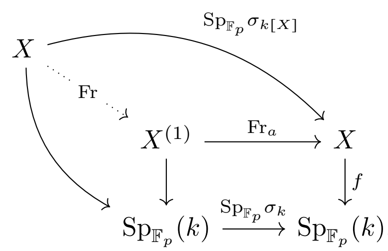
The universally induced $X^{(1)}$ is called the Frobenius twist of X; it has

$$k[X^{(1)}] = k^{(1)} \otimes_k k[X],$$

where $k^{(1)}$ is the 1-dimensional k-algebra afforded by σ_k. The map denoted Fr_a is known as the *arithmetic Frobenius*. By pullback, the morphisms

$$\mathrm{Sp}_{\mathbb{F}_p}\sigma_{k[X]} : X \to X \quad \text{and} \quad X \to \mathrm{Sp}_{\mathbb{F}_p}(k)$$

induce a *geometric Frobenius* morphism $\mathrm{Fr} : X \to X^{(1)}$:



Exercise 3.1. If X is an affine $\mathbb{F}_p$-scheme, then $X^{(1)} \cong X$.

Exercise 3.2. Suppose X is a closed subvariety of $\mathbb{A}^n$ defined by

$$f_1, \ldots, f_m \in k[\mathbb{A}^n].$$

Establish defining equations for $X^{(1)}$ as a subvariety of $\mathbb{A}^n$, and explicitly describe the geometric Frobenius morphism in terms of this embedding.

Iterating the above construction, we get a chain of morphisms

$$X \to X^{(1)} \to X^{(2)} \to \cdots;$$

the composite $X \to X^{(n)}$ is denoted Fr^n. Importantly, if G is a k-group scheme, then so are its Frobenius twists and Fr^n is a homomorphism of k-group schemes. Pulling back along these homomorphisms yields Frobenius twist functors

$$\mathrm{Rep}(G^{(n)}) \to \mathrm{Rep}(G), \quad V \mapsto V^{Fr^n}.$$

Example 3.3. Identifying $\mathbb{G}_a^{(1)} \cong \mathbb{G}_a$ (as in Exercise 3.1), we have $V_1^{Fr} \cong k \oplus kz^p$, in the notation of Example 2.8(4).

Definition 3.4. The *n-th Frobenius kernel* of a k-group scheme G is its subgroup scheme

$$G_n = \ker \mathrm{Fr}^n \leqslant G.$$

Exercise 3.5. (1) Verify that $k[\mathbb{G}_{a,n}] = k[z]/(z^{p^n})$.

(2) Show that a finite-dimensional representation of $\mathbb{G}_{a,n}$ is equivalent to the data $(V, \phi_1, \ldots, \phi_n)$, where V is finite dimensional and the ϕ_i are commuting operators on V with $\phi_i^p = 0$.

To conclude this lecture, we indicate the theoretical significance of Frobenius kernels. Let G be an k-group scheme and let

$$\mathfrak{g} = T_1 G = \mathrm{Der}(k[G], k)$$

denote the Lie algebra of $\mathfrak{g}$. We can identify $X \in \mathrm{Der}(k[G], k)$ with left-invariant k-derivations from $k[G]$ to itself, i.e. those for which the following square commutes:

$$\begin{array}{ccc} k[G] & \xrightarrow{\Delta} & k[G] \otimes k[G] \\ \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle 1\otimes D} \\ k[G] & \xrightarrow{\Delta} & k[G] \otimes k[G] \end{array}$$

The bracket on $\mathfrak{g}$ is then the commutator of derivations, and furthermore we can see $\mathfrak{g}$ is a p-Lie algebra with $X^{[p]} = X^p$.

Regardless of the characteristic of k, there is a functor of "differentiation",

$$D : \mathrm{Rep}(G) \to \mathrm{Rep}(\mathfrak{g}),$$

obtained according to the following recipe: Given a $k[G]$-comodule V with action map $a : V \to V \otimes k[G]$, we define

$$X \cdot v = (1 \otimes X)(a(v)), \quad X \in \mathfrak{g} = \mathrm{Der}(k[G], k), \quad v \in V.$$

We then have the following useful proposition.

Proposition 3.6. Assume G is connected.

(1) In characteristic 0, D is fully faithful.
(2) In characteristic p, D induces an equivalence

$$\mathrm{Rep}(G_1) \cong u(\mathfrak{g})\text{-mod},$$

where $u(\mathfrak{g}) = U(\mathfrak{g})/(X^p - X^{[p]})$.

Remark 3.7. Let us outline the proof of Prop. 3.6(2). If A is a finite-dimensional Hopf algebra over k, then the dual k-vector space A^* is naturally a Hopf algebra: all the structure maps are tranposes of structure maps of A. For instance, the multiplication of A^* is the image of the comultiplication of A under the isomorphism

$$\mathrm{Hom}(A, A \otimes A) \cong \mathrm{Hom}(A^* \otimes A^*, A^*).$$

The correspondence $A \leftrightarrow A^*$ hence defines a self-duality on the category of finite-dimensional Hopf algebras over k, with the additional property that

$$A\text{-mod} \cong A^*\text{-comod}. \tag{3.1}$$

Now, it can be shown that

$$k[G_1] \cong u(\mathfrak{g})^*, \tag{3.2}$$

while we have seen in Proposition 2.4 that for every algebraic k-group H there is an equivalence of categories

$$\mathrm{Rep}(H) \cong k[H]\text{-comod}. \tag{3.3}$$

The equivalence $\mathrm{Rep}(G_1) \cong u(\mathfrak{g})$-mod is then a corollary of (3.1), (3.2), and (3.3).

The intuition behind Proposition 3.6 is that the underlying field's characteristic has a strong bearing on the size of an algebraic group's subgroups, and so on how much of the group's representation theory is "seen around the identity" by $\mathfrak{g}$. In characteristic zero, G has "no small subgroups", while in characteristic $p > 0$, G has "many small subgroups" (particularly the Frobenius kernels). To be precise, the property of *having small subgroups* means that every neighbourhood U of the identity in G contains a subgroup $H \leqslant G$.

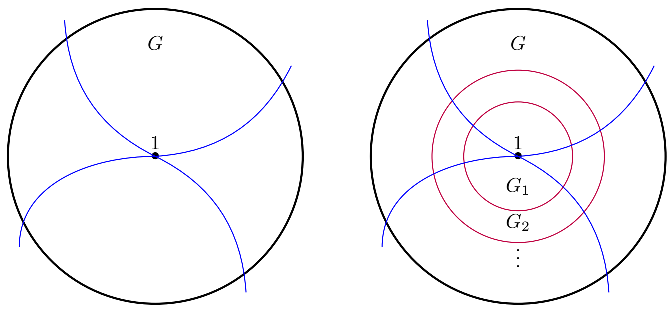


FIGURE 1. Left: the characteristic zero picture, with "no small subgroups". Right: the modular picture, with "many small subgroups". Closed subgroups, indicated with curved lines, are generally abundant in both contexts.

We also have a result that $\mathrm{Rep}(G) \hookrightarrow 2\text{-}\varprojlim \mathrm{Rep}(G_m)$; here we refer to a 2-limit of categories, viewing them as objects in some appropriate 2-category. Practically speaking, this means that for any $V, V' \in \mathrm{Rep}(G)$, there is $n \geqslant 1$ such that

$$\mathrm{Hom}_G(V, V') = \mathrm{Hom}_{G_n}(V, V').$$

In this sense, the family of Frobenius kernels of G controls the representation theory of G.

Exercise 3.8. Show that in characteristic p,

$$\mathrm{Rep}(\mathbb{G}_a) \cong \{(V, \phi_n)_{n \geqslant 1} : V \text{ a } k\text{-vector space}, \ \phi_i \in \mathrm{End}_k(V) \text{ with } \phi^p = 0\}.$$

This description is visibly the direct limit of the description in Exercise 3.5. On the other hand, show that in characteristic zero the right-hand side should instead consist of pairs (V, ϕ) with $\phi : V \to V$ nilpotent.

Lecture II

4. Reductive groups and root data

In this lecture, we will restrict our attention to a class of algebraic groups whose specification demands several new adjectives. We approach this as directly as possible, recommending sources such as [Hum75] or [Spr81] for much more detail.

Definition 4.1. An algebraic group is *unipotent* if it is isomorphic to a closed subgroup scheme of U_n.

Definition 4.2. Let G be a (topologically) connected algebraic group over k.

(1) G is *semi-simple* if the only smooth connected solvable normal subgroup of G is trivial.
(2) G is *reductive* if the only smooth connected unipotent normal subgroup of G is trivial.

It can be shown that any unipotent group over $k = \overline{k}$ admits a composition series in which each quotient is isomorphic to $\mathbb{G}_a$. In particular, all unipotent groups are solvable, so all semi-simple groups are reductive.

Example 4.3. The archetypal reductive group is GL_n. It contains many tori, which are also reductive. A maximal such torus, i.e. one contained in no other, is the subgroup of diagonal matrices $D_n \cong \mathbb{G}_m^n$.

Let G be a reductive, connected algebraic group over the algebraically closed field k. The group's action on itself by conjugation defines a homomorphism of k-group functors $G \to \mathrm{Aut}(G)$, and automormophisms of G can be differentiated to elements of $\mathrm{Aut}(\mathfrak{g})$. Thus we obtain the adjoint action of G on $\mathfrak{g}$.

With respect to the adjoint action of a maximal torus $T \subseteq G$, there is a decomposition

$$\mathfrak{g} = \mathrm{Lie}(G) = \mathrm{Lie}(T) \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha.$$

Here $R \subseteq \mathfrak{X} = X(T)$ are the *roots* relative to T, and $\mathfrak{g}_\alpha$ is the subspace upon which T acts with character α; by definition, $\mathfrak{g}_\alpha \neq 0$ for $\alpha \in R$. Pulling back through $\alpha : T \to \mathbb{G}_m$, the natural action of $\mathbb{G}_m$ on $\mathbb{G}_a$ by multiplication yields an action of T on $\mathbb{G}_a$. Up to scalar, there is a unique *root homomorphism*

$$x_\alpha : \mathbb{G}_a \to G$$

which intertwines the actions of T and induces an isomorphism

$$dx_\alpha : \mathrm{Lie}(\mathbb{G}_a) \cong \mathfrak{g}_\alpha;$$

we denote its image subgroup by U_α. After normalising x_α and $x_{-\alpha}$ suitably, we can construct $\varphi_\alpha : \mathrm{SL}_2 \to G$ such that

$$\varphi_\alpha \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = x_\alpha(a) \quad \text{and} \quad \varphi_\alpha \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = x_{-\alpha}(a).$$

Then, we get $\alpha^\vee \in \mathfrak{X}^\vee = Y(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ by defining

$$\alpha^\vee(\lambda) = \varphi_\alpha \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

and we write $R^\vee = \{\alpha^\vee : \alpha \in R\} \subseteq \mathfrak{X}^\vee$.

Definition 4.4. (1) A *root datum* consists of a quadruple $(R \subseteq \mathfrak{X}, R^\vee \subseteq \mathfrak{X}^\vee)$, along with a map $R \to R^\vee, \alpha \mapsto \alpha^\vee$, satisfying the following conditions:
- $\mathfrak{X}$, $\mathfrak{X}^\vee$ are free abelian groups of finite rank, equipped with a perfect pairing $\langle -, - \rangle : \mathfrak{X} \times \mathfrak{X}^\vee \to \mathbb{Z}$.
- R and $R^\vee$ are finite and $\alpha \mapsto \alpha^\vee$ is bijective.
- For all $\alpha \in R$, we have $\langle \alpha, \alpha^\vee \rangle = 2$, and the function s_α defined by

$$s_\alpha(x) = x - 2\langle x, \alpha^\vee \rangle \alpha$$

permutes R and induces an action on $\mathfrak{X}^\vee$ which restricts to a permutation of $R^\vee$.

