

“Lusztig’s exceptional mathematical ability became evident at an early stage of his career at Warwick. . . . His early experience as a mathematician was not without certain difficulties. There was a period during which, for financial reasons, he preferred to live in a tent outside the Mathematics Research Centre houses at Warwick University rather than in the houses themselves.” – Roger Carter.

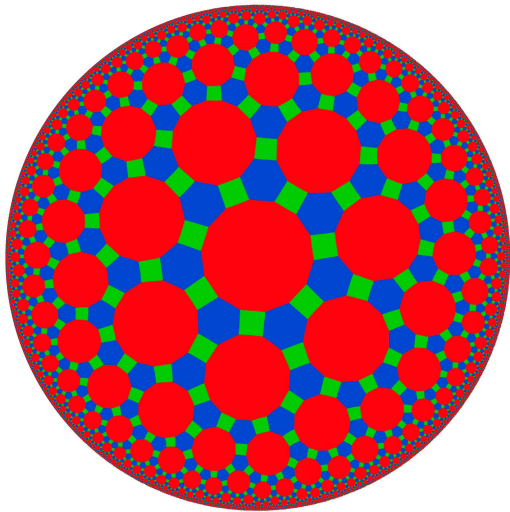
Throughout (W, S) will denote a Coxeter system:

$$\begin{aligned} W &= \langle s \in S \mid s^2 = 1, (st)^{m_{st}} = 1 \rangle \\ &= \langle s \in S \mid s^2 = 1, \underbrace{st \dots}_{m_{st} \text{ terms}} = \underbrace{ts \dots}_{m_{st} \text{ terms}} \rangle \end{aligned}$$

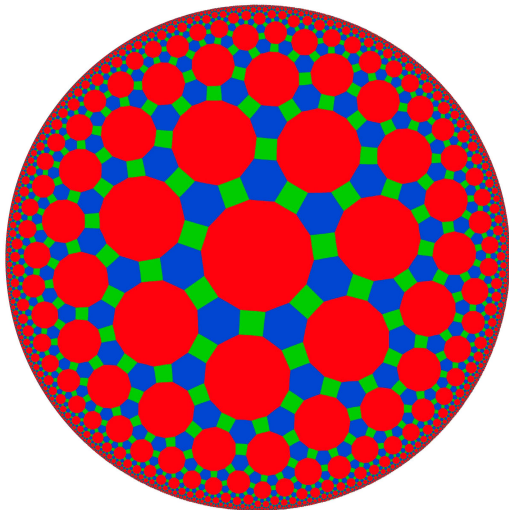
(where $m_{st} \in \{2, 3, \dots, \infty\}$).

For example, we could take W to be a real reflection group...

...or the symmetries of this tessellation of the hyperbolic plane:



...or the symmetries of this tessellation of the hyperbolic plane:



$$\bullet \text{ --- } \bullet \text{ --- } \frac{7}{\bullet}$$

To a Coxeter system (W, S) one may associate a simplicial complex $| (W, S) |$ called the *Coxeter complex* of W .

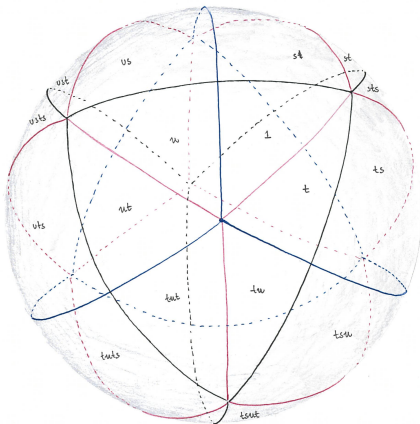
To a Coxeter system (W, S) one may associate a simplicial complex $| (W, S) |$ called the *Coxeter complex* of W .

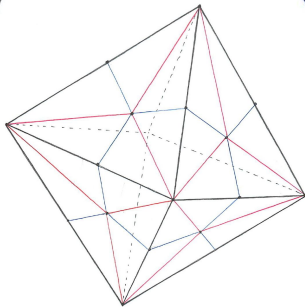
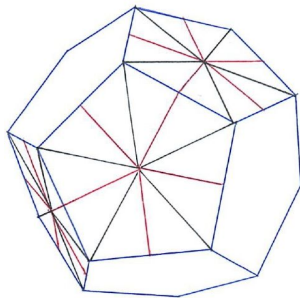
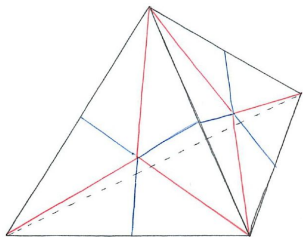
The Coxeter complex is an object of geometric group theory: it allows us to “imagine” (W, S) by providing a nice space on which it acts.

If W is a finite reflection group, then $|(W, S)|$ is a triangulation of the unit sphere in the space where W acts.

If W is a finite reflection group, then $|(W, S)|$ is a triangulation of the unit sphere in the space where W acts.

For example, $W = S_4$ acting on \mathbb{R}^3 :

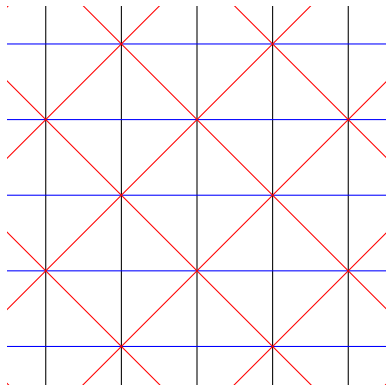




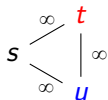
If W is a discrete group of affine transformations of V , then $|(W, S)|$ is a triangulation of V .

If W is a discrete group of affine transformations of V , then $|(W, S)|$ is a triangulation of V .

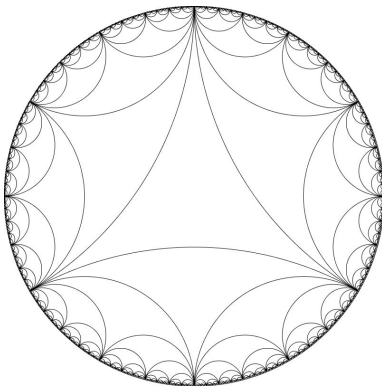
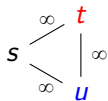
$$s \xrightarrow{4} \textcolor{red}{t} \xrightarrow{4} \textcolor{blue}{u}$$



For a hyperbolic reflection group we get a triangulation of hyperbolic space:

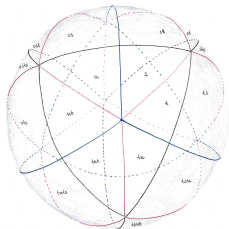


For a hyperbolic reflection group we get a triangulation of hyperbolic space:

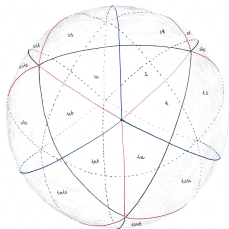


;

Key properties of Coxeter complex:

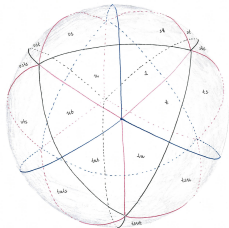


Key properties of Coxeter complex:



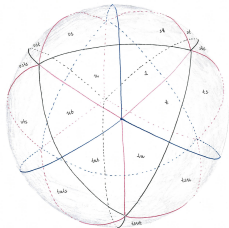
- 1 Simplices of maximal dimension (“alcoves”) are labelled by W .

Key properties of Coxeter complex:



- 1 Simplicies of maximal dimension (“alcoves”) are labelled by W .
- 2 Codimension 1 simplicies (“walls”) are coloured by S , the simple reflections.

Key properties of Coxeter complex:



- 1 Simplicies of maximal dimension (“alcoves”) are labelled by W .
- 2 Codimension 1 simplicies (“walls”) are coloured by S , the simple reflections.
- 3 x and y share an s -coloured wall if and only if $xs = y$.

The Coxeter complex has a (continuous) left action of W .

W also acts on the alcoves of $|(W, S)|$ on the right by

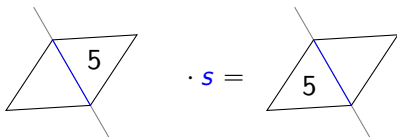
$$\Delta_w \cdot s = \Delta_{ws}.$$

This action is *not* continuous, but is “local”: cross the wall coloured by s .

The Coxeter complex provides a convenient way of visualising the group algebra $\mathbb{Z}W$ of W . Recall that the group algebra $\mathbb{Z}W$ consists of finite formal linear combinations $\sum \lambda_w w$ of elements of W . The product in W induces a multiplication in $\mathbb{Z}W$.

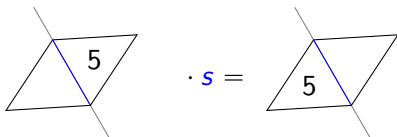
The Coxeter complex provides a convenient way of visualising the group algebra $\mathbb{Z}W$ of W . Recall that the group algebra $\mathbb{Z}W$ consists of finite formal linear combinations $\sum \lambda_w w$ of elements of W . The product in W induces a multiplication in $\mathbb{Z}W$.

Hence we can picture an element of $\mathbb{Z}W$ as the assignment of integers to each alcove, such that only finitely many are non-zero. If we view $\mathbb{Z}W$ as a right module over itself it is easy to picture the action of the elements of S :

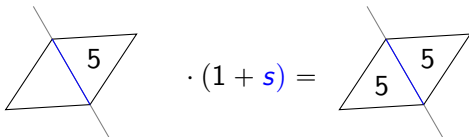


The Coxeter complex provides a convenient way of visualising the group algebra $\mathbb{Z}W$ of W . Recall that the group algebra $\mathbb{Z}W$ consists of finite formal linear combinations $\sum \lambda_w w$ of elements of W . The product in W induces a multiplication in $\mathbb{Z}W$.

Hence we can picture an element of $\mathbb{Z}W$ as the assignment of integers to each alcove, such that only finitely many are non-zero. If we view $\mathbb{Z}W$ as a right module over itself it is easy to picture the action of the elements of S :



Similarly (“ s averaging operator”)

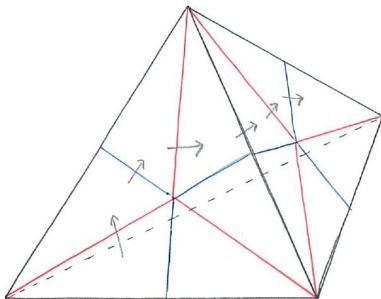


Let $\ell : W \rightarrow \mathbb{N}$ denote the length function on W :

$\ell(w)$ = length of a minimal expression for w in the generators s
= number of walls crossed in a minimal path $id \rightarrow w$ in $|(W, S)|$.