Members of R (resp. $R^\vee$) are called *roots* (resp. *coroots*).

(2) A *morphism* of root data

$$(R \subseteq \mathfrak{X}, R^\vee \subseteq \mathfrak{X}^\vee) \to (R_0 \subseteq \mathfrak{X}_0, R_0^\vee \subseteq \mathfrak{X}_0^\vee)$$

is an abelian group homomorphism $\mathfrak{X} \to \mathfrak{X}_0$ which sends $R \to R_0$, induces a map $\mathfrak{X}^\vee \to \mathfrak{X}_0^\vee$ which sends $R^\vee \to R_0^\vee$, and commutes with the bijections between roots and coroots. This gives sense to the notion of an isomorphism of root data.

Above, we explained how to construct a root datum from a reductive algebraic group G. It turns out this defines a bijection on isomorphism classes (see [GM20, Section 1.3]).

Theorem 4.5 (Chevalley)**.** There is a canonical one-to-one correspondence,

$$\{\text{reductive algebraic groups over } k\}/\cong \quad \leftrightarrow \quad \{\text{root data}\}/\cong .$$

Remark 4.6. (1) The bijection in the theorem is independent of k (but crucially depends on the property of being algebraically closed). It turns out that for any root datum there exists a corresponding "Chevalley group scheme" over $\mathbb{Z}$, whose base change to k gives the corresponding reductive group over k.

(2) Interchanging $R \leftrightarrow R^\vee$ and $\mathfrak{X} \leftrightarrow \mathfrak{X}^\vee$ defines an obvious involution on the set of root data. On the other side of the bijection, this is a deep operation $G \leftrightarrow G^\vee$ on algebraic groups known as the *Langlands dual*.

A root datum $(R \subseteq \mathfrak{X}, R^\vee \subseteq \mathfrak{X}^\vee)$ yields a finite Weyl group

$$W_{\mathrm{f}} = \langle s_\alpha : \alpha \in R \rangle$$

and also an *abstract root system* within the subspace V spanned by R in the Euclidean space $\mathfrak{X}_\mathbb{R} = \mathfrak{X} \otimes_\mathbb{Z} \mathbb{R}$. In particular, there exists a choice of *simple roots* $\Sigma \subseteq R$, which is a basis for V such that any element of R is a non-negative or non-positive integral linear combination from Σ. Then we obtain positive roots

$$R^+ = \{\alpha \in R : \alpha = \sum_{\sigma \in \Sigma} c_\sigma \sigma \text{ for } c_\sigma \in \mathbb{Z}_{\geqslant 0}\},$$

and simple reflections $S_{\mathrm{f}} = \{s_\alpha : \alpha \in \Sigma\}$. Assume from now on that we are working with the root datum corresponding to a reductive group G, and fix choices $\Sigma \subseteq R^+ \subseteq R$ of positive (simple) roots. Corresponding to the choice of R^+ is a *Borel subgroup* $T \subseteq B^+ = TU^+ \subseteq G$, where U^+ is the subgroup of G generated by the U_α for $\alpha \in R^+$.

Examples 4.7. (1) Take $G = \mathrm{GL}_n$ with maximal torus $T = D_n$. Then

$$\mathfrak{X} = \bigoplus_i \mathbb{Z}\varepsilon_i, \quad \mathfrak{X}^\vee = \bigoplus_i \mathbb{Z}\varepsilon_i^\vee,$$

where $\varepsilon_i(\mathrm{diag}(\lambda_1, \dots, \lambda_n)) = \lambda_i$. The roots are

$$R = \{\varepsilon_i - \varepsilon_j : i \neq j\},$$

and if we choose $R^+ = \{\varepsilon_i - \varepsilon_j : i < j\}$ then $\Sigma = \{\varepsilon_i - \varepsilon_{i+1}\}$ and B^+ is the set of upper triangular matrices.

(2) If $G = \mathrm{SL}_n \leqslant \mathrm{GL}_n$ is the subgroup of matrices with determinant 1, then it contains a maximal torus $T \cong D_{n-1}$ consisting of the diagonal matrices with non-zero entries whose product is 1. We get

$$\mathfrak{X} = \left(\bigoplus_{i=1}^n \mathbb{Z}\varepsilon_i\right)/(\varepsilon_1 + \cdots + \varepsilon_n) \cong \bigoplus_{i=1}^{n-1} \mathbb{Z}(\varepsilon_i - \varepsilon_{i+1}),$$

where by abuse of notation we conflate $\varepsilon_i - \varepsilon_{i+1}$ with its image in the indicated quotient. On the other hand, $\mathfrak{X}^\vee$ naturally identifies with the subgroup of $\oplus_i \mathbb{Z}\varepsilon_i^*$ whose coefficients in $\{\varepsilon_i^*\}$ sum to zero. Then

$$R = \{\alpha_{ij} = \varepsilon_i - \varepsilon_j : i \neq j\}$$

and we can choose $\Sigma = \{\varepsilon_i - \varepsilon_{i+1}\}$, for which B^+ is again the set of upper triangular matrices in SL_2. Up to scalar,

$$x_{\alpha_{ij}}(\lambda) = I_n + \lambda e_{ij},$$

where e_{ij} denotes the matrix with 1 in position (i, j) and zeroes elsewhere, so we can take

$$\varphi_{\alpha_{ij}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a e_{ii} + b e_{ij} + c e_{ji} + d e_{jj}$$

for $i < j$. Thus we see $\alpha_{ij}^\vee = \varepsilon_i^* - \varepsilon_j^*$.

(3) Let $G = \mathrm{PGL}_n$, the quotient of GL_n by its centre. If $D_n = T \leqslant \mathrm{GL}_n$ is chosen as a maximal torus, then $q(T)$ is a maximal torus in G, where $q : \mathrm{GL}_n \to G$ is the defining quotient map. We obtain that

$$\mathfrak{X} = X(q(T)) = \left\{ \sum_{i=1}^n a_i \varepsilon_i : \sum a_i = 0 \right\} \subseteq X(T).$$

The cocharacter lattice $\mathfrak{X}^\vee$ is isomorphic to $(\oplus_i \mathbb{Z}\varepsilon_i^*)/(\varepsilon_1^* + \cdots + \varepsilon_n^*)$, where the image of ε_i^* corresponds to the cocharacter $\lambda \mapsto I + (\lambda - 1)e_{ii}$. After determining roots and coroots, it becomes clear from our descriptions that PGL_n is the Langlands dual of SL_n.

Exercise 4.8. Calculate the root data of Sp_{2n}, SO_{2n}, and SO_{2n+1}. Identify the Langlands dual in each case.

5. Flag varieties

5.1. **Geometric realisations of simple modules.** We have seen that every representation of G embeds into a direct sum of copies of $k[G]$, or in other words, is "seen by $k[G]$". However, for reductive groups G, much can be gleaned by studying the flag variety G/B^+. In particular, we will see that simple representations of G arise in spaces of global sections of sheaves on G/B^+.

Examples 5.1. (1) For $G = \mathrm{SL}_2$ we have $G/B^+ = \mathbb{P}^1$. To see this, notice that we can identify $\mathbb{P}^1$ with the set L of lines $0 \subseteq \ell \subseteq V = k^2$. There is an obvious transitive action of G on L, under which the unique line $\ell \in L$ containing $e_1 = (1, 0)$ has stabiliser B^+. Hence, the action map yields the stated isomorphism.

(2) For entirely similar reasons, $G = \mathrm{GL}_n$ is such that

$$G/B^+ = \{0 \subseteq V_1 \subseteq \cdots \subseteq k^n : V_i \text{ is an } i\text{-dimensional subspace}\}.$$

Indeed, let $\mathscr{F}$ denote the right-hand side. Notice that if $F_0 \in \mathscr{F}$ is the *standard flag* corresponding to an ordered basis relative to which B^+ consists of upper triangular matrices, then the map

$$G \to \mathscr{F}, \quad g \mapsto g \cdot F_0$$

induces a morphism $G/B^+ \to \mathscr{F}$ by the definition of quotient varieties; this turns out to be an isomorphism.

Definition 5.2. Suppose G acts on a k-scheme X through $\sigma : G \times X \to X$. A *$G$-equivariant sheaf* $\mathcal{F}$ on X is a sheaf of $\mathcal{O}_X$-modules together with an isomorphism of $\mathcal{O}_{G\times X}$-modules

$$\phi : \sigma^*\mathcal{F} \to p_2^*\mathcal{F}$$

which satisfies the cocycle condition

$$p_{23}^*\phi \circ (1_G \times \sigma)^*\phi = (m \times 1_X)^*\phi.$$

Here we refer to the obvious projections $p_{23} : G \times G \times X \to G \times X$, $p_2 : G \times X \to X$, and multiplication $m : G \times G \to G$.

Remark 5.3. On the stalk level, the first of these conditions ensures that $\mathcal{F}_{gx} \cong \mathcal{F}_x$ for all $x \in X$, and the second that the isomorphism $\mathcal{F}_{ghx} \cong \mathcal{F}_x$ coincides with $\mathcal{F}_{ghx} \cong \mathcal{F}_{hx} \cong \mathcal{F}_x$.

Given a G-equivariant sheaf $\mathcal{F}$ on X with G-action σ, the space of global sections $\Gamma(X, \mathcal{F})$ admits a natural G-module structure. This is prescribed by the formula

$$g \cdot w = (\phi_{G\times X} \circ \sigma_X^{\#})(w)(g^{-1}), \quad g \in G, \quad w \in \Gamma(X, \mathcal{F}),$$

where ϕ is the $\mathcal{O}_{G\times X}$-module isomorphism required by Definition 5.2 and $\sigma_X^{\#}$ is the map $\Gamma(X, \mathcal{F}) \to \Gamma(G \times X, \sigma^*\mathcal{F})$ induced by σ.

Suppose that V is a simple G-module. Then G acts on $\mathbb{P}(V^*)$ and $\mathcal{O}(1)$ is an equivariant line bundle for the action. In particular, we recover the representation V from the action on global sections,

$$\Gamma(\mathbb{P}(V^*), \mathcal{O}(1)) = V.$$

We now want to transfer this realisation to the flag variety. Either of the following facts may be adduced to prove that B^+ has a fixed point in its action on $\mathbb{P}(V^*)$.

Theorem 5.4 (Borel)**.** If H is a connected, solvable algebraic group acting through regular functions on a non-empty complete variety W over an algebraically closed field, then there is a fixed point of H on W.

Proposition 5.5. Suppose U is a unipotent group and M is a non-zero U-module. Then $M^U \neq 0$.

Exercise 5.6. Prove Proposition 5.5.

In light of this fixed point, we obtain a diagram

$$\begin{array}{ccc} f^*\mathcal{O}(1) & \longrightarrow & \mathcal{O}(1) \\ \downarrow & & \downarrow \\ G/B^+ & \xrightarrow{f} & \mathbb{P}(V^*). \end{array}$$

Taking global sections, we have a non-zero map $V \to \Gamma(G/B^+, f^*\mathcal{O}(1))$. Since V is simple, it is injective. Hence we can conclude that simple representations of G occur in global sections of line bundles on the flag variety.

Examples 5.7. Let $G = \mathrm{SL}_2$. On $\mathbb{P}^1 = G/B^+$, the line bundles $\mathcal{O}(n)$ have a unique equivariant structure. Recall that we can identify

$$\nabla_n = \Gamma(\mathbb{P}^1, \mathcal{O}(n)) = k[x,y]_{\deg\, n} = ky^n \oplus ky^{n-1}x \oplus \cdots \oplus kx^n$$

if $n \geqslant 0$, and is zero otherwise.

If $p = 0$, then the ∇_n are exactly the simple SL_2-modules; this follows (for example) from Lie algebra considerations and leads, for example, to the theory of spherical harmonics.

If $p > 0$, then ∇_n is simple for $0 \leqslant n < p$, but ∇_p is not. Indeed, there is a G-submodule

$$L_p = kx^p \oplus ky^p \subseteq \nabla_p,$$

which is the Frobenius twist of ∇_1, the natural representation of SL_2 on k^2. This is clear from the formula for the action of an arbitrary matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x^p = a^p x^p + c^p y^p; \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot y^p = c^p x^p + d^p y^p.$$

In general, $L_n = G \cdot x^n \subseteq \nabla_n$ is simple; hence the simple modules are indexed by the same set as in the characteristic zero case, although their dimensions are different in general.

5.2. **Line bundles on the flag variety.** Having located simple G-modules within the global sections of line bundles on G/B^+, it remains for us to construct and study those line bundles. Let B be the Borel subgroup of G corresponding to $-R^+$ (the opposite Borel subgroup to B^+). In the sequel, it will be more convenient to work with G/B (which is isomorphic to G/B^+, since B and B^+ are conjugate). Importantly, there is an open embedding

$$\mathbb{A}^{|R^+|} \cong U^+ \hookrightarrow G/B, \quad u \mapsto uB/B, \tag{5.1}$$

whose image is dense and often called the (opposite) open *Bruhat cell.*

The following definition is really also a proposition; see [Jan03], I.5.8.

Definition 5.8. Let H be a flat group scheme acting freely on a k-scheme X in such a way that X/H is a scheme; let $\pi : X \to X/H$ be the canonical morphism. There is an *associated sheaf* functor

$$\mathscr{L} = \mathscr{L}_{X,H} : \{H\text{-modules}\} \to \{\text{vector bundles on } X/H\},$$

defined on objects as follows: if $U \subseteq X/H$ is open, then

$$\mathscr{L}(M)(U) = \{f \in \mathrm{Hom}_{\mathrm{Sch}}(\pi^{-1}(U), M_a) : f(xh) = h^{-1}f(x)\}.$$

In case that $\pi^{-1}(U)$ is affine, these sections coincide with $(M \otimes k[\pi^{-1}U])^H$.

The associated sheaf functor has a number of useful properties, including exactness. Much of its theoretical importance derives from its relation to *induction*: whenever H_2 is a subgroup scheme of H_1 such that H_1/H_2 is a scheme, there is an isomorphism

$$R^n \mathrm{ind}_{H_1}^{H_2} M \cong H^n(H_1/H_2, \mathscr{L}(M)), \quad n \geqslant 0, \tag{5.2}$$

where R^n refers to the n-th right derived functor. In fact, many results concerning induction are most readily proved geometrically via (5.2). The interested reader is referred to [Jan03, Section I.5].

Notation 5.9. For $\lambda \in \mathfrak{X}$, let k_λ be the corresponding representation of B, arising from pullback along

$$B \to B/[B,B] \cong T.$$

Then define the sheaf $\mathcal{O}(\lambda) = \mathscr{L}_{G,B}(k_{-\lambda})$ on G/B. Because any character is of rank 1, $\mathcal{O}(\lambda)$ is a locally free sheaf of rank 1. (For more detail on this point, see [Jan03, I.5.16(2)] and [Jan03, II.1.10(2)].)

Exercise 5.10. Let $G = \mathrm{SL}_2$ and let ϖ denote the fundamental weight, i.e. the weight such that $\langle \varpi, \alpha^\vee \rangle = 1$, where α is a positive root. Verify that $\mathcal{O}(n\varpi)$ agrees with the invertible sheaf $\mathcal{O}(n) \in \mathbb{P}^1_k$.

Restricting along the open embedding 5.1, we find

$$\Gamma(G/B, \mathcal{O}(\lambda)) \hookrightarrow \Gamma(U^+, \mathcal{O}_{U^+}) \cong k[U^+];$$

here we are using that line bundles on affine k-space, including $\mathcal{O}(\lambda)|_{U^+}$, are trivial.

Proposition 5.11. (1) There is an action of T on $\Gamma(U^+, \mathcal{O}_{U^+})$ such that 1 has weight λ.
(2) The following are equivalent:
   (a) 1 extends to a section $v_\lambda \in \Gamma(G/B, \mathcal{O}(\lambda))$.
   (b) $\lambda \in \mathfrak{X}_+$ is dominant.
   (c) $\Gamma(G/B, \mathcal{O}(\lambda)) \neq 0$.

In light of the above, let us write $\nabla_\lambda = \Gamma(G/B, \mathcal{O}(\lambda))$ in case $\lambda \in \mathfrak{X}_+$. There is an inclusion

$$\nabla_\lambda^{U^+} \hookrightarrow k[U^+]^{U^+} = k \cdot 1,$$

which implies that ∇_λ is indecomposable and that it has a simple socle

$$L_\lambda = \mathrm{soc}\ \nabla_\lambda.$$

Indeed, if there were a non-trivial decomposition

$$\nabla_\lambda = M \oplus N,$$

then we would obtain $\nabla_\lambda^{U^+} = M^{U^+} \oplus N^{U^+}$, which is at least two-dimensional by the non-triviality of each summand (see Prop. 5.5); similar considerations prove that the socle is simple. Now we are in a position to generalise our findings for SL_2.

Theorem 5.12 (Chevalley)**.** There is a bijection,

$$\mathfrak{X}_+ \to \{\text{simple } G\text{-modules}\}/\cong, \quad \lambda \mapsto L_\lambda.$$

Exercise 5.13. Prove Theorem 5.12 using the ideas in this section.

6. Kempf vanishing theorem

Definition 6.1. Let M be a finite-dimensional representation of G. We define the *character* of M to be

$$\text{ch } M = \sum_{\lambda \in \mathfrak{X}_+} (\dim M_\lambda) e^\lambda \in \mathbb{Z}[\mathfrak{X}],$$

where $M_\lambda = \{m \in M : tm = \lambda(t)m \text{ for all } t \in T\}$ is the λ-eigenspace of M.

The characters of the modules ∇_λ admit remarkably elegant expressions.

Theorem 6.2 (Weyl)**.** Let W be the Weyl group of G and $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$. Then

$$\text{ch } \nabla_\lambda = \frac{\sum_{w \in W} (-1)^\ell (w) e^{w(\lambda+\rho)}}{\sum_{w \in W} (-1)^{\ell(w)} e^{w\rho}}.$$

Many proofs exist for this formula in the case $k = \mathbb{C}$; see [Hal15, 10.4] for one account. For arbitrary k, the result is derived as a consequence of the next theorem, which is of fundamental interest for us.

Theorem 6.3 (Kempf)**.** Let $\lambda \in \mathfrak{X}_+$. Then

$$H^i(G/B, \mathcal{O}(\lambda)) = 0$$

for all $i > 0$.

We now sketch a proof of this theorem, assuming two black boxes; the original paper is [Kem76], and another account is available in [Jan03, II.4]. To begin, we introduce some of the main characters in our story, the *Steinberg modules*

$$\text{St}_m = \nabla_{(p^m-1)\rho}, \quad m \geqslant 1. \tag{6.1}$$

These are simple modules whose dimensions are $p^{m|R^+|}$. The fact that these induced modules are simple is our first black box; a beautiful proof is given in [Kem81].

Recall the Frobenius morphism Fr from the previous lecture, particularly

$$\text{Fr} : G/B \to G/B.$$

The following isomorphism (our second black box) is due to Anderson [And80a] and Haboush [Hab80]:

$$\text{Fr}_*^m(\mathcal{O}((p^m-1)\rho) \cong \text{St}_m \otimes \mathcal{O}. \tag{6.2}$$

Now, for any $\gamma \in \mathfrak{X}_+$,

$$(\text{Fr}^m)^* \mathcal{O}(\gamma) = \mathcal{O}(p^m\gamma).$$

Using the projection formula [Har77, Exercise II.8.3] in combination with (6.2), we find

$$(\text{Fr}^m)_* \mathcal{O}((p^m-1)\rho + p^m\gamma) = \text{Fr}_*^m \mathcal{O}((p^m-1)\rho) \otimes \mathcal{O}(\gamma) \cong \text{St}_m \otimes \mathcal{O}(\gamma).$$

Taking cohomology yields

$$H^i(G/B, \mathcal{O}((p^m-1)\rho + p^m\gamma)) \cong \text{St}_m \otimes H^i(G/B, \mathcal{O}(\gamma)).$$

But $\mathcal{O}(\rho)$ is ample, so by Serre's vanishing theorem [Har77, III.5.2], the left-hand side is zero for $i \neq 0$ and sufficiently large m; hence the right-tensor factor on the right-hand side is necessarily zero.

Exercise 6.4. We have a (variant of) the Bruhat decomposition,

$$G/B = \bigsqcup_{w \in W} B^+ \cdot xB/B,$$

where each $B^+ \cdot xB/B \cong \mathbb{A}^{\ell(w_o)-\ell(x)}$.

(1) Use this decomposition to determine $\mathrm{Pic}(G/B)$.
(2) Determine the class of $\mathcal{O}(\lambda)$ in the Picard group in terms of the previous description.
(3) All equivariant line bundles on G/B have the form $\mathcal{O}(\lambda)$. Use this to determine when a line bundle on G/B admits an equivariant lift, in terms of the root datum of G.

Lecture III

7. Steinberg tensor product theorem

7.1. **Motivation from finite groups.** Suppose momentarily that G is a finite group with a normal subgroup N:

$$1 \to N \to G \to G/N. \tag{7.1}$$

Recall that if $g \in G$, then conjugation by g defines an automorphism of groups $\sigma_g : G \to G$. Pulling back along σ_g defines a functor $V \mapsto V^g$ on G-modules, whose image we call the *twist* of V by g.

Part of Clifford's theorem for finite groups states that if V is a simple G-module, then $V|_N$ is a semi-simple N-module and all of its irreducible summands are G-conjugate [Web16, Section 5.3]. With this fact in mind, let us take one additional assumption.

Assumption 7.1. All simple N-modules extend to G-modules.

A consequence of Assumption 7.1 is that every simple N-module W is fixed by G, in the sense that $W^g \cong W$ for all $g \in G$. In particular, all the irreducible summands of $V|_N$ are isomorphic when V is a simple G-module.

So, suppose in this setting that $V' \subseteq V$ is an irreducible summand of V as an N-module. It then decomposes into copies of V' with some multiplicity:

$$V \cong V' \oplus \cdots \oplus V'.$$

Then

$$\mathrm{Hom}_N(V', V) \otimes V' \to V, \quad f \otimes v' \mapsto f(v')$$

is an isomorphism of G-modules. (Indeed, it is easily seen to be surjective, and then we can compare dimensions.) Hence, in this scenario, we can conclude that every simple G-module arises as the tensor product of a simple G/N-module and an irreducible N-module extending to G. We will now witness a similar phenomenon in the setting of reductive groups.

7.2. **Back to reductive groups.** Let us return now to our usual level of generality, where G is a reductive algebraic k-group for $k = \overline{k}$ of characteristic $p > 0$. Unless otherwise stated, the following assumption will be in force from here on.

Assumption 7.2. *G is semi-simple and simply connected.*

We call a weight $\lambda \in \mathfrak{X}_+$ *p-restricted* in case $\langle \lambda, \alpha^\vee \rangle < p$ for all simple roots α; their subset is denoted $\mathfrak{X}_{<p} \subseteq \mathfrak{X}$. Recall also the Frobenius exact sequence

$$1 \to G_1 \to G \to G^{(1)} \to 1.$$

We would like to view this sequence as an analogue of (7.1); in this light, the analogue of Assumption 7.1 for reductive groups is the following result:

Theorem 7.3 (Curtis [Cur60])**.** If $\lambda \in \mathfrak{X}_{<p}$ then $L_\lambda|_{G_1}$ is simple, and moreover all simple G_1-modules occur in this way. Hence all simple G_1-modules extend to G.