Let $\ell : W \rightarrow \mathbb{N}$ denote the length function on W :

$\ell(w)$ = length of a minimal expression for w in the generators s
= number of walls crossed in a minimal path $id \rightarrow w$ in $|(W, S)|$.



The Hecke algebra H is a quantization of $\mathbb{Z}W$. It is an algebra over $\mathbb{Z}[v^{\pm 1}]$ with basis $\{H_x \mid x \in W\}$ parametrised by W . If we write $\underline{H}_s := H_s + vH_{id}$ then the multiplication in H is determined by

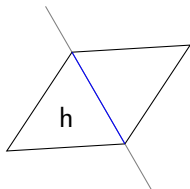
$$H_x \underline{H}_s = \begin{cases} H_{xs} + vH_x & \text{if } \ell(xs) > \ell(x), \\ H_{xs} + v^{-1}H_x & \text{if } \ell(xs) < \ell(x). \end{cases}$$

The Hecke algebra H is a quantization of $\mathbb{Z}W$. It is an algebra over $\mathbb{Z}[v^{\pm 1}]$ with basis $\{H_x \mid x \in W\}$ parametrised by W . If we write $\underline{H}_s := H_s + vH_{id}$ then the multiplication in H is determined by

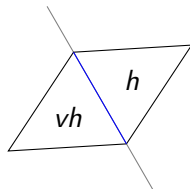
$$H_x \underline{H}_s = \begin{cases} H_{xs} + vH_x & \text{if } \ell(xs) > \ell(x), \\ H_{xs} + v^{-1}H_x & \text{if } \ell(xs) < \ell(x). \end{cases}$$

We can visualise this as follows: (“quantized averaging operator”)

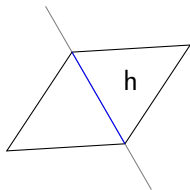
id



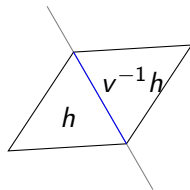
$\cdot \underline{H}_s =$



id



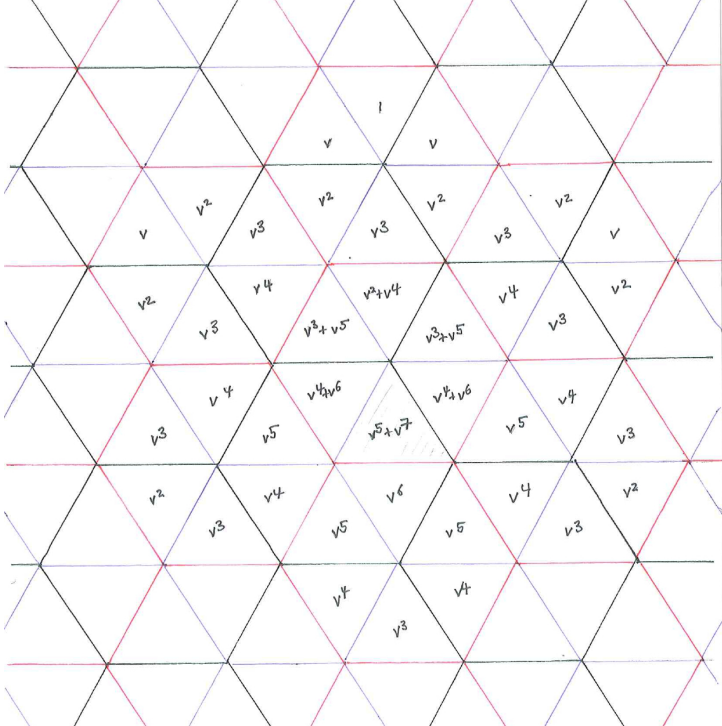
$\cdot \underline{H}_s =$



In 1979 Kazhdan and Lusztig defined a new basis for the Hecke algebra using the combinatorial structure of W . We denote this new basis by $\{\underline{H}_x \mid x \in W\}$. It satisfies

$$\underline{H}_x := H_x + \sum_{\substack{y \in W \\ \ell(y) < \ell(x)}} h_{y,x} H_y$$

with $h_{y,x} \in v\mathbb{Z}[v]$. These polynomials are the *Kazhdan-Lusztig polynomials*.



The definition is inductive. The first few Kazhdan-Lusztig basis elements are easily defined:

$$\underline{H}_{id} := H_{id}, \quad \underline{H}_s := H_s + vH_{id} \quad \text{for } s \in S.$$

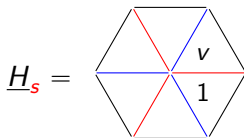
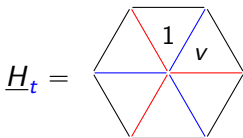
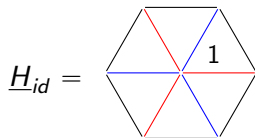
Now the work begins. Suppose that we have calculated \underline{H}_y for all y with $\ell(y) \leq \ell(x)$. Choose $s \in S$ with $\ell(xs) > \ell(x)$ and write

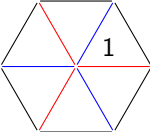
$$\underline{H}_x \underline{H}_s = H_{xs} + \sum_{\ell(y) < \ell(xs)} g_y H_y.$$

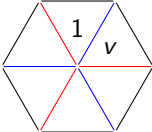
The formula for the action of \underline{H}_s shows that $g_y \in \mathbb{Z}[v]$ for all $y < \ell(xs)$. If all $g_y \in v\mathbb{Z}[v]$ then $\underline{H}_{xs} := \underline{H}_x \underline{H}_s$. Otherwise we set

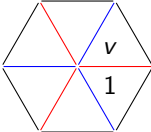
$$\underline{H}_{xs} = \underline{H}_x \underline{H}_s - \sum_{\substack{y \\ \ell(y) < \ell(x)}} g_y(0) \underline{H}_y.$$

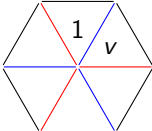
$$\bullet \text{---} \overset{3}{\text{---}} \bullet$$

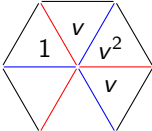


$$\underline{H}_{id} =$$


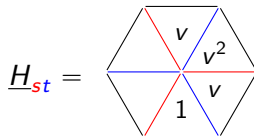
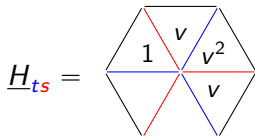
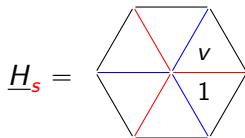
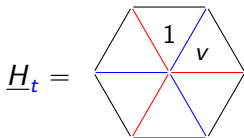
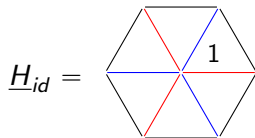
$$\underline{H}_{\textcolor{blue}{t}} =$$


$$\underline{H}_{\textcolor{red}{s}} =$$


$$\underline{H}_{\textcolor{blue}{t}}\underline{H}_{\textcolor{red}{s}} =$$


$$\cdot \underline{H}_{\textcolor{red}{s}} =$$


$$= \underline{H}_{\textcolor{blue}{t}\textcolor{red}{s}}$$



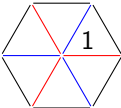
$$\underline{H}_{id} =$$

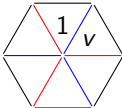
$$\underline{H}_t =$$

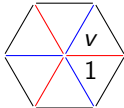
$$\underline{H}_s =$$

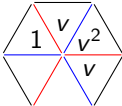
$$\underline{H}_{ts} =$$

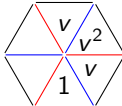
$$\underline{H}_{st} =$$

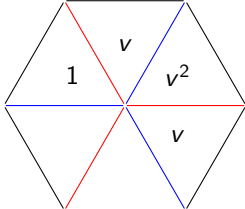
$$\underline{H}_{id} =$$


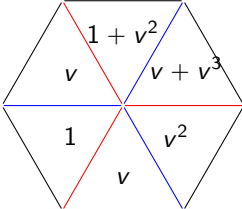
$$\underline{H}_t =$$


$$\underline{H}_s =$$


$$\underline{H}_{ts} =$$


$$\underline{H}_{st} =$$


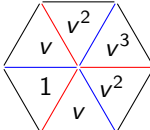
$$\underline{H}_{ts}\underline{H}_t =$$


$$\cdot \underline{H}_t =$$


$$\begin{aligned}
 \underline{H}_{id} &= \text{Hexagon with red diagonal (top-left to bottom-right), blue diagonal (top-right to bottom-left), and horizontal red line. Top-right triangle contains '1'.} \\
 \underline{H}_t &= \text{Hexagon with red diagonal (top-left to bottom-right), blue diagonal (top-right to bottom-left), and horizontal red line. Top-right triangle contains '1', top-left triangle contains 'v'.} \\
 \underline{H}_s &= \text{Hexagon with red diagonal (top-left to bottom-right), blue diagonal (top-right to bottom-left), and horizontal red line. Top-right triangle contains 'v', top-left triangle contains '1'.} \\
 \underline{H}_{ts} &= \text{Hexagon with red diagonal (top-left to bottom-right), blue diagonal (top-right to bottom-left), and horizontal red line. Top-right triangle contains 'v', top-left triangle contains '1', bottom-right triangle contains 'v^2', bottom-left triangle contains 'v'.} \\
 \underline{H}_{st} &= \text{Hexagon with red diagonal (top-left to bottom-right), blue diagonal (top-right to bottom-left), and horizontal red line. Top-right triangle contains 'v', top-left triangle contains '1', bottom-right triangle contains 'v^2', bottom-left triangle contains 'v'.}
 \end{aligned}$$

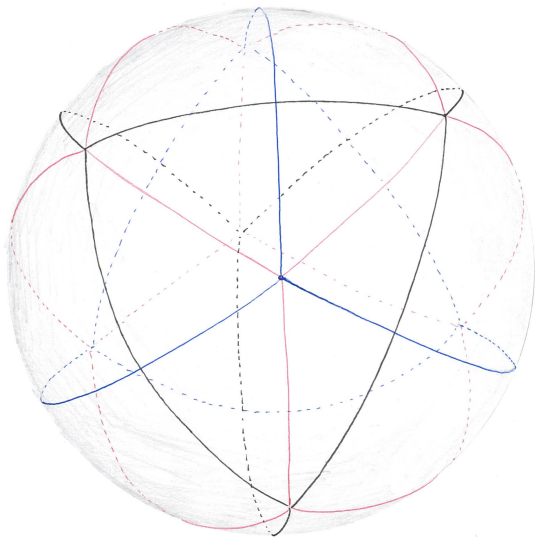
$$\begin{aligned}
 \underline{H}_{ts}\underline{H}_t &= \text{Hexagon with red diagonal (top-left to bottom-right), blue diagonal (top-right to bottom-left), and horizontal red line. Top-right triangle contains 'v', top-left triangle contains '1', bottom-right triangle contains 'v^2', bottom-left triangle contains 'v'.} \\
 \cdot \underline{H}_t &= \text{Hexagon with red diagonal (top-left to bottom-right), blue diagonal (top-right to bottom-left), and horizontal red line. Top-right triangle contains '1 + v^2', top-left triangle contains 'v', bottom-right triangle contains 'v + v^3', bottom-left triangle contains '1', and bottom triangle contains 'v^2'.}
 \end{aligned}$$