Example 7.4. The p simple SL_2-modules $L_0, \dots, L_{p-1}$ remain simple when considered over $\mathfrak{g} = \mathfrak{sl}_2$.

Theorem 7.5. All simple G-modules are of the form $L_\lambda \otimes L_\mu^{(1)}$, for $\lambda \in \mathfrak{X}_{<p}$ and $\mu \in \mathfrak{X}_+$.

Notice this theorem is nicely in analogy to the conclusion of Section 1.1: as there, it expresses simple G-modules as tensor products of simple modules over a quotient (namely $L_\mu^{(1)}$ over $G/G_1 \cong G^{(1)}$) and simple modules over a normal subgroup which admit an extension to G (namely L_λ over G_1). By induction on Theorem 7.5, we obtain a well-known and beautiful result:

Theorem 7.6 (Steinberg [Ste74])**.** Let $\lambda \in \mathfrak{X}_+$ and write $\lambda = \lambda_0 + p\lambda_1 + \cdots + p\lambda_m$, for $\lambda_i \in \mathfrak{X}_{<p}$. Then

$$L_\lambda \cong L_{\lambda_0} \otimes L_{\lambda_1}^{(1)} \otimes \cdots L_{\lambda_m}^{(m)}.$$

Importantly, it is a consequence of Assumption 7.2 that any $\lambda \in \mathfrak{X}_+$ admits the decomposition into p-restricted digits as described.

Remark 7.7. One of the great uses of Steinberg's $\otimes$-theorem is that it reduces many questions (for instance, concerning characters) to a finite set of modules: the L_γ for $\gamma \in \mathfrak{X}_{<p}$.

Example 7.8. The theorem provides us a complete answer to the question of characters for $G = \mathrm{SL}_2$. Define

$$\mathrm{Fr} : \mathbb{Z}[\mathfrak{X}] \to \mathbb{Z}[\mathfrak{X}], \quad e^\lambda \mapsto e^{p\lambda}.$$

Each $\lambda = n \in \mathbb{N}$ can be written $n = \sum_{i \geqslant 0} \lambda_i p^i$ with $0 \leqslant \lambda_i < p$. Then, decomposing L_n into a tensor product by Steinberg's theorem and taking characters, we obtain

$$\mathrm{ch}\, L_n = \prod_{i \geqslant 0} \mathrm{ch}\, L_{\lambda_i}^{(\mathrm{Fr})^i} = \prod_{i \geqslant 0} (e^{-\lambda_i} + e^{-\lambda_i + 2} + \cdots + e^{\lambda_i})^{(\mathrm{Fr})^i}.$$

For instance, we have $p^m - 1 = (p-1) + (p-1)p + \cdots + (p-1)p^{m-1}$, so

$$\begin{aligned} \mathrm{ch}\, \mathrm{St}_m = \mathrm{ch}\, L_{p^m - 1} &= \left(\frac{e^p - e^{-p}}{e - e^{-1}}\right) \left(\frac{e^p - e^{-p}}{e - e^{-1}}\right)^{(\mathrm{Fr})} \cdots \left(\frac{e^p - e^{-p}}{e - e^{-1}}\right)^{(\mathrm{Fr})^{m-1}} \\ &= \frac{e^{p^m} - e^{-p^m}}{e - e^{-1}}; \end{aligned}$$

here we refer to the Steinberg module defined in (6.1).

Exercise 7.9. Let $G = \mathrm{SL}_2$.

(1) Explicitly write out the characters of L_m for $0 \leqslant m \leqslant p^2 - 1$.
(2) Hence express the characters of these L_m in terms of the modules ∇_n.
(3) Repeat this for p^2 and record your observations. What changes?
(4) For bonus credit, repeat the first part for $0 \leqslant m \leqslant p^3 - 1$.

8. Kazhdan–Lusztig conjecture

Our next main goal is to state the Lusztig conjecture on $\mathrm{ch}\, L_\lambda$. This formula was motivated by the earlier Kazhdan–Lusztig conjecture, which we will describe in this section after recalling certain elements of the theory of complex semi-simple Lie algebras. A comprehensive reference for the background theory is [Hum08].

8.1. **Background on complex semi-simple Lie algebras.** Fix $\mathfrak{g}$ a complex semi-simple Lie algebra, containing

$$\mathfrak{h} \subseteq \mathfrak{b}^+ \supseteq \mathfrak{n}^+$$

Cartan and Borel subalgebras, along with its nilpotent radical, respectively. Recall that $\mathfrak{b}^+ \cong \mathfrak{h} \oplus \mathfrak{n}^+$ as vector spaces, that $\mathfrak{n}^+ = [\mathfrak{b}, \mathfrak{b}]$, and that

$$\mathrm{Hom}(\mathfrak{b}^+, \mathbb{C}) = \mathrm{Hom}(\mathfrak{b}^+/[\mathfrak{b}^+, \mathfrak{b}^+], \mathbb{C}) = \mathfrak{h}^*.$$

To any $\lambda \in \mathfrak{h}^*$ we associate the *standard* or *Verma* $\mathfrak{g}$-module $\Delta_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} \mathbb{C}_\lambda$. By the PBW theorem, we can write $U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \otimes U(\mathfrak{b}^+)$, so that

$$\Delta_\lambda \cong (U(\mathfrak{n}^-) \otimes U(\mathfrak{b}^+)) \otimes_{U(\mathfrak{b}^+)} \mathbb{C}_\lambda \cong U(\mathfrak{n}^-) \otimes \mathbb{C}_\lambda$$

as $\mathfrak{n}^-$-modules.

Definition 8.1 (BGG)**.** The *BGG category* $\mathcal{O}$ is the full subcategory of $\mathfrak{g}$-mod consisting of objects M satisfying the following conditions:

- M is $\mathfrak{h}$-diagonalisable:
$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda,$$
where $M_\lambda = \{m \in M : h \cdot m = \lambda(h)m \text{ for all } h \in \mathfrak{h}\}$.
- M is locally finite for the action of $\mathfrak{b}^+$: every $m \in M$ is contained in a finite-dimensional $\mathbb{C}$-vector space stable for the action of $\mathfrak{b}^+$.
- M is finitely generated over $\mathfrak{g}$.

Conspicuous objects in the category $\mathcal{O}$ are the Δ_λ and the finite-dimensional simple modules. In fact, Δ_λ has a unique simple quotient, since it has a unique maximal submodule; we denote this simple module by L_λ.

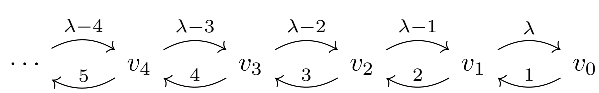
Proposition 8.2. There is a bijection,

$$\mathfrak{h}^* \to \{\text{simple objects in } \mathcal{O}\}/\cong, \quad \lambda \mapsto [L_\lambda].$$

Example 8.3. Consider the simplest non-trivial example, $\mathfrak{g} = \mathfrak{sl}_2$. We can be very explicit about the structure of the standard module Δ_λ in this case. It admits an infinite basis

$$v_0 = 1 \otimes 1, \quad v_1 = f \otimes 1, \quad \ldots, \quad v_m = \frac{1}{m!} f^m \otimes 1, \quad \ldots$$

such that the action of $\mathfrak{sl}_2$ is given by the following quiver:



In terms of a standard basis (h, e, f) for $\mathfrak{sl}_2$, arrows to the right represent the action of e, arrows to the left represent the action of f, and labels represent weights. It is visible from this description that if $\lambda \notin \mathbb{Z}_{\geqslant 0}$, then Δ_λ is simple; otherwise, if $\lambda \in \mathbb{Z}_{\geqslant 0}$, there is a short exact sequence

$$0 \to L_{-\lambda-2} \to \Delta_\lambda \to L_\lambda \to 0.$$

Notation 8.4. Recall the element $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$. The *dot action* of the finite Weyl group W_{f} on $\mathfrak{X}$ is given by

$$x \bullet \lambda = x(\lambda + \rho) - \rho.$$

In words, this shifts the standard action of W_{f} to have centre $-\rho$.

Theorem 8.5 (Harish-Chandra)**.** There is an isomorphism

$$Z(U(\mathfrak{g})) \to S(\mathfrak{h})^{(W_{\mathrm{f}},\bullet)} \cong \mathcal{O}(X)$$

where X denotes the quotient space $\mathfrak{h}^*/(W_{\mathrm{f}},\bullet)$.

Because every object of $\mathcal{O}$ admits a central character, the category $\mathcal{O}$ decomposes as

$$\mathcal{O} = \bigoplus_{\lambda\in\mathfrak{h}^*/(W_{\mathrm{f}},\bullet)} \mathcal{O}_\lambda,$$

where the *block* $\mathcal{O}_\lambda$ consists of modules with *generalised central character* λ. By definition, such modules consist of generalised χ_λ-eigenvectors for $Z(\mathfrak{g})$, where

$$\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$$

is the *central character* associated to $\lambda \in \mathfrak{h}^*$; see [Hum08, Sect. 1.7-1.11] for further details. Through this decomposition of $\mathcal{O}$ and other considerations, particularly Jantzen's *translation principle* (to be discussed soon in the analogous setting of algebraic groups), the problem of calculating characters can be reduced to the *principal block* $\mathcal{O}_0$.

Remark 8.6. As used here, the term *block* is a misnomer: blocks of a category are usually understood to be indecomposable, which need not hold for the $\mathcal{O}_\lambda$. In fact, the relevant condition for $\mathcal{O}_\lambda$ to be a genuine block is that λ is integral.

Notation 8.7. For $x \in W_{\mathrm{f}}$, let

$$L_x = L_{xw_0\bullet 0}, \quad \Delta_x = \Delta_{xw_0\bullet 0} \in \mathcal{O}_0,$$

where $w_0 \in W_{\mathrm{f}}$ is the longest element. For example, $L_{\mathrm{id}} = L_{-2\rho}$, $L_{w_0} = L_0$.

8.2. **The conjecture and its proof.** We are now prepared to state the Kazhdan–Lusztig conjecture.

Conjecture 8.8 (Kazhdan–Lusztig)**.** In the Grothendieck group of $\mathcal{O}$, we have

$$[L_x] = \sum_{y\in W_{\mathrm{f}}} (-1)^{\ell(x)-\ell(y)} P_{y,x}(1)[\Delta_y], \tag{8.1}$$

where the $P_{y,x} \in \mathbb{Z}[q, q^{-1}]$ are the Kazhdan–Lusztig polynomials.

We omit a detailed introduction to the Kazhdan–Lusztig polynomials, instead directing the reader to [Soe97]. In what follows, we shall have occasion to refer to the *Hecke algebra* $H = H(W_0, S_0)$ over $\mathbb{Z}[v^{\pm 1}]$ associated to a Coxeter system (W_0, S_0), with *standard basis* $\{h_w\}_{w\in W_0}$ and *Kazhdan–Lusztig basis* $\{b_w\}_{w\in W_0}$. An introduction to all these objects can be found in [EMTW].

Example 8.9. Consider the example of $\mathfrak{sl}_2$. Here we have

$$b_1 = h_1, \quad b_s = h_s + v,$$

so that $P_{1,1} = 1 = P_{s,s}$, $P_{1,s} = v$, and $P_{s,1} = 0$ are the relevant Kazhdan–Lusztig polynomials. Hence the conjecture predicts

$$[L_{\mathrm{id}}] = [\Delta_{\mathrm{id}}], \quad [L_s] = -[\Delta_{\mathrm{id}}] + [\Delta_s],$$

which we know by simplicity of the Verma module $\Delta_{w_0\bullet 0} = \Delta_{-2}$ and by considering the exact sequence

$$0 \to L_{-2} = L_{\mathrm{id}} \to \Delta_0 \to L_0 = L_s \to 0$$

for the Verma module $\Delta_0 = \Delta_s$ (see Example 8.3).

Let us make some remarks on the proof of Conjecture 8.8. Doing so will require us to work with perverse sheaves, but will omit to describe this topic, beyond introducing some necessary notation.

Notation 8.10. Suppose $Y = \bigsqcup_{\lambda\in\Lambda} Y_\lambda$ is a $\mathbb{C}$-variety stratified by subvarieties Y_λ isomorphic to affine spaces. Then we have the following perverse sheaves on Y:

$$\Delta_\lambda^{\mathrm{geom}} = j_{\lambda !}(\underline{\mathbb{C}}_{Y_\lambda}[d_\lambda]), \quad \mathrm{IC}_\lambda = j_{\lambda !*}(\underline{\mathbb{C}}_{Y_\lambda}[d_\lambda]), \quad \nabla_\lambda^{\mathrm{geom}} = j_{\lambda *}(\underline{\mathbb{C}}_{Y_\lambda}[d_\lambda]),$$

where $j_\lambda : Y_\lambda \hookrightarrow Y$ is the inclusion, d_λ is the complex dimension of Y_λ, and underlines denote constant sheaves. Referring instead to its support, IC_λ is sometimes written as $\mathrm{IC}(\overline{Y_\lambda})$.

We will also need to recall briefly the notion of a differential operator on a commutative k-algebra A; see [MR01, Ch. 15] for a more detailed exposition of this topic. The following is an inductive definition of differential operators on A.

Definition 8.11. A k-linear endomorphism $P \in \mathrm{End}(A)$ is a *differential operator of order* $\leqslant n \in \mathbb{Z}$ if either

- P is a differential operator of order zero, i.e. multiplication by some $a \in A$.
- $[P, a]$ is a differential operator of order $\leqslant n-1$ for all $a \in A$.

We write $D^n(A)$ for the ring of differential operators of order $\leqslant n$, and

$$D(A) = \bigcup_n D^n(A).$$

If X is an affine k-scheme, we set $D(X) = D(k[X])$. For more general k-schemes X, this construction sheafifies to give a *sheaf of differential operators* $\mathscr{D}_X$ on X.

We are ready to return to the Kazhdan–Lusztig conjecture. Consider $Y = G/B$ over $\mathbb{C}$, stratified according to the Bruhat decomposition with $\Lambda = W_{\mathrm{f}}$ and $Y_w = BwB/B$. In the Grothendieck group of G/B, the theory of perverse sheaves gives us a formula which is similar in appearance to (8.1):

$$[\mathrm{IC}_x] = \sum_{y\in W_{\mathrm{f}}} (-1)^{\ell(x)-\ell(y)} P_{y,x}(1)[\Delta_y^{\mathrm{geom}}]; \tag{8.2}$$

here the ground ring is $A = \mathbb{C}$ and $\mathrm{IC}_x = \mathrm{IC}_x^{\mathbb{C}}$, etc. Equation (8.2) is a consequence of Kazhdan–Lusztig's calculation of the stalks of intersection cohomology complexes via Kazhdan–Lusztig polynomials [KL80].

On the other hand, the Beilinson–Bernstein localisation theorem (introduced in [BB81]) posits an equivalence of categories,

$$(U(\mathfrak{g})/(Z^+))\text{-mod} \cong \mathscr{D}_{G/B}\text{-mod}, \tag{8.3}$$

where Z^+ is the kernel of the map $Z \to \mathrm{End}(\mathbb{C})$ given by action on the trivial module. In one direction of this equivalence, we *localise* modules for $U(\mathfrak{g})/(Z^+)$ to construct sheaves of $\mathscr{D}_{G/B}$-modules; in the other, we take global sections of $\mathscr{D}_{G/B}$-modules. Requiring certain good behaviour cuts out a *regular holonomic* subcategory $\mathcal{H} \subseteq \mathscr{D}_{G/B}$-mod, and there is then an equivalence

$$\mathcal{H} \to \mathrm{Perv}(G/B, \mathbb{C}); \tag{8.4}$$

this is a version of the Riemann–Hilbert correspondence, which in its classical form states that certain differential equations are determined by their monodromy. Under the composite of (8.3) and (8.4), L_x and Δ_x correspond to IC_x and Δ_x^{geom}, respectively, and hence we deduce the Kazhdan–Lusztig conjecture by combining (8.2), (8.3), and (8.4).

9. Lusztig conjecture

9.1. **Affine Weyl group.** The key point of this lecture is to state Lusztig's conjecture for the group G over the field k, which parallels Conjecture 8.8. Recall the finite Weyl group W_{f} introduced above.

Definition 9.1. (1) The *affine Weyl group* associated to the dual root system $(R^\vee \subseteq \mathfrak{X}^\vee, R \subseteq \mathfrak{X})$ is

$$W = W_{\mathrm{f}} \ltimes \mathbb{Z}R.$$

We may view this semidirect product naturally as a subgroup of the affine transformations of $\mathfrak{X}_{\mathbb{R}}$. Denote by $t_\gamma \in W$ the translation by $\gamma \in \mathbb{Z}R$.

(2) The *p-dilated dot action* of W on $\mathfrak{X}_{\mathbb{R}}$ is prescribed as follows:

$$x \bullet_p \lambda = x(\lambda + \rho) - \rho, \quad t_\gamma \bullet_p \lambda = \lambda + p\gamma;$$

here $x \in W_{\mathrm{f}}$ and $\gamma \in \mathbb{Z}R$. In words, this action shifts the centre to $-\rho$ and dilates translations by a factor of p.

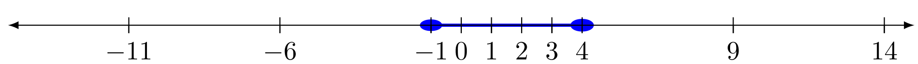
(3) The *fundamental alcove* is

$$A_{\mathrm{fund}} = \{\lambda \in \mathfrak{X}_{\mathbb{R}} : 0 < \langle \lambda + \rho, \alpha^\vee \rangle < p \text{ for all } \alpha \in R^+\}.$$

Its closure in $\mathfrak{X}_{\mathbb{R}}$ is a fundamental domain for the p-dilated action of W.

Notation 9.2. We let W^{f} and ${}^{\mathrm{f}}W$ denote fixed sets of minimal coset representatives for W/W_{f} and $W_{\mathrm{f}}\backslash W$, respectively.

Example 9.3. The following picture for $G = \mathrm{SL}_2$ and $p = 5$ indicates some of the reflection 0-hyperplanes (points) for the dot action of W on $\mathfrak{X}_{\mathbb{R}}$, along with the closure of the fundamental alcove (in blue). Notice the shift by $-\rho = -1$.



Exercise 9.4. If E is an $\mathbb{R}$-vector space, let $\mathrm{Aff}(E)$ denote the set of affine transformations of E. Show that the p-dilated dot action of W on $\mathfrak{X}_{\mathbb{R}}$ corresponds to the subgroup of $\mathrm{Aff}(\mathfrak{X}_{\mathbb{R}})$ generated by the reflections in the hyperplanes

$$\langle \lambda + \rho, \alpha^\vee \rangle = p^m, \quad \text{for all } \alpha \in R^+, m \in \mathbb{Z}.$$

Definition 9.5. Suppose $\mathcal{A}$ is an abelian category. A *Serre subcategory* of $\mathcal{A}$ is a non-empty full subcategory $\mathcal{C} \subseteq \mathcal{A}$ such that for any exact sequence

$$0 \to A' \to A \to A'' \to 0$$

in $\mathcal{A}$, $A \in \mathcal{C}$ if and only if $A', A'' \in \mathcal{C}$. Equivalently, $\mathcal{C}$ is closed under taking subobjects, quotients, and extensions in $\mathcal{A}$.

Proposition 9.6 (Linkage Principle)**.** The category $\mathrm{Rep}(G)$ is the direct sum of its *blocks* $\mathrm{Rep}_\lambda(G)$, for $\lambda \in \mathfrak{X}_{\mathbb{R}}/(W, \bullet_p)$. Here $\mathrm{Rep}_\lambda(G)$ is the Serre subcategory generated by simple modules L_μ for $\mu \in (W \bullet_p \lambda) \cap \mathfrak{X}_+$.

Remark 9.7. The appropriate analogue of Remark 8.6 applies here: our blocks $\mathrm{Rep}_\lambda(G)$ need not be indecomposable. The abuse is not too severe for $p \geqslant h$, where $\mathrm{Rep}_\lambda(G)$ is indecomposable unless $\langle \mu + \rho, \alpha^\vee \rangle$ is divisble by p for every $\alpha \in R_+$. The 'true' block decomposition is laid out in [Don80].

If $p \geqslant h$, the Coxeter number, then 0 is a *p-regular* element in A_{fund} (that is, it has trivial stabiliser under the p-dilated dot action of W). When x is such that $x \bullet_p 0 \in \mathfrak{X}_+$, set

$$L_x = L_{x \bullet_p 0};$$

this is a simple module in the *principal block* $\text{Rep}_0(G)$.

Exercise 9.8. Suppose $p \geqslant h$. By explicit calculation, work out how many weights of the form $W \bullet_p 0$ are p-restricted for the root systems A_1, A_2, B_2, G_2. On the basis of these calculations, formulate a conjecture for the answer in general.

Now we are ready to state the Lusztig conjecture, at least in a simplified form.

Conjecture 9.9 (Lusztig [Lus80])**.** Under certain assumptions on p and x, the following equation holds in the Grothendieck group $[\text{Rep}_0(G)]$:

$$[L_x] = \sum_y (-1)^{\ell(x)+\ell(y)} P_{w_0 y, w_0 x}(1)[\Delta_y].$$

The key point to note here is the independence of the formula from the prime p, subject to the assumptions mentioned; the parallel to the Kazhdan–Lusztig conjecture 8.8 should also be apparent. In the next lecture we shall be explicit about these assumptions, and say more about the current status of this conjecture.

9.2. **Distribution algebras.** To conclude, we introduce distribution algebras and relate them to the representation theory of G. This is not technology we will rely on in later lectures, but it would be remiss to omit them entirely from our story. Moreover, consideration of distribution algebras allows one to see why the Linkage Principle holds (at least when p is not too small).

Definition 9.10. Extending our previously discussed notion of left-invariant derivations on G, we define the k-algebra of *distributions* on G to be

$$\text{Dist}\, G = \text{left-invariant differential operators on } G.$$

In terms of terminology already available to us, this is the most convenient definition of $\text{Dist}(G)$. For a definition and discussion of $\text{Dist}(G)$ as a certain subalgebra of $k[G]^*$, see [Jan03, Chapter I.7].

Examples 9.11.
(1) $\text{Dist}\, \mathbb{G}_a$ has a countably infinite k-basis given by the elements

$$\frac{\partial_z^n}{n!}, \quad n \geqslant 0,$$

where ∂_z is the left-invariant k-linear derivation of $k[\mathbb{G}_a] = k[z]$ specified by $\partial_z(z) = 1$.

(2) $\text{Dist}\, \mathbb{G}_m$ has a k-basis in the elements

$$\binom{z\partial_z}{n} = \frac{\partial_z(\partial_z - 1)\cdots(\partial_z - n + 1)}{n!}, \quad n \geqslant 0,$$

where $k[\mathbb{G}_m] = k[z, z^{-1}]$.

The inclusion $\mathfrak{g} = \text{Lie}(G) \hookrightarrow \text{Dist}(G)$ induces an algebra homomorphism

$$\gamma : U(\mathfrak{g}) \to \text{Dist}(G).$$

For groups defined over a ground field of characteristic zero, γ is an isomorphism; for k of characteristic $p > 0$, all one can say is that γ factors over an embedding

$$u(\mathfrak{g}) \hookrightarrow \mathrm{Dist}(G).$$

In fact, $\mathrm{Dist}(G)$ turns out to be the best replacement for $U(\mathfrak{g})$ in characteristic p. Any G-module M gives rise to a locally finite $\mathrm{Dist}(G)$-module, and the induced functor is fully faithful:

$$\mathrm{Hom}_G(M, M') = \mathrm{Hom}_{\mathrm{Dist}(G)}(M, M').$$

Conversely, if G is semi-simple and simply connected, then a theorem of Sullivan [Sul78] establishes that any locally finite $\mathrm{Dist}(G)$-module arises from a G-module.