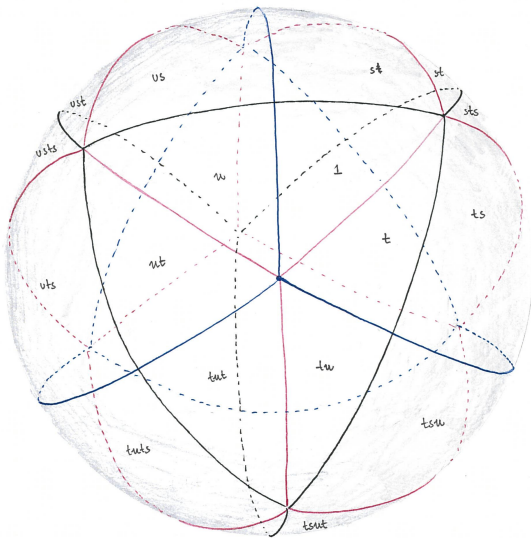
Hence: $\underline{H}_{tst} = \underline{H}_{ts}\underline{H}_t - \underline{H}_t =$



For dihedral groups (rank 2) we always have $h_{y,x} = v^{\ell(x)-\ell(y)}$
(Kazhdan-Lusztig basis elements are *smooth*.)

However in higher rank the situation quickly becomes more interesting...