We may assume that $G = G_k$ arises via base change from an algebraic group $G_{\mathbb{Z}}$ defined over $\mathbb{Z}$. Base extension to $\mathbb{C}$ yields $G_{\mathbb{C}}$, with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. This Lie algebra is spanned by Chevalley elements f_α, e_α, and $h_\alpha = [e_\alpha, f_\alpha]$, $\alpha \in R^+$; see for instance [Hum72]. Consider the following $\mathbb{Z}$-subalgebra of $U(\mathfrak{g}_{\mathbb{C}})$:

$$U_{\mathbb{Z}} = \mathbb{Z}\left[\frac{f_\alpha^\ell}{\ell!}, \binom{h_\alpha}{\ell}, \frac{e_\alpha^\ell}{\ell!}\right].$$

We then have $\mathrm{Dist}\ G_{\mathbb{Z}} = U_{\mathbb{Z}}$ and $\mathrm{Dist}\ G_k = U_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$. The algebra $U_{\mathbb{Z}}$ is known as a *Kostant* $\mathbb{Z}$*-form* of $U(\mathfrak{g}_{\mathbb{C}})$.

Remark 9.12. If $Z(G)$ is reduced and p is good, the Linkage Principle 9.6 can be established by considering $Z(\mathrm{Dist}(G))$. Since $\mathrm{Dist}(G)$ replaces $U(\mathfrak{g})$ in characteristic p, this proof strategy is analogous to the usual approach to the Linkage Principle for category $\mathcal{O}$ of a complex semi-simple Lie algebra $\mathfrak{g}$. In that setting, consideration of central characters yields that L_λ and L_μ are in the same block[3] if and only if $\lambda = \mu$ in

$$\mathfrak{h}^*/(W_{\mathrm{f}}, \bullet) = \mathrm{Spec}\ Z(U(\mathfrak{g})).$$

In characteristic p, the reduction modulo p of $Z(U(\mathfrak{g}))$ defines a subalgebra in $Z(\mathrm{Dist}\ G)$. Considerations of central characters for this subalgebra gives the analogous condition that $\lambda = \mu$ in

$$\mathfrak{h}^*_{\mathbb{F}_p}/(W_{\mathrm{f}}, \bullet) = \mathfrak{h}^*/((W, \bullet) \rtimes p\mathfrak{X}).$$

This is almost the Linkage Principle; to conclude, we need to pass from $(W_{\mathrm{f}}, \bullet) \rtimes p\mathfrak{X}$ to $(W_{\mathrm{f}}, \bullet) \rtimes p\mathbb{Z}R$. This is achieved by consideration of the centre $Z(G)$, which—if reduced—agrees with $p\mathfrak{X}/p\mathbb{Z}R$. The proof of the Linkage Principle for small p has a rather different complexion, first provided by Andersen [And80b].

[3] Caution: do not forget about Remark 8.6!

Lecture IV

10. The Linkage Principle

10.1. **Recollections.** As previously, assume G is a semi-simple and simply connected algebraic k-group. The affine Weyl group (of the dual root system) is $W = W_{\mathrm{f}} \ltimes \mathbb{Z}R$. We also have the p-dilated affine Weyl group, $W_p = W_{\mathrm{f}} \ltimes p\mathbb{Z}R$, i.e.

$$W_p = \langle \text{reflections in hyperplanes } \langle \lambda + \rho, \alpha^\vee \rangle = mp, \text{ for } \alpha \in R,\ m \in \mathbb{Z} \rangle.$$

Evidently the p-dilated dot action of W corresponds to the regular dot action of W_p, so the choice to speak in terms of $\bullet_p$ or W_p is mostly a matter of taste. We saw that a fundamental domain for the $(W_p, \bullet)$-action on $\mathfrak{X}_{\mathbb{R}}$ is the closure of

$$A_{\text{fund}} = \{\lambda \in \mathfrak{X}_{\mathbb{R}} : 0 < \langle \lambda + \rho, \alpha^\vee \rangle < p \text{ for all } \alpha \in R^+\}$$

in $\mathfrak{X}_{\mathbb{R}}$. Now (W_p, S) is a Coxeter system, where $S = \{\text{reflections in the walls of } A_{\text{fund}}\}$. We will assume $p \geqslant h$, the Coxeter number, so that $0 \in A_{\text{fund}}$ is a *regular element* in the sense that $\mathrm{Stab}_{(W_p,\bullet)}(0) = \{1\}$.

Recall that the *facet* containing $\lambda \in \mathfrak{X}_{\mathbb{R}}$ is the subset of all $\mu \in \mathfrak{X}_{\mathbb{R}}$ sharing the same stabiliser as λ under $(W, \bullet_p)$.

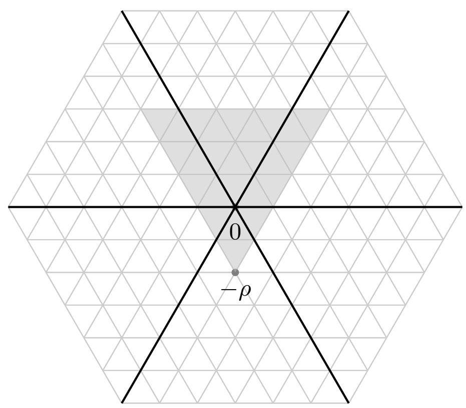


Figure 2. Here is a picture for $G = \mathrm{SL}_3$ and $p = 5$, with root hyperplanes in bold and A_{fund} shaded.

10.2. **Statement; translation functors.** One way to state the Linkage Principle for G is by the decomposition

$$\mathrm{Rep}\ G = \bigoplus_{\lambda \in \mathfrak{X}/(W_p,\bullet)} \mathrm{Rep}_\lambda(G);$$

as usual, we call $\mathrm{Rep}_0(G)$ the principal block.

Most questions about the representation theory of G can be reduced to questions about $\mathrm{Rep}_0(G)$ using *translation functors*; let us describe these briefly (for more detail, see [Jan03, Ch. II.7]). For $\lambda \in \mathfrak{X}$, let pr_λ denote the projection functor from $\mathrm{Rep}\ G$ onto $\mathrm{Rep}_\lambda(G)$. Given $\lambda, \mu \in A_{\text{fund}}$, let ν be the unique dominant weight in the W-orbit of $\mu - \lambda$. We then define the translation functor $T_\lambda^\mu : \mathrm{Rep}(G) \to \mathrm{Rep}(G)$ by the formula

$$T_\lambda^\mu(V) = \mathrm{pr}_\mu(L_\nu \otimes \mathrm{pr}_\lambda V).$$

This functor is exact and $(T_\lambda^\mu, T_\mu^\lambda)$ is an adjoint pair. By restriction, T_λ^μ induces a functor $\mathrm{Rep}_\lambda(G) \to \mathrm{Rep}_\mu(G)$. The *translation principle* states this is an equivalence whenever λ, μ belong to the same facet. Roughly speaking, blocks associated to weights in the closure of a facet are "simpler"; this is the essence of why considering the principal block is sufficient for many purposes.

Notation 10.1. Write $L_x = L_{x\bullet 0}$, $\nabla_x = \nabla_{x\bullet 0}$, where $x \in {}^{\mathrm{f}}W_p$ is a minimal coset representative.

11. Elaborations on the Lusztig Conjecture

11.1. **Explicit statement.** On our second pass, we will be precise in stating the Lusztig Conjecture.

Conjecture 11.1 (Lusztig)**.** Suppose $p \geqslant h$ and $x \bullet 0$ is an element of $\mathfrak{X}_+$, where $x \in W$ satisfies *Jantzen's condition*: $\langle x \bullet_p 0 + \rho, \alpha^\vee \rangle \leqslant p(p - h + 2)$ for all $\alpha \in R^+$. Then

$$[L_x] = \sum (-1)^{\ell(x)+\ell(y)} P_{w_0 y, w_0 x}(1)[\nabla_y].$$

The key feature to observe is the independence from p, or in other words that the formula is uniform over all $p \geqslant h$.

11.2. **History.** The conjecture was made in 1980 and proved in the mid 1990's for $p \geqslant N$, where N is a non-explicit bound depending only the root system; this was work of Lusztig ([Lus94], [Lus95]), Kashiwara–Tanisaki ([KT95], [KT96]), Kazhdan–Lusztig ([KL93], [KL94a], [KL94b]), and Andersen–Jantzen–Soergel ([AJS94]). In the mid-2000's, a new proof was provided by Arkhipov–Bezrukavnikov–Ginzburg [ABG04]. Early in the next decade, Fiebig gave another new proof [Fie11] and an explicit but enormous lower bound N [Fie12]; for instance, $N = 10^{100}$ for GL_{10}. Most recently, the second author [Wil17b], [Wil17c] (with help from Elias, He, Kontorovich, and McNamara variously in [EW16], [HW15], and the appendix to [Wil17c]) proved that the conjecture is not true for GL_n for many p on the order of exponential functions of n.

	∇_0	∇_8	∇_{10}	∇_{18}	∇_{20}	∇_{28}	∇_{30}
L_0	1						
L_8	−1	1					
L_{10}	1	−1	1				
L_{18}	−1	1	−1	1			
L_{20}	1	−1	1	−1	1		
L_{28}					−1	1	
L_{30}				−1	1	−1	1

FIGURE 3. Plot of multiplicities in the principal block of SL_2 for $p = 5$. Shaded is the region in which Lusztig's conjecture is valid.

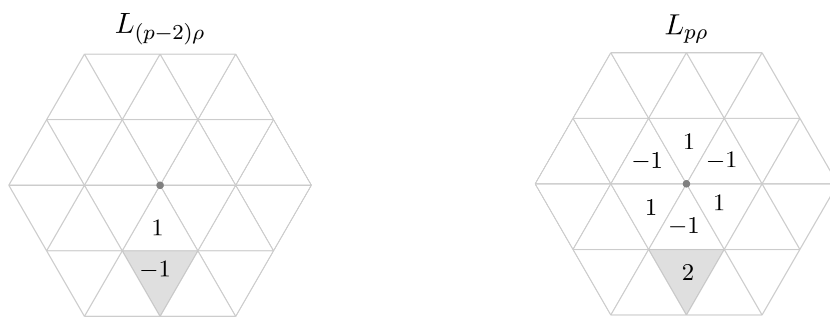


FIGURE 4. Similar plots for SL_3 and the highest weights $(p-2)\rho$ and $p\rho$, respectively ($p \geqslant 3$). The gray regions are A_{fund}. The second example is the first one featuring a 2; its multiplicities can be checked using the Steinberg tensor product theorem.

Exercise 11.2. Verify that the multiplicities given in Figure 4 are correct.

Exercise 11.3. Let $W = \widetilde{A}_1$ denote the infinite dihedral group with Coxeter generators s_0, s_1, and let

$$w_m = s_0 s_1 \cdots, \quad w'_m = s_1 s_0 \cdots$$

be the elements given by the unique reduced expressions starting with s_0 and s_1, respectively, of length m. Note that any non-identity $x \in W$ is equal to a unique w_m or w'_m.

(1) Compute the Bruhat order on W.
(2) Prove inductively that one has

$$b_{w_m} = h_{w_m} + \sum_{0<n<m} v^{m-n} h_{w_n} + \sum_{0<n'<m} v^{m-n'} h_{w'_{n'}} + v^m h_{\mathrm{id}}.$$

(3) Deduce that Lusztig's character formula holds for $x \bullet_p 0 \in \mathfrak{X}_+$ if and only if $x \bullet_p 0$ has two p-adic digits. (Hint: This part will require use of an exercise from Lecture III.)