● — ● — ●
 s — t — u

utsut

tsut

utsu

lut

tu

tsu

ut

uts

t

ts

u

1

suts

su

s

sts

sut

st

stuts

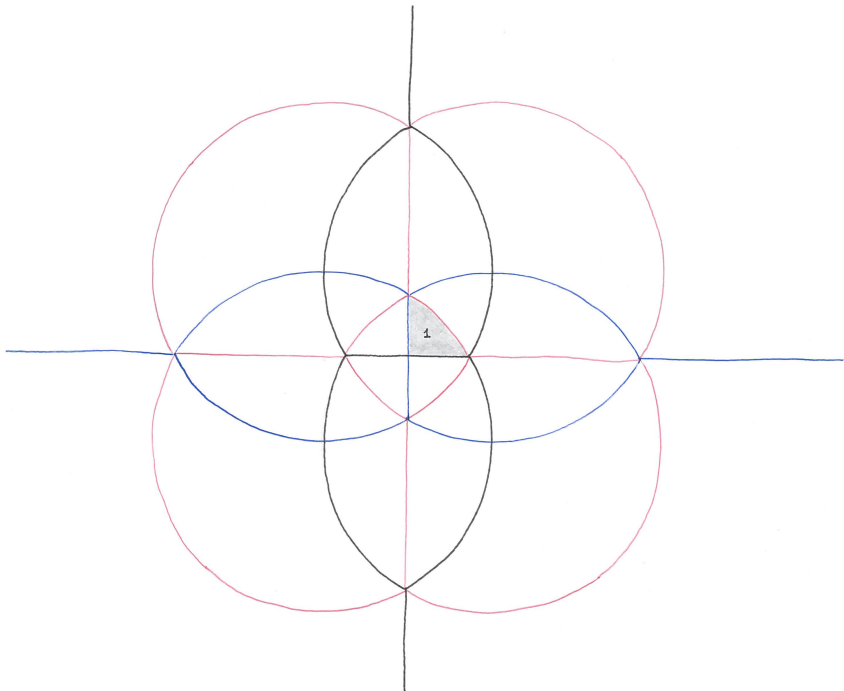
stut

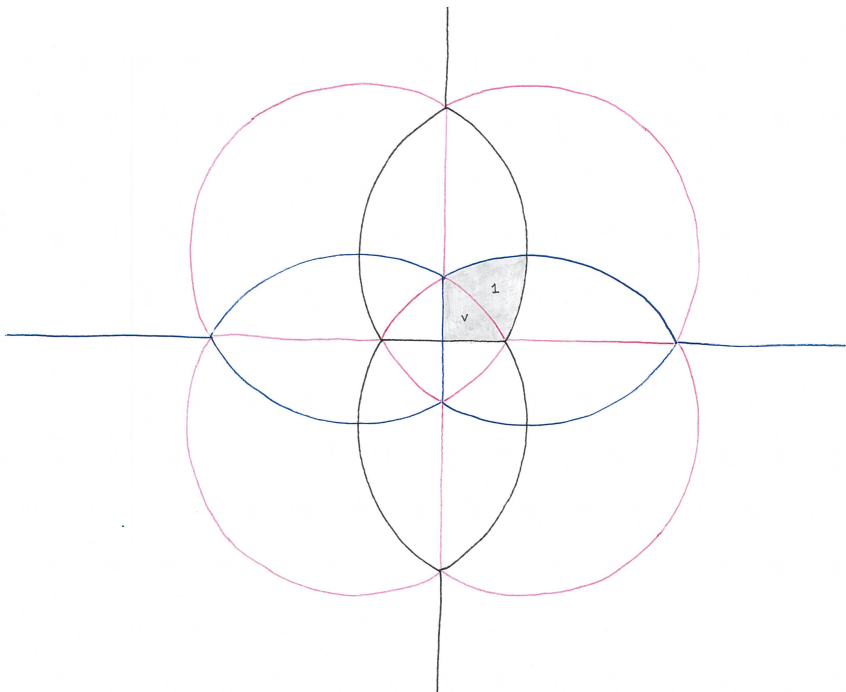
stu

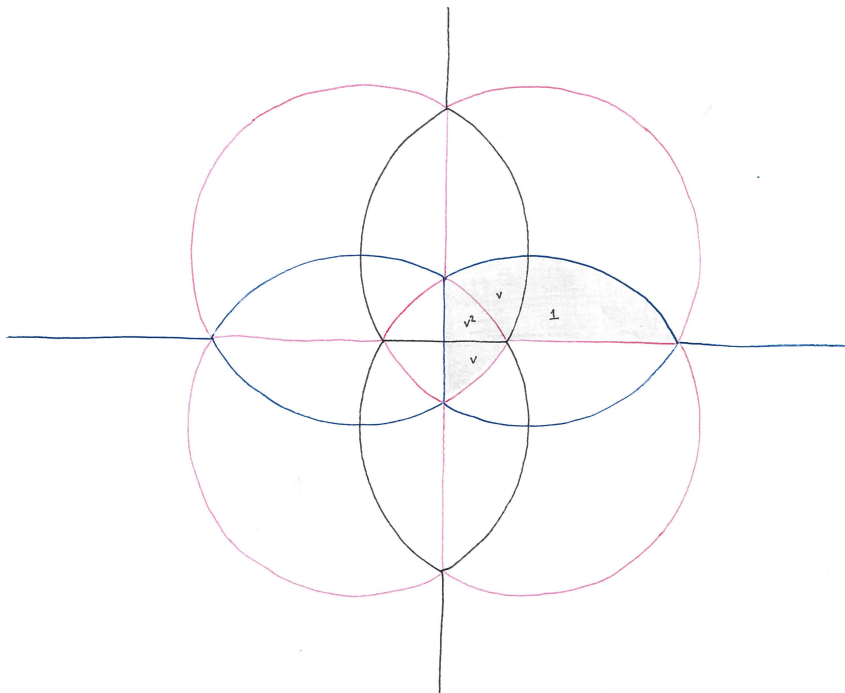
stus

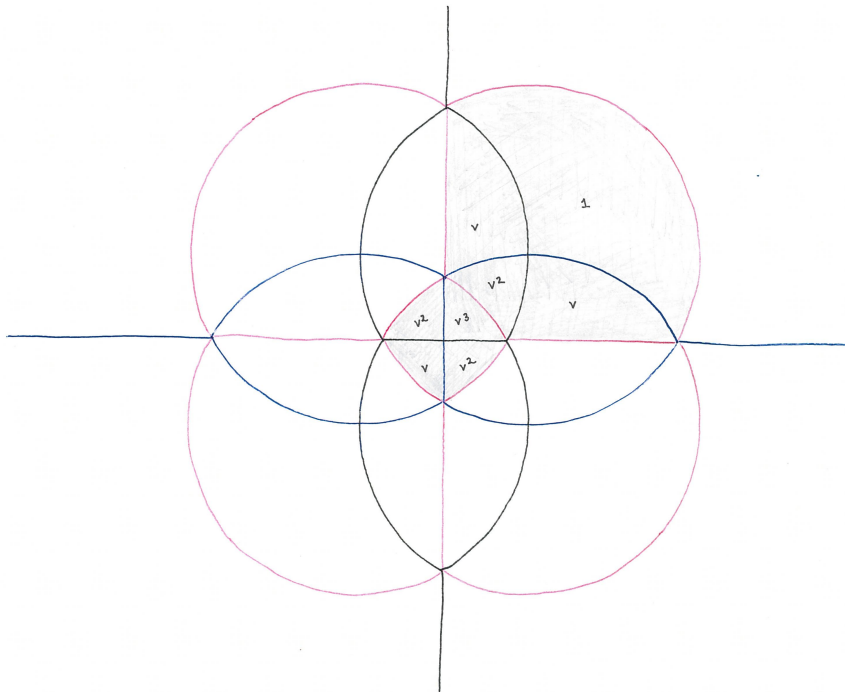
$w_0 = \text{stutst}$

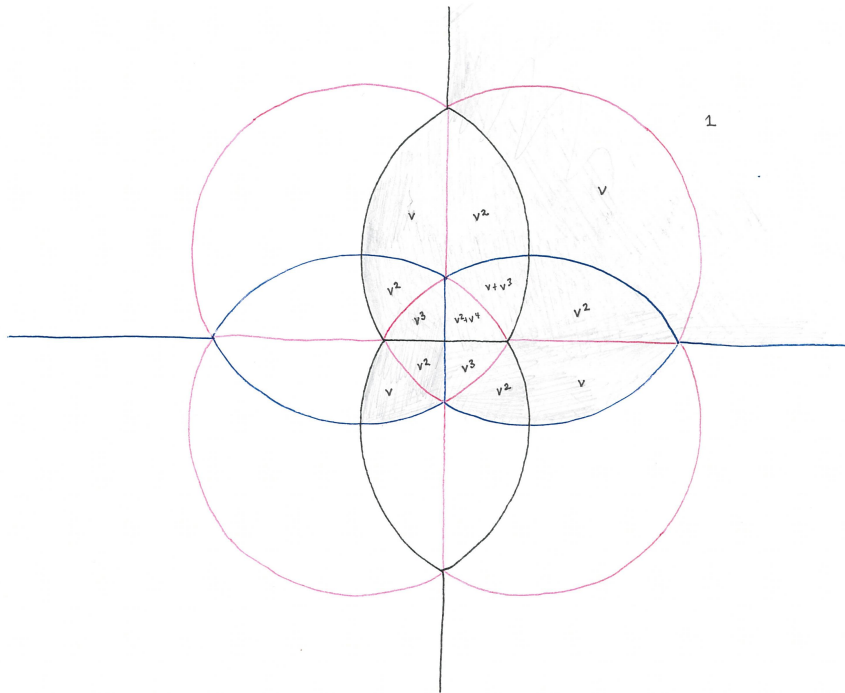
stust



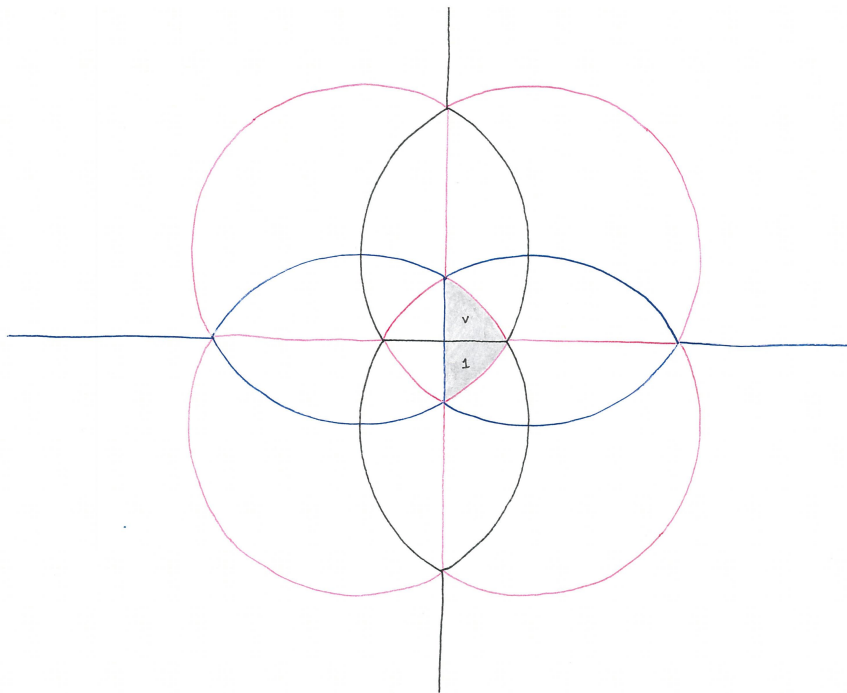


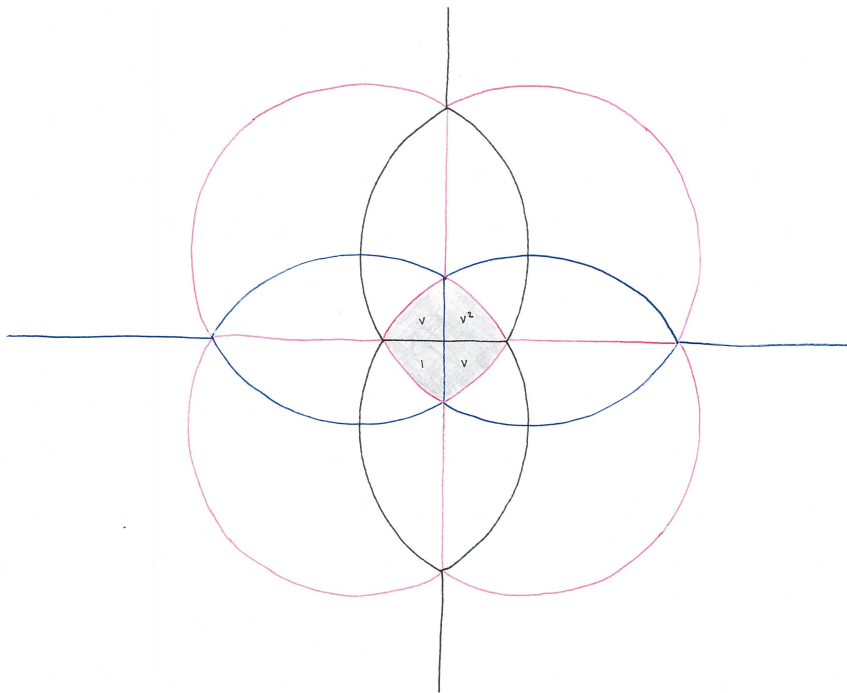


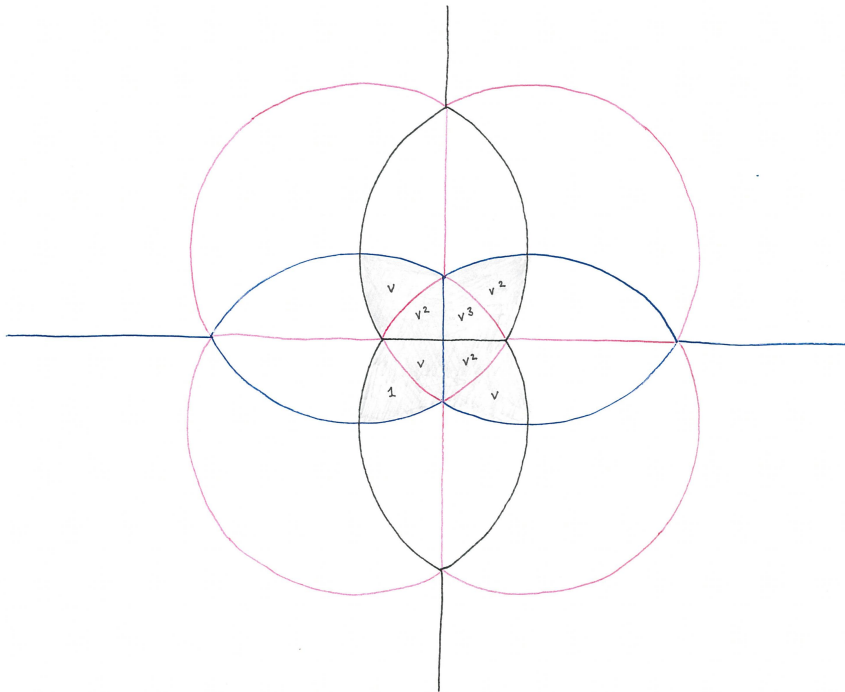


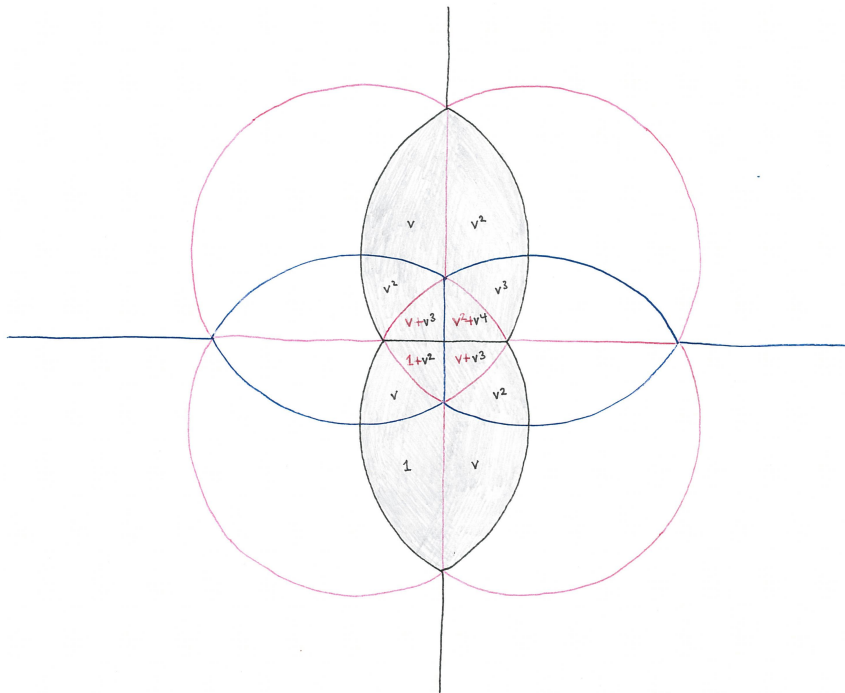


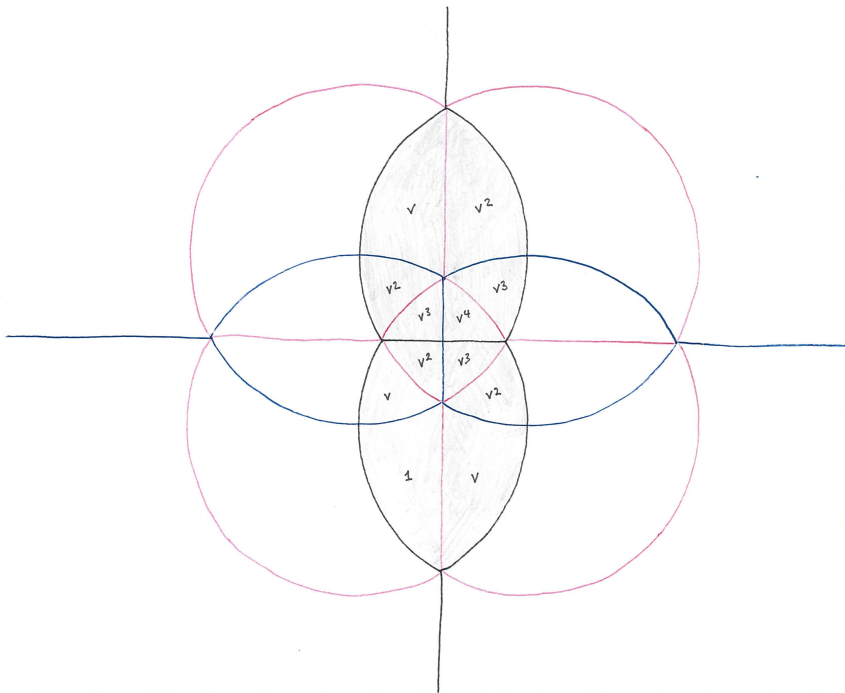
1

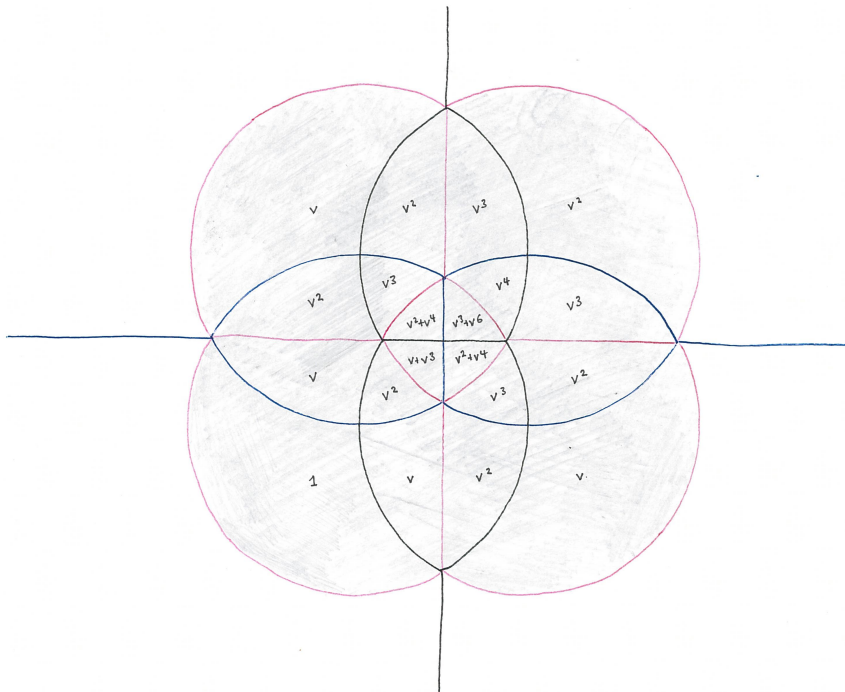




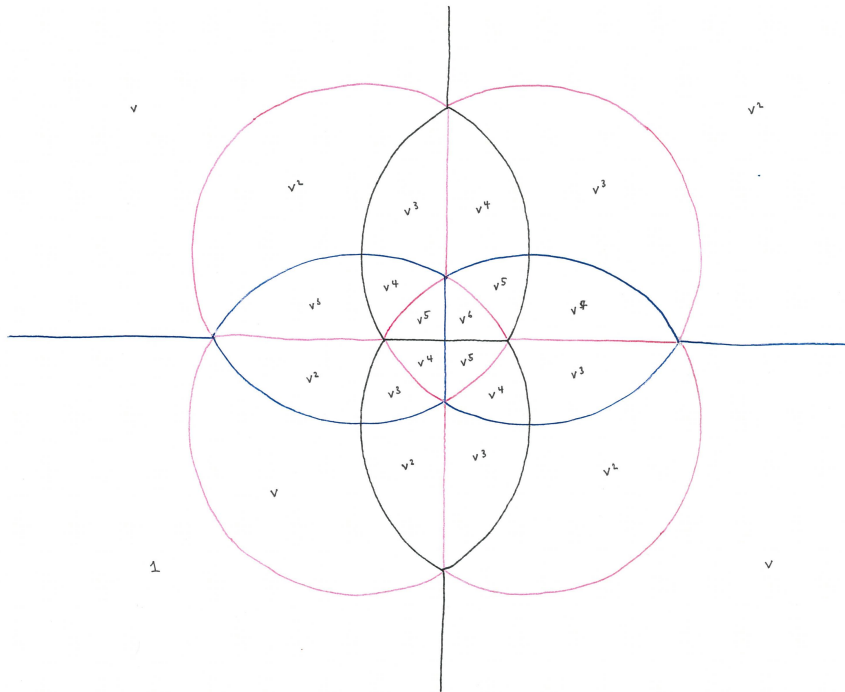