12. THE FINKELBERG–MIRKOVIĆ CONJECTURE

12.1. **Objects in affine geometry.** Let $G = G_k$ be a connected semi-simple group, obtained by base change from a group $G_{\mathbb{Z}}$ over $\mathbb{Z}$. Recall that the *adjoint representation* of G on $\mathfrak{g} = \mathrm{Lie}(G)$ is given by differentiating inner automorphisms at the identity $e \in G$:

$$\mathrm{Ad} : G \to \mathfrak{gl}(\mathfrak{g}), \quad g \mapsto d(\mathrm{Int}\ g)_e,$$

where $\mathrm{Int}\ g(h) = ghg^{-1}$ for $h \in G$. The *adjoint group* of G is then the image

$$G_{\mathrm{ad}} = \mathrm{Ad}\ G \subseteq \mathrm{Aut}(\mathfrak{g}).$$

Exercise 12.1. Show that the character and root lattices of G_{ad} coincide. Hence deduce that for general semi-simple G, one always has an equivalence:

$$\mathrm{Rep}_0(G) \cong \mathrm{Rep}_0(G_{\mathrm{ad}}).$$

For the sake of simplicity, and in light of Exercise 12.1, we will be content to operate with the following assumption from now on.

Assumption 12.2. G is of *adjoint type*, meaning that Ad is faithful: $G \cong G_{\mathrm{ad}}$.

The point of making this assumption is that Frobenius twist then yields a functor

$$(-)^{\mathrm{Fr}} : \mathrm{Rep}\, G \to \mathrm{Rep}_0(G).$$

Indeed, the Frobenius twist of the simple module L_λ is $L_{p\lambda}$, and $p\lambda \in p\mathfrak{X} = p\mathbb{Z}R$ by Assumption 12.2.

Let us denote by $G^\vee$ the dual group to G over the *complex* numbers. Let $F = \mathbb{C}((t))$ with ring of integers $\mathcal{O} = \mathbb{C}[[t]]$. Then

$$G^\vee(F) \supseteq K = G^\vee(\mathcal{O});$$

this K is analogous to a maximal compact subgroup of $G^\vee(F)$. The assignment $t = 0$ defines an evaluation homomorphism

$$\mathrm{ev} : K \to G^\vee(\mathbb{C}) = G^\vee;$$

consider then the preimage $\mathrm{Iw} = \mathrm{ev}^{-1}(B^\vee) \subseteq K$ of a Borel subgroup $B^\vee \subseteq G^\vee$. Now we can introduce some geometric objects:

$$\mathrm{Fl} = G^\vee(F)/\mathrm{Iw} = \bigsqcup_{x \in W} \mathrm{Iw} \cdot x\mathrm{Iw}/\mathrm{Iw},$$

the affine flag variety, which is a $K/\mathrm{Iw} = G^\vee/B^\vee$-bundle over

$$\mathrm{Gr} = G^\vee(F)/K = \bigsqcup_{x \in W^{\mathrm{f}}} \mathrm{Iw} \cdot xK/K,$$

the affine Grassmannian; here we view $W^{\mathrm{f}} \subseteq W$ as a set of minimal coset representatives for W/W_{f}. In these two decompositions, each Iw-orbit is isomorphic to an affine space of dimension $\ell(x)$; we refer to these orbits as *Bruhat cells*.

The affine Grassmannian Gr and affine flag variety Fl are *ind-varieties* (that is, colimits of varieties under closed embeddings). An in-depth treatment of their geometric properties would require at least another lecture; since time does not permit us to deliberate, we recommend [Kum02], [BR18], and [Zhu17] for further information on this fascinating topic.

Example 12.3. For $G = \mathrm{SL}_2$, we have $G^\vee = \mathrm{SL}_2$. The (complex points of the) affine Grassmannian can be written as the disjoint union

$$\mathbb{C}^0 \sqcup \mathbb{C}^1 \sqcup \mathbb{C}^2 \sqcup \cdots$$

along complicated gluing maps. These strata are in bijective correspondence with $W/W_{\mathrm{f}} \cong \mathbb{Z}_{\geqslant 0}$ and coincide with the Bruhat cells above.

The following exercise is very beautiful and due to Lusztig [Lus81, Section 2].

Exercise 12.4. Let V denote an n-dimensional $\mathbb{C}$-vector space and consider

$$E = V^{\oplus n}$$

equipped with the nilpotent operator

$$t : E \to E, \quad (v_1, \cdots, v_n) \mapsto (0, v_1, \cdots, v_{n-1}).$$

Denote by Y the variety of t-stable n-dimensional subspaces of E.

(1) Prove that Y is a projective variety.

(2) Let $U \subseteq Y$ be the open subvariety of t-stable subspaces transverse to $V^{\oplus(n-1)} \oplus 0$. Show that a point $X \in U$ is uniquely determined by maps $f_i : V \to V$, $1 \leqslant i \leqslant n-1$, such that

$$X = \{(f_{n-1}(v), f_{n-2}(v), \cdots, f_1(v), v) : v \in V\}.$$

(3) Now use that X is t-stable to deduce that $f_i = f_1^i$ and that $f_1^n = 0$. Conclude that

$$U \cong \mathfrak{N}(\mathrm{End}(V)),$$

the subvariety of nilpotent endomorphisms of V.

(4) Prove that $Y \cong \overline{\mathrm{Gr}_{n\varpi_1}}$, a Schubert variety in the affine Grassmannian of GL_n.

12.2. **Statement of the conjecture.** The following theorem is one of the most important geometric tools in the theory.

Theorem 12.5 (Geometric Satake equivalence)**.** There is an equivalence of monoidal categories,

$$\mathrm{Sat} : (\mathrm{Rep}(G), \otimes) \to (\mathrm{Perv}_{(G^\vee(\mathcal{O}))}(\mathrm{Gr}, k), *),$$

where $*$ is the *convolution product* on perverse sheaves.

This theorem was established by Mirković –Vilonen [MV07]. Their proof is non-constructive and relies on the Tannakian formalism: one shows that the category of perverse sheaves is Tannakian, and hence is equivalent to the representations of some group scheme. One then works hard to show this group scheme is G. In this way, an equivalence of categories is established without explicitly providing functors in either direction!

Remark 12.6. In fact, the theorem can be used to *construct* the dual group, without knowing its existence a priori.

Notation 12.7. From this point onwards, we will sometimes refer to (co)standard and IC sheaves with coefficients in a general commutative ring A (generalising Notation 8.10). If A is not clear from context, we will use notation such as IC_λ^A or $\mathrm{IC}(\overline{Y_\lambda}, A)$; commonly, A will be $\mathbb{C}$, k, or $\mathbb{Z}$.

Conjecture 12.8 (Finkelberg–Mirković)**.** There is an equivalence of abelian categories FM fitting into a commutative diagram:

$$\begin{array}{ccc} \mathrm{Rep}_0(G) & \xrightarrow[FM]{\cong} & \mathrm{Perv}_{(\mathrm{Iw})}(\mathrm{Gr}, k) \\ {\scriptstyle (-)^{\mathrm{Fr}}}\uparrow & & \uparrow \\ \mathrm{Rep}(G) & \xrightarrow[\mathrm{Sat}]{\cong} & \mathrm{Perv}_{(G^\vee(\mathcal{O}))}(\mathrm{Gr}, k). \end{array}$$

Moreover, under the equivalence FM,

$$L_x \mapsto \mathrm{IC}_{x^{-1}}^k \quad \text{and} \quad \nabla_x \mapsto \nabla_{x^{-1}}^{\mathrm{geom}}.$$

Assumption 12.9. Through the remainder of these notes, we will assume Conjecture 12.8 holds; in fact, a proof was recently announced by Bezrukavnikov–Riche. The conjecture provides a useful guiding principle in geometric representation theory. All the consequences that we will draw from it below can be established by other means, but with proofs that are much more roundabout.

Application 12.10. As a first application, let us explain why the Finkelberg–Mirković conjecture helps us understand Lusztig's character formula. Recall that we want to find expressions of the form:

$$[L_x] = \sum a_{y,x}[\nabla_y].$$

If we apply the Finkelberg–Mirković equivalence, this becomes

$$[\mathrm{IC}^k_{x^{-1}}] = \sum a_{y,x}[\nabla^{\mathrm{geom}}_{y^{-1}}].$$

Taking Euler characteristics of costalks at $y^{-1}\mathrm{Iw}/\mathrm{Iw}$ yields

$$\chi((\mathrm{IC}^k_{x^{-1}})^!_{y^{-1}}) = (-1)^{\ell(y)}a_{y,x}.$$

Now, there exist "integral forms" $\mathrm{IC}^{\mathbb{Z}}_{x^{-1}}$ such that the perverse shaves $\mathrm{IC}^{\mathbb{Z}}_{x^{-1}} \otimes^L_{\mathbb{Z}} k$ are isomorphic to $\mathrm{IC}^k_{x^{-1}}$ if (certain) stalks and costalks of the $\mathrm{IC}^{\mathbb{Z}}_{x^{-1}}$ are free of p-torsion; suppose this holds and let $\mathrm{IC}^{\mathbb{Q}}_{x^{-1}} = \mathrm{IC}^{\mathbb{Z}}_{x^{-1}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Then

$$(-1)^{\ell(y)}a_{y,x} = \chi((\mathrm{IC}^k_{x^{-1}})^!_{y^{-1}}) = \chi((\mathrm{IC}^{\mathbb{Q}}_{x^{-1}})^!_{y^{-1}}) \tag{12.1}$$
$$= (-1)^{\ell(x)}P_{y^{-1}w_0,x^{-1}w_0}(1), \tag{12.2}$$

from which it follows that $a_{y,x} = (-1)^{\ell(x)+\ell(y)}P_{y^{-1}w_0,x^{-1}w_0}(1)$. Note it is the final equality on line (12.1) which depends on the p-torsion assumption, while the equation on line (12.2) follows from a classical formula of Kazhdan–Lusztig [KL80] for $P_{y,x}$ in terms of IC sheaf cohomology.

In conclusion, then, we can see that if $\mathrm{IC}^{\mathbb{Z}}_{x^{-1}} \otimes^L_{\mathbb{Z}} k$ stays simple for all $x \in {}^{\mathrm{f}}W_p$ satisfying Jantzen's condition, then the Lusztig conjecture holds. An induction shows that this implication is in fact an "if and only if".

Lecture V

13. Torsion explosion

Assume G is a Chevalley scheme over $\mathbb{Z}$, with $k = \overline{k}$ of characteristic p fixed as before. It is a 2017 result of Achar–Riche [AR18], expanding on earlier work of Fiebig [Fie11], that the Lusztig conjecture for G_k is equivalent to the absence of p-torsion in the stalks and costalks of $\mathrm{IC}(\overline{\mathrm{Gr}_x}, \mathbb{Z})$ for $x \in {}^{\mathrm{f}}W_p$ satisfying Jantzen's condition. This provided a clear topological approach to deciding the validity of Lusztig's character formula; we will discuss this in some detail momentarily.

Let us first resketch the historical picture. In the mid-1990s, the character formula was proved for large $p > N$; this was work of many authors, continued into the late 2000s by Fiebig's discovery of an effective (enormous) bound for N in terms of just the root system of G [Fie11]. It remained to determine the soundness of more modest estimates for the best possible N (e.g. linear or polynomial in h).