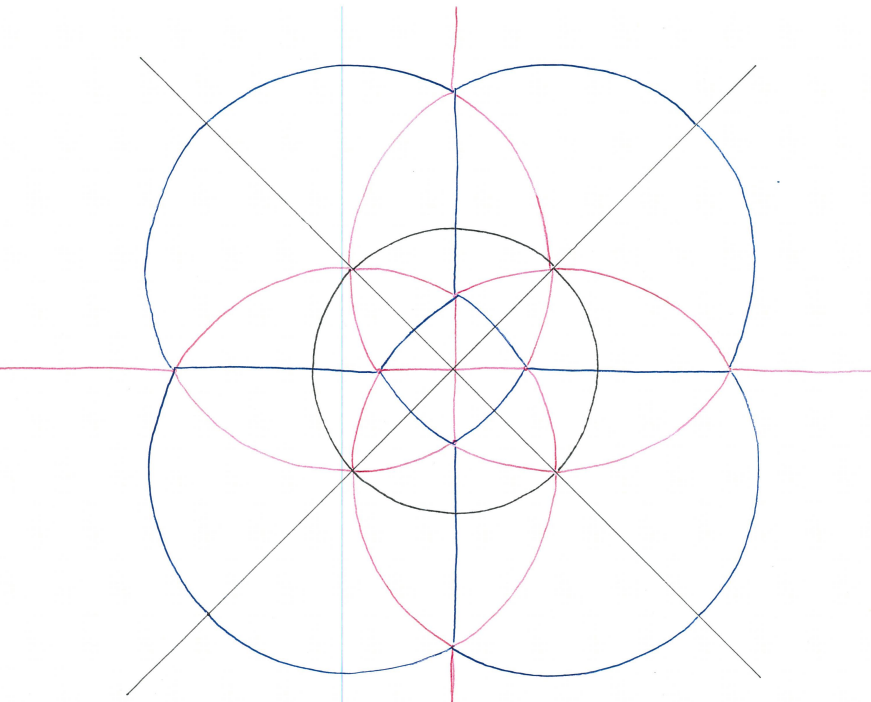


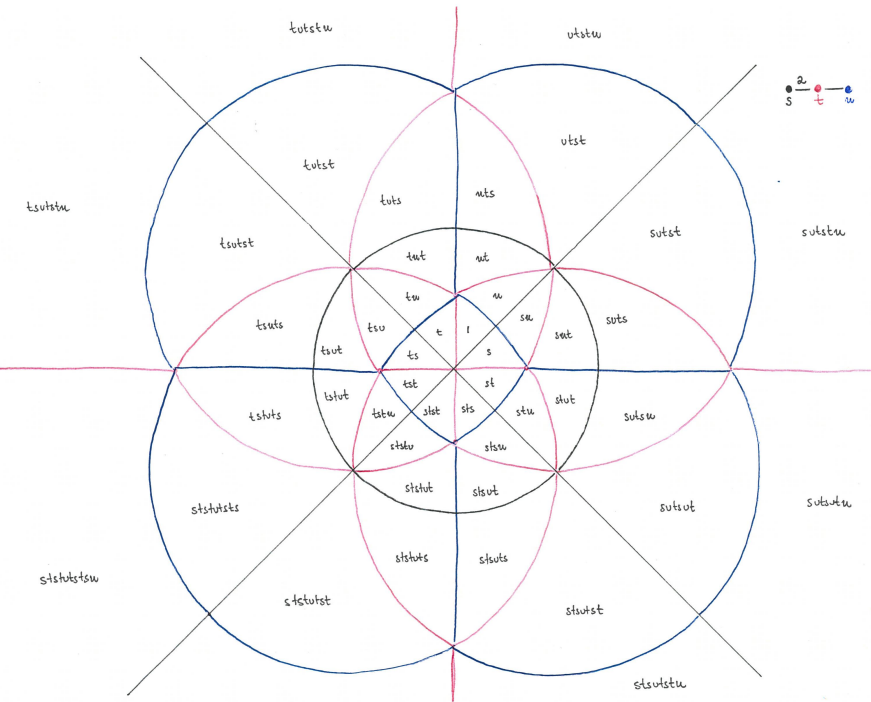


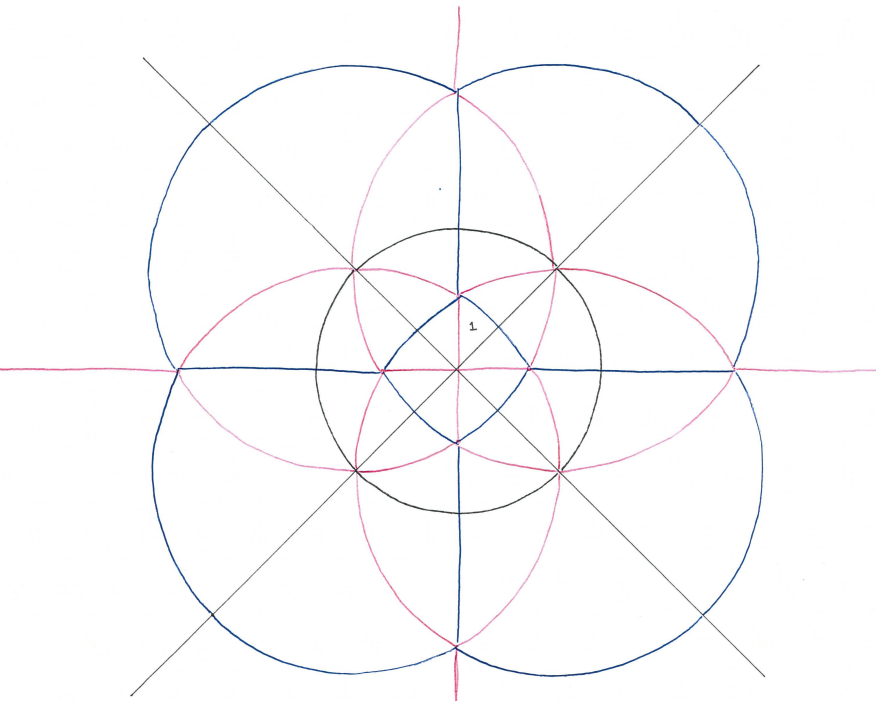


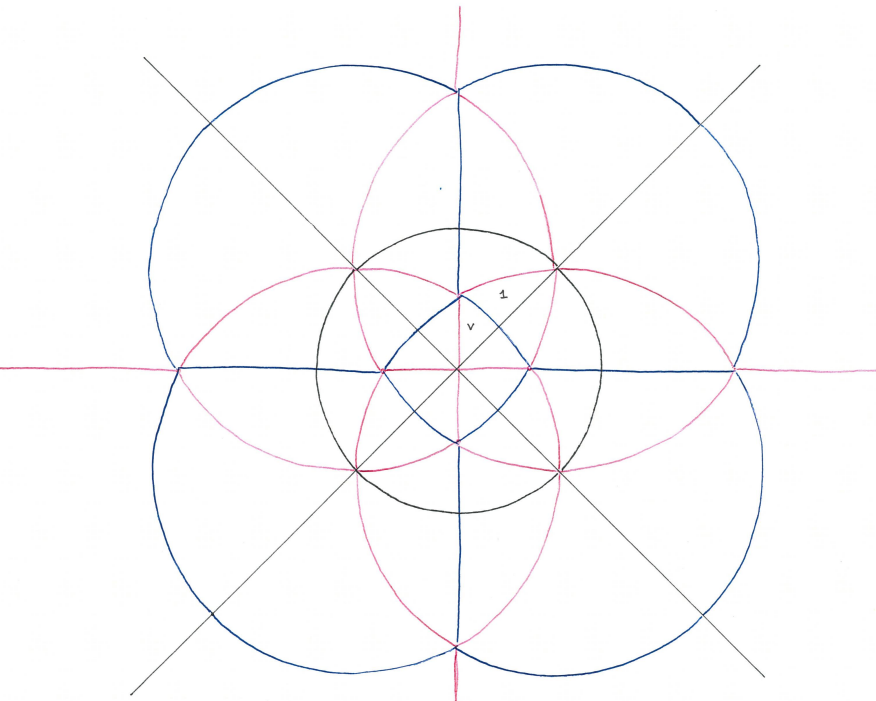


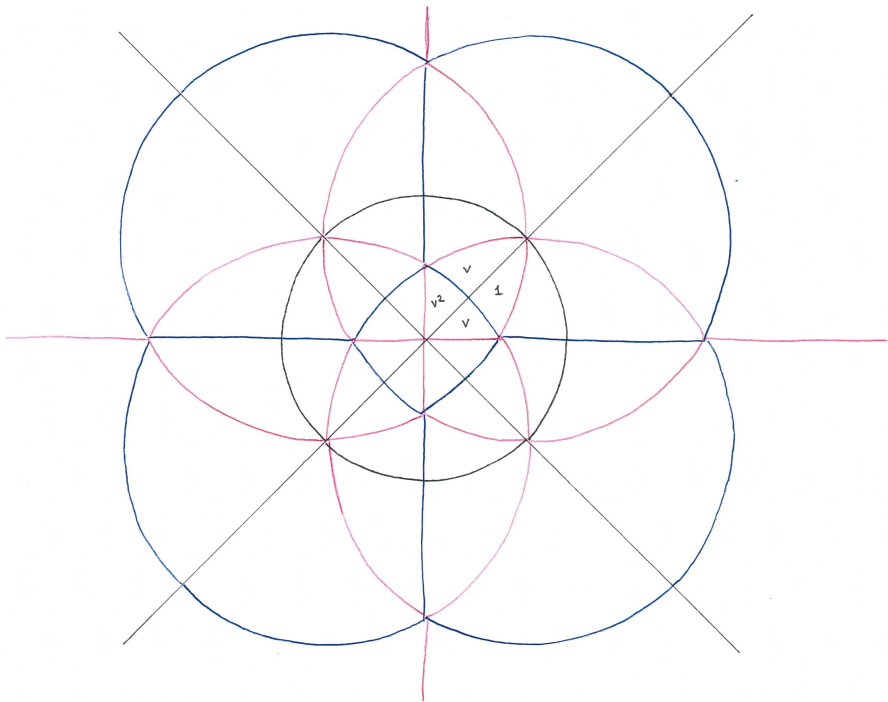


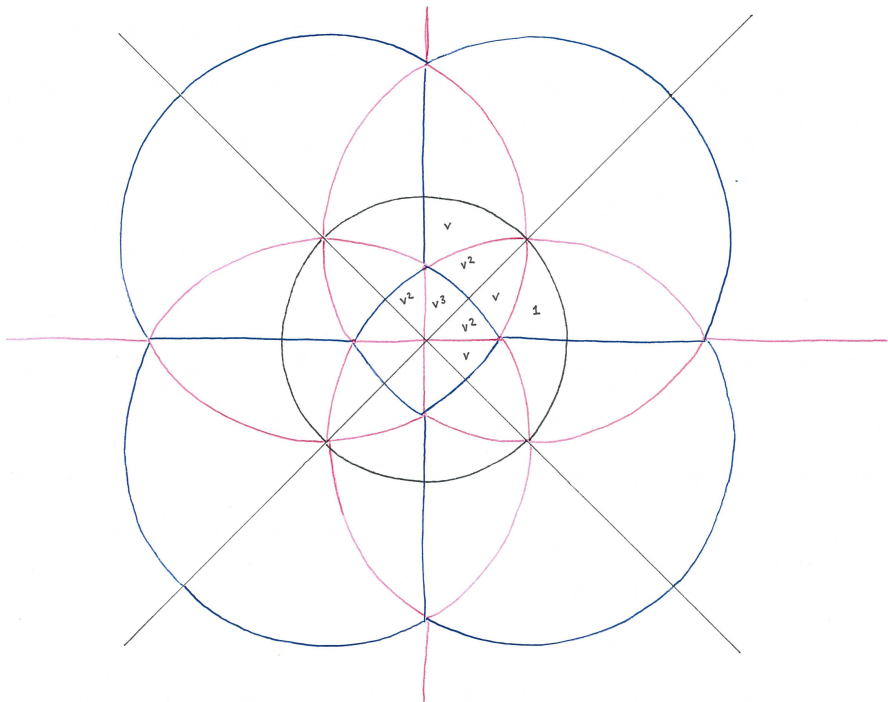


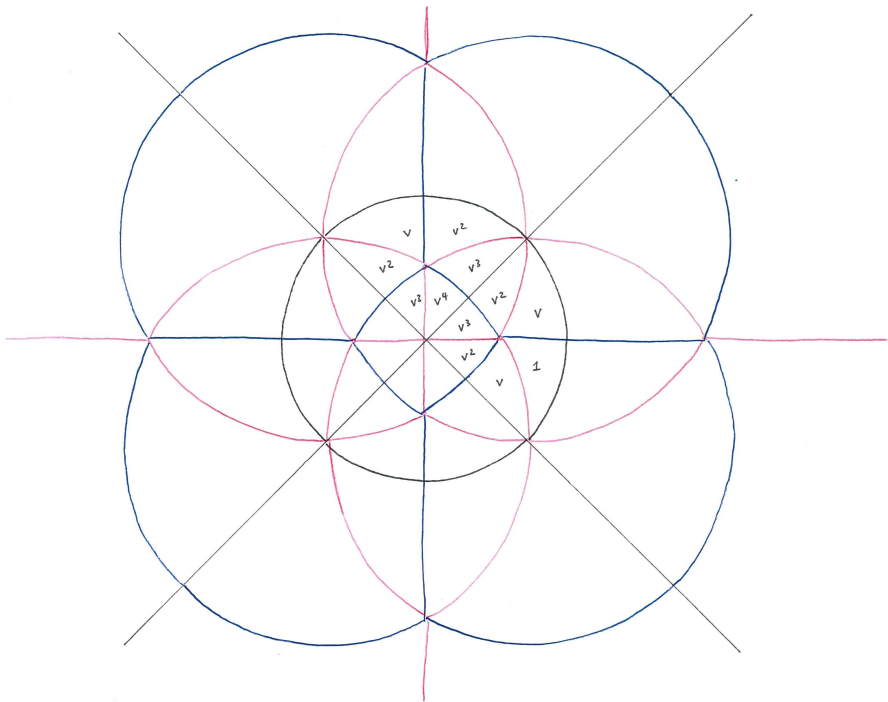


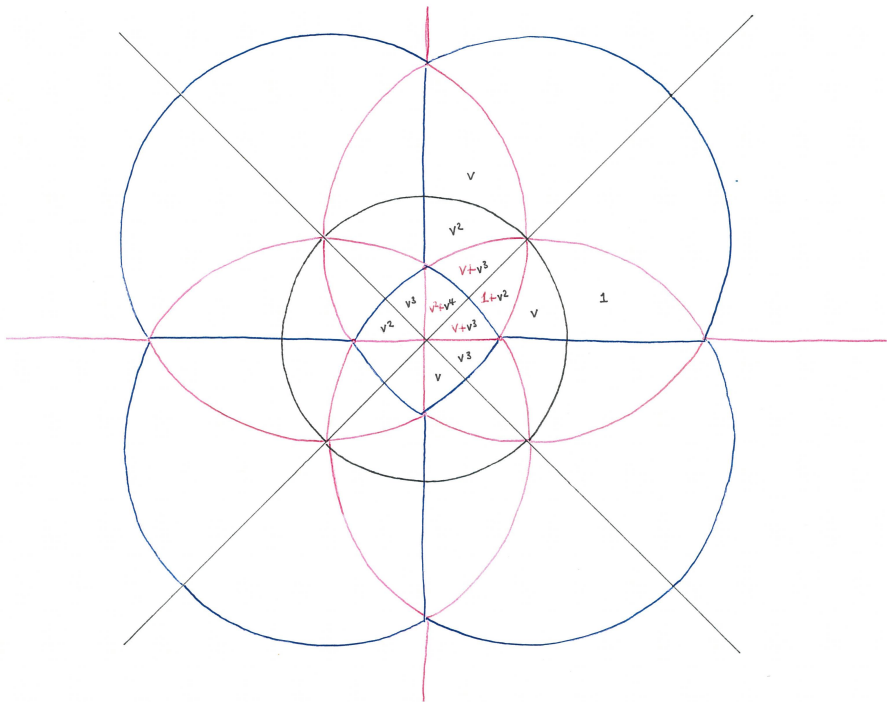


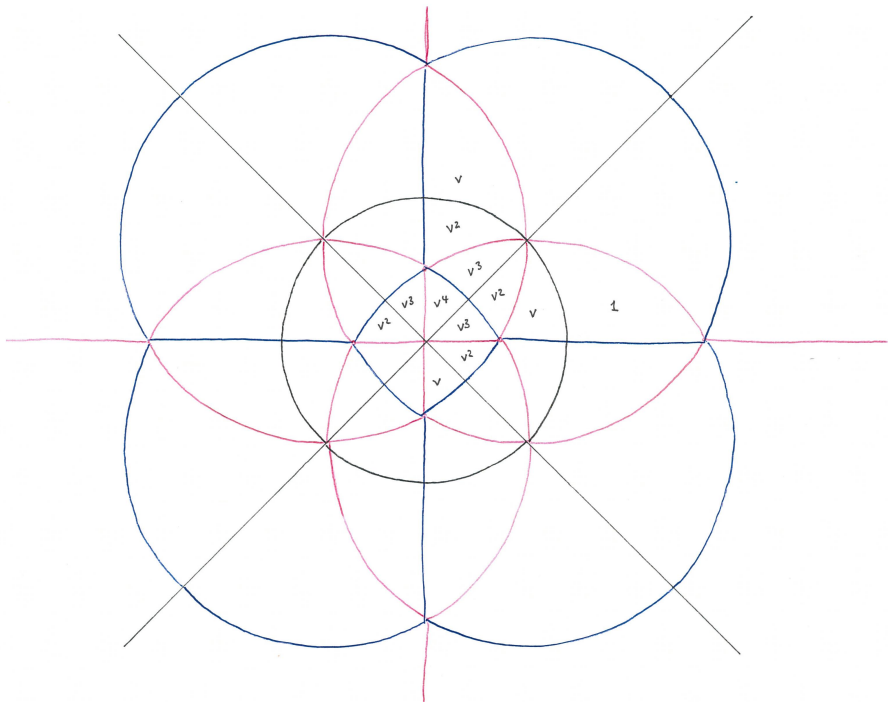


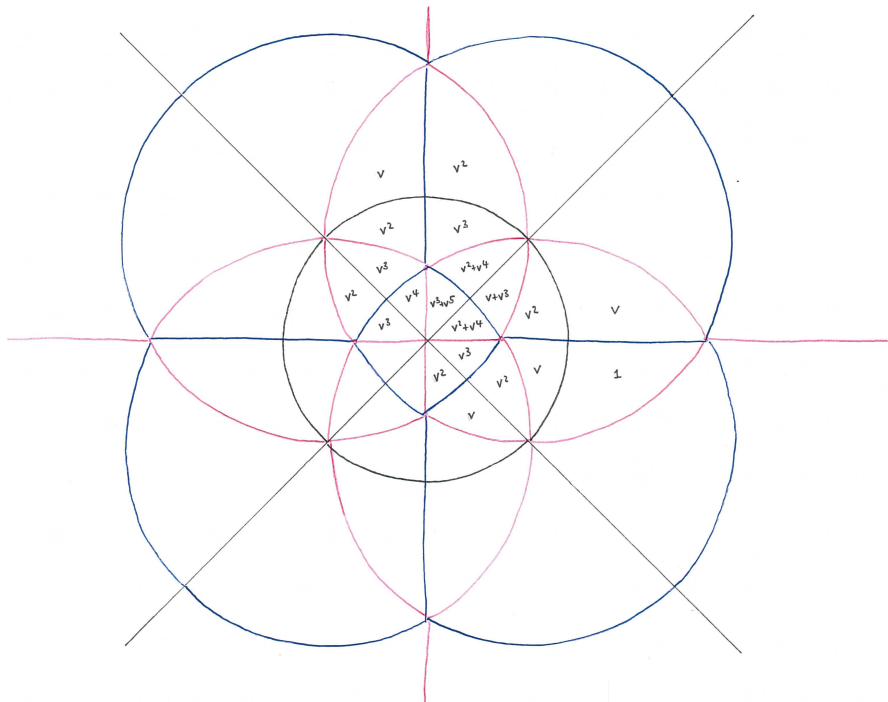


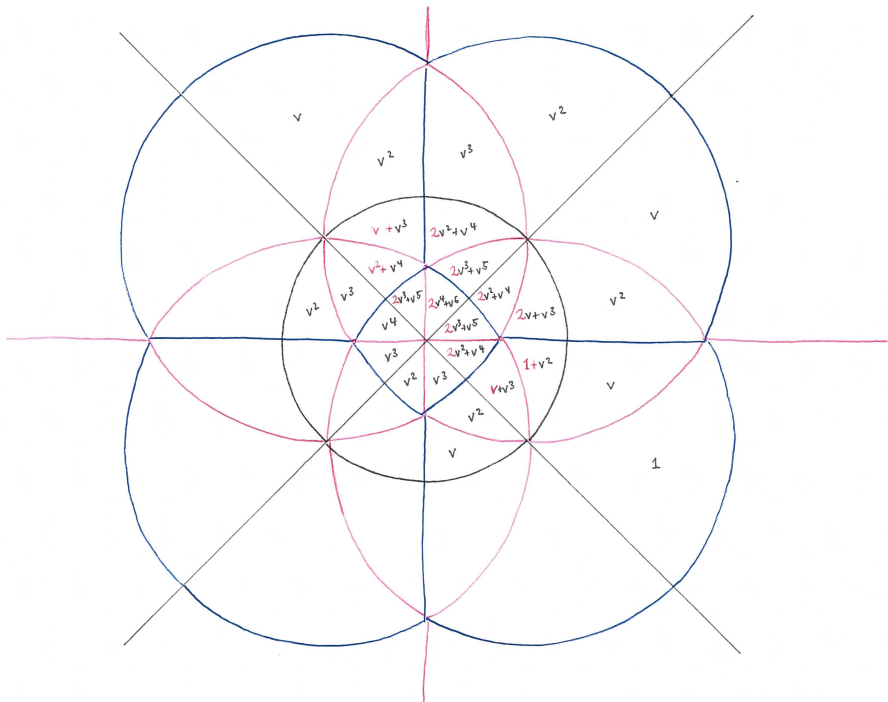


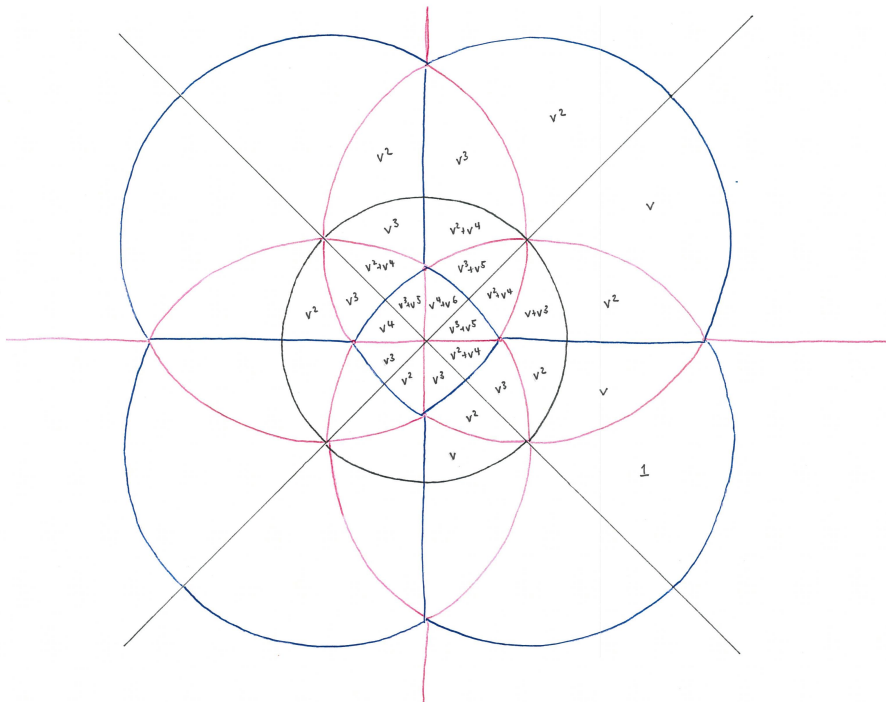


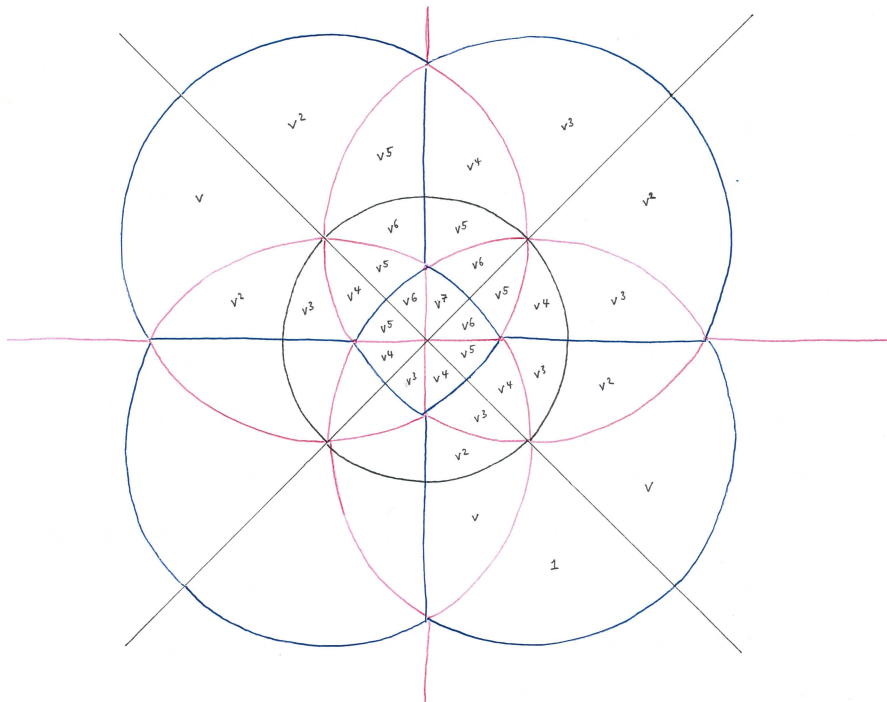


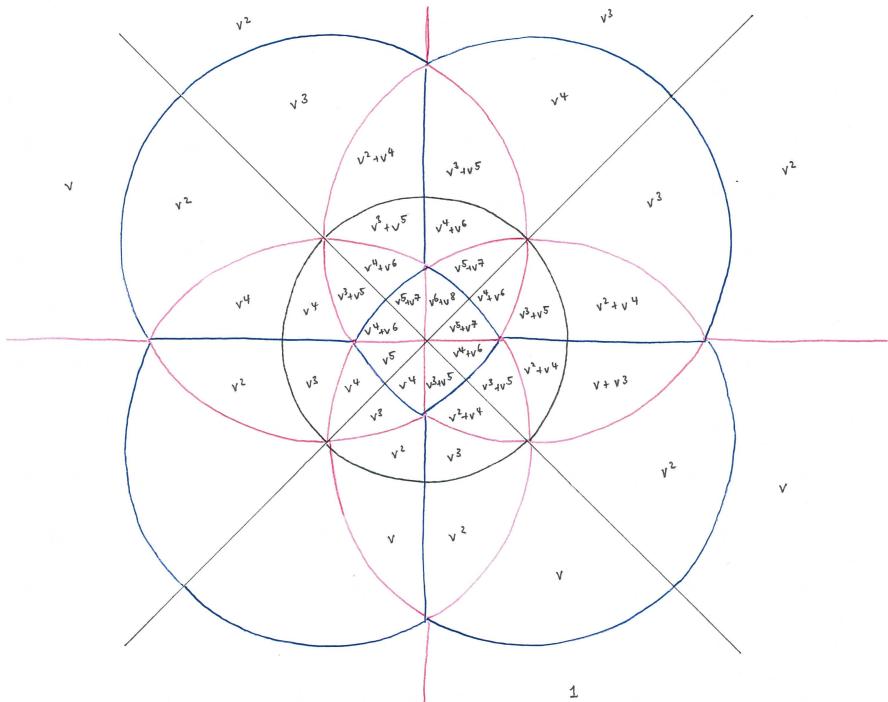


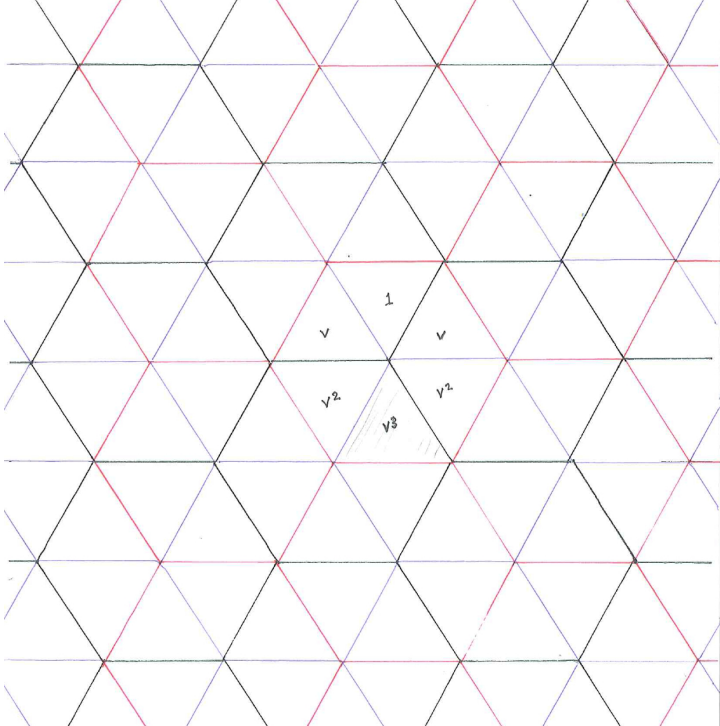


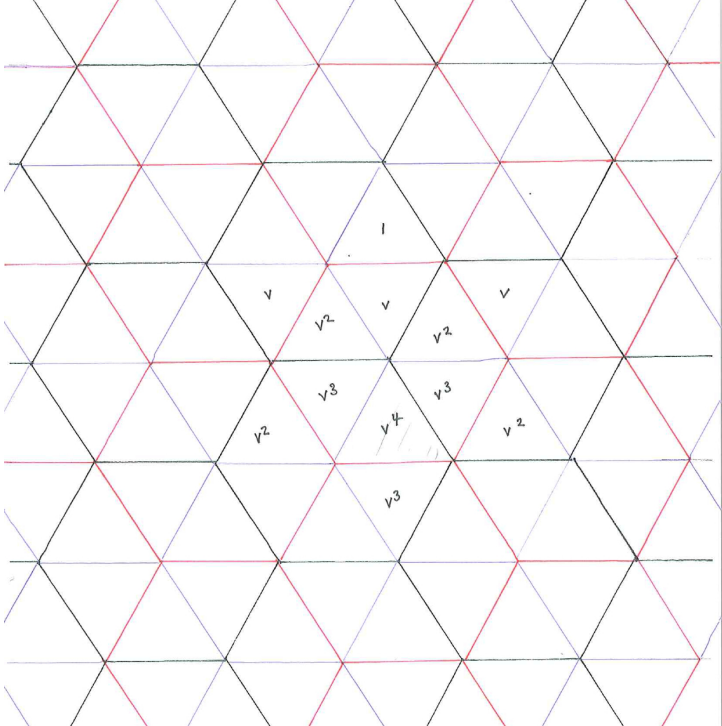


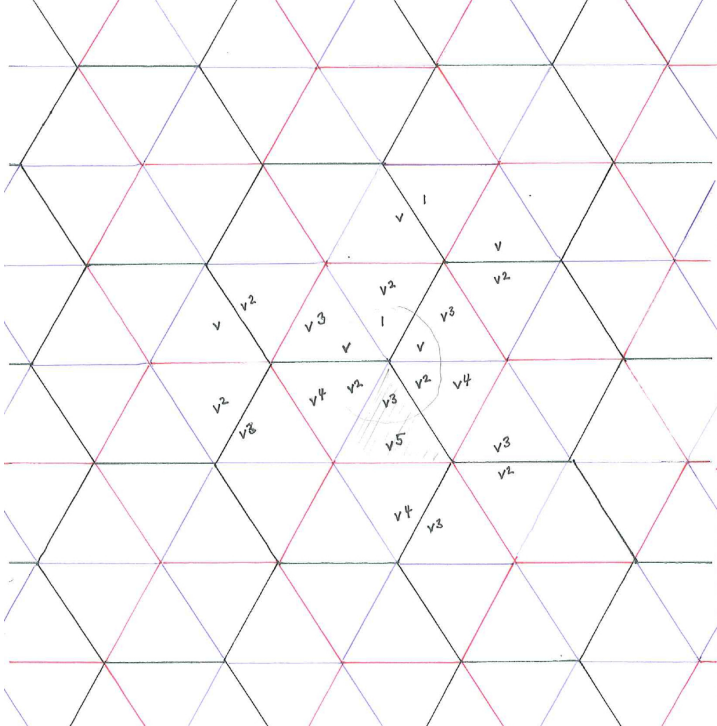


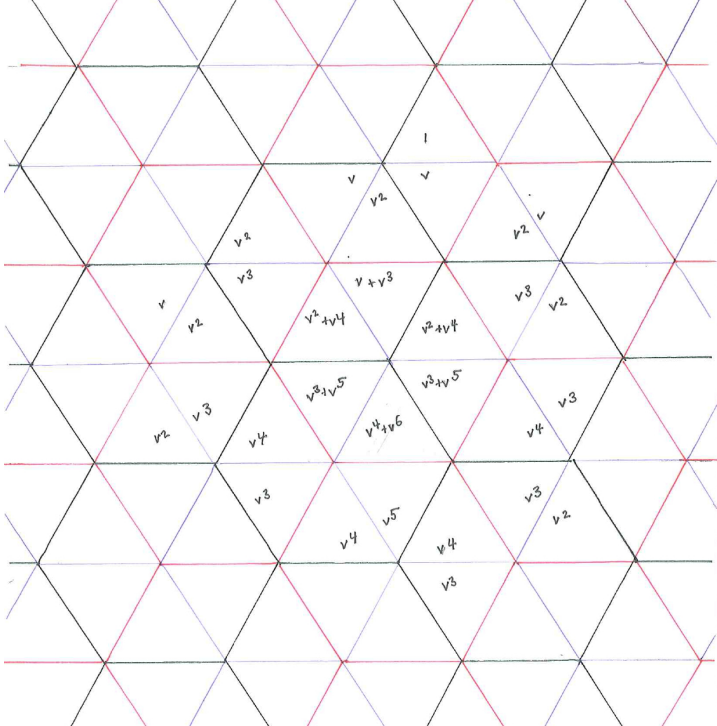




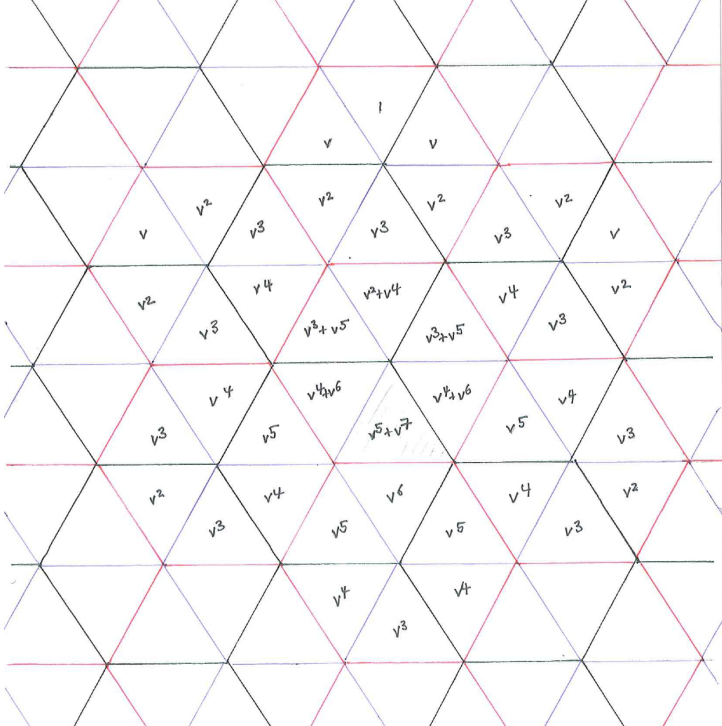














Kazhdan-Lusztig positivity conjecture (1979):

$$h_{x,y} \in \mathbb{Z}_{\geq 0}[v]$$

Kazhdan-Lusztig positivity conjecture (1979):

$$h_{x,y} \in \mathbb{Z}_{\geq 0}[v]$$

Established for crystallographic W by Kazhdan and Lusztig in 1980, using Deligne's proof of the Weil conjectures.

Crystallographic: $m_{st} \in \{2, 3, 4, 6, \infty\}$.

Why are Kazhdan-Lusztig polynomials hard?

Why are Kazhdan-Lusztig polynomials hard?

Polo's Theorem (1999)