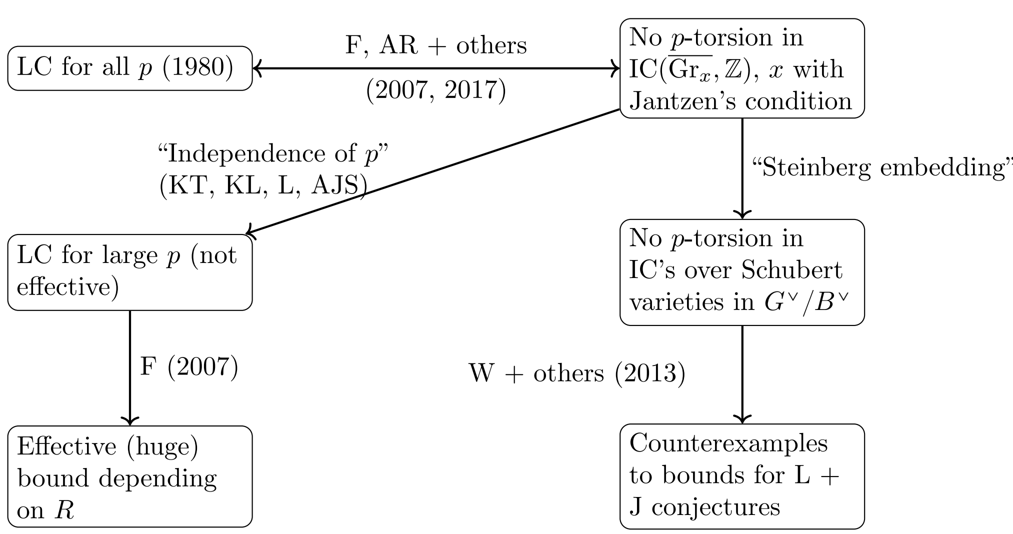
An important consequence of the topological formulation is the absence of p-torsion ($p > h$) in IC sheaves over Schubert varieties lying inside $G^\vee/B^\vee$ (the *finite flag variety*). This was first observed by Soergel in an influential paper [Soe00]; it follows by considering the "Steinberg embedding" associated to any dominant regular $\lambda \in \mathfrak{X}_+$,

$$G^\vee/B^\vee \hookrightarrow \mathrm{Gr}, \quad g \mapsto g \cdot t^\lambda,$$

which is stratum-preserving and induces an equivalence of categories,

$$\mathrm{Perv}_{(B^\vee)}(G^\vee/B^\vee) \cong \mathrm{Perv}_{(\mathrm{Iw})}(U),$$

where $U = \bigsqcup_{x \in W_{\mathrm{f}}} \mathrm{Iw} \cdot t^{x\lambda}$ (a locally closed subset of Gr). Using this property of the IC sheaves, the second author (in 2013, with help from several colleagues) was able to construct counter-examples to the expected bounds in Lusztig's conjecture and the James conjecture for irreducible mod p representations of symmetric groups. In particular, torsion was shown to grow at least exponentially, as opposed to linearly (as implied by the Lusztig conjecture) or quadratically (as implied by the James conjecture); in other words, a phenomenon of "torsion explosion".



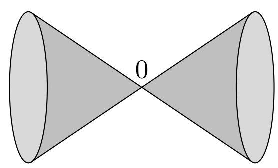
14. Geometric example

In this section we discuss a simple geometric example, where the phenomenon of torsion in IC sheaves is clearly visible. For more details on this example, the reader is referred to [JMW12].

Denote by X the quadric cone

$$\begin{aligned}\mathbb{C}^2/(\pm 1) \cong \operatorname{Spec} \mathbb{C}[X,Y]^{(\pm 1)} &= \operatorname{Spec} \mathbb{C}[X^2, XY, Y^2] \\ &= \operatorname{Spec} \mathbb{C}[a,b,c]/(ab - c^2) \\ &= \left\{ x = \begin{pmatrix} c & -a \\ b & -c \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C}) : x \text{ is nilpotent} \right\}.\end{aligned}$$

This variety has a stratification into two pieces, $X = X^{\mathrm{reg}} \sqcup \{0\}$. Its real points can be pictured as follows:



Suppose $\mathcal{F}$ is a perverse sheaf on X with respect to the given stratification. The following table indicates the degrees i in which $\mathcal{H}^i(\mathcal{F}|_{X'})$ can be non-zero for a stratum $X' = X_{\mathrm{reg}}$ or $X' = \{0\}$.

	-3	-2	-1	0	1
X_{reg}	0	$\star$	0	0	0
$\{0\}$	0	$\star$	$\star$	$\star$	0

To compute the intersection cohomology sheaf $\mathrm{IC}(X,k) = \mathrm{IC}(\overline{X_{\mathrm{reg}}}, k)$, we can use the *Deligne construction*:

$$\mathrm{IC}(X,k) = \tau_{<0} j_* \underline{k}_{X^{\mathrm{reg}}}[2],$$

where $j : X^{\mathrm{reg}} \hookrightarrow X$ is the inclusion and j_* denotes the right-derived functor Rj_*. First compute

$$(j_* \underline{k}_{X^{\mathrm{reg}}}[2])_0 = \lim_{\varepsilon \to 0} H^{i+2}(B(0,\varepsilon) \cap X^{\mathrm{reg}}, k);$$

since $B(0,\varepsilon) \cap X^{\mathrm{reg}}$ is homotopic to $S^3_\varepsilon \cap X^{\mathrm{reg}} = S^3/(\pm 1) = \mathbb{RP}^3$, we reduce to computing $H^*(\mathbb{RP}^3, k)$, or, by the universal coefficient theorem, $H^*(\mathbb{RP}^3, \mathbb{Z})$:

$H^*(\mathbb{RP}^3, \mathbb{Z}) =$

0	1	2	3
$\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$

$\overset{\text{UCF}}{\rightsquigarrow}$ $H^*(\mathbb{RP}^3, k) =$

0	1	2	3
k	$(k)_2$	$(k)_2$	k

.

Here $(k)_2 = k$ if $2 = 0$ in k, and 0 otherwise. Thus we find

stalks of $j_* \underline{k}_{X^{\mathrm{reg}}}[2] =$

-2	-1	0	1
k	0	0	0
k	$(k)_2$	$(k)_2$	k

and hence, applying $\tau_{<0}$:

stalks of $\mathrm{IC}(X,k) =$

-2	-1	0	1
k	0	0	0
k	$(k)_2$	0	0

.

Similarly, $\mathrm{IC}(X,\mathbb{Z}) = \underline{\mathbb{Z}}[2]$ and $\mathbb{D}(\mathrm{IC}(X,\mathbb{Z})) = \mathrm{IC}^+(X,\mathbb{Z})$ have stalks as follows:

stalks of IC =

-2	-1	0
$\mathbb{Z}$	0	0
$\mathbb{Z}$	0	0

Verdier duality $\rightsquigarrow$

-2	-1	0
$\mathbb{Z}$	0	0
$\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$

= stalks of IC^+.

Note particularly the 2-torsion in the lower right of the preceding table; this is what causes the aforementioned complications with torsion in this example. It turns out $\mathrm{IC}(X,\mathbb{Z}) \otimes^L_{\mathbb{Z}} k$ is simple if the characteristic of k is not 2; otherwise, it has composition factors $\mathrm{IC}(X,k)$ and $\mathrm{IC}(0,k)$.

Remark 14.1. By Exercise 12.4, X occurs as an open piece of the Schubert variety $\overline{\mathrm{Gr}_{2\varpi_1}} \subseteq \mathrm{Gr}_{\mathrm{SL}_2}$. Under the geometric Satake correspondence, our above analysis then translates into the fact that

$$\nabla_{2\varpi_1} \text{ is irreducible} \quad \Leftrightarrow \quad p \neq 2.$$

On the other hand, under the Finkelberg–Mirković conjecture,

$$\text{LCF for } L_{2p} \quad \Leftrightarrow \quad 2p \text{ has} \leqslant \text{two } p\text{-adic digits} \quad \Leftrightarrow \quad \mathrm{IC}(X,\mathbb{Z}) \otimes k \text{ is simple.}$$

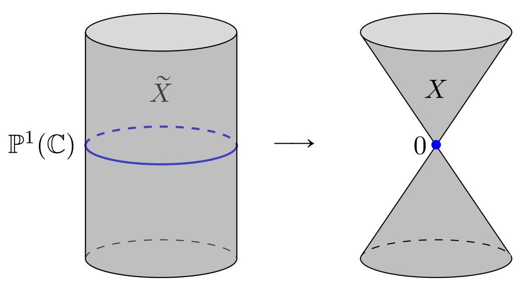
15. Intersection forms

A key reference for this section is [dCM09]. Practically speaking, a substantial problem is that the Deligne construction cannot be computed except in the very simplest cases, but looking at resolutions provides a way forward.

In the case considered above, the Springer resolution is

$$f : \widetilde{X} = T^*\mathbb{P}^1(\mathbb{C}) \to X,$$

or pictorially:



As we will see momentarily, there is an *intersection form* $[-,-]$ on

$$H^2(\mathbb{P}^1) = \mathbb{Z}[\mathbb{P}^1\mathbb{C}]$$

with values in $\mathbb{Z}$. Moreover,

$$[\mathbb{P}^1]^2 \text{ invertible in } k \quad \Leftrightarrow \quad f_*\underline{k}_X[2] \text{ is semi-simple.}$$

In this case $[\mathbb{P}^1]^2 = -2$, so this falls in line with our earlier findings. (Recall: the self-intersection of any variety inside its cotangent bundle is the negative of its Euler characteristic!)

Let X now be general, with stratification

$$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$$

and resolution $f : \widetilde{X} \to X$. We now have a schematic

$$\begin{array}{ccccc} \widetilde{X} & \longleftarrow & \widetilde{N_\lambda} & \longleftarrow & F_\lambda = f^{-1}(x_\lambda) \\ \big\downarrow{\scriptstyle f} & & \big\downarrow & & \big\downarrow \\ X = \bigsqcup_\lambda X_\lambda & \longleftarrow & N_\lambda & \longleftarrow & \{x_\lambda\} \end{array}$$

where $x_\lambda \in X_\lambda$ is an arbitrary point and $\varnothing \neq N_\lambda \subseteq X$ is a *normal slice* meeting the stratum X_λ transversely at the point x_λ. Assume f is *semi-small*, so that

$$\dim F_\lambda \leqslant \frac{1}{2} \dim \widetilde{N_\lambda} = n_\lambda.$$

This ensures the existence of a $\mathbb{Z}$-valued intersection form IF_λ on top homology

$$H_{2n_\lambda}(F_\lambda) = \bigoplus_{C \in \mathcal{C}_\lambda} \mathbb{Z}[C],$$

where $\mathcal{C}_\lambda$ is the set of irreducible components of F_λ of dimension n_λ.

Proposition 15.1 ([JMW14])**.** *$f_* \underline{k}[\dim \widetilde{X}]$ decomposes into a direct sum of IC sheaves if and only if every $IF_\lambda \otimes_{\mathbb{Z}} k$ is non-degenerate.*

Remark 15.2. The IF_λ are usually still difficult to calculate, since one must first find the fibres F_λ, compute components, and so on. A "miracle situation" arises when the F_λ are smooth, since for $p : F_\lambda \to \mathrm{pt}$,

$$IF_\lambda = p_!(\text{Euler class of normal bundle of } F_\lambda).$$

The following result underpins the idea of torsion explosion, by implying that torsion in the (co)stalks of Schubert varieties in SL_n/B grows at least exponentially with n.

Theorem 15.3 (Williamson [Wil17c])**.** *For any entry γ of any word of length ℓ in the generators $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, one can associate a Schubert variety $X_x \subseteq \mathrm{SL}_{3\ell+5}/B$, a Bott-Samelson resolution $\widetilde{X_x} \to X$, and a point $w_I \in X$, such that the miracle situation holds and the intersection form is $(\pm\gamma)$.*

For further discussion of the connections between torsion explosion and the bounds required for Lusztig's conjecture, see [Wil17a, Section 2.7].

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