For any $P \in 1 + q\mathbb{Z}_{\geq 0}[q]$ there exists an m such that $v^m P(v^{-2})$ occurs as a Kazhdan-Lusztig polynomial in some symmetric group.

Why are Kazhdan-Lusztig polynomials hard?

Polo's Theorem (1999)

For any $P \in 1 + q\mathbb{Z}_{\geq 0}[q]$ there exists an m such that $v^m P(v^{-2})$ occurs as a Kazhdan-Lusztig polynomial in some symmetric group.

Roughly: all positive polynomials are Kazhdan-Lusztig polynomials!

The most complicated Kazhdan-Lusztig-Vogan polynomial computed by the *Atlas of Lie groups and Representations* project:

$$\begin{aligned} &152q^{22} + 3\,472q^{21} + 38\,791q^{20} + 293\,021q^{19} + 1\,370\,892q^{18} + \\ &+ 4\,067\,059q^{17} + 7\,964\,012q^{16} + 11\,159\,003q^{15} + \\ &+ 11\,808\,808q^{14} + 9\,859\,915q^{13} + 6\,778\,956q^{12} + \\ &+ 3\,964\,369q^{11} + 2\,015\,441q^{10} + 906\,567q^9 + \\ &+ 363\,611q^8 + 129\,820q^7 + 41\,239q^6 + \\ &+ 11\,426q^5 + 2\,677q^4 + 492q^3 + 61q^2 + 3q \end{aligned}$$

(This polynomial is associated to the reflection group of type E_8 .
See www.liegroups.org.)

Why are Kazhdan-Lusztig polynomials useful?

Infinite dimensional highest weight representations of semi-simple Lie algebras.

Infinite dimensional highest weight representations of semi-simple Lie algebras.

Kazhdan-Lusztig character formula (conjectured in 1979):

$$\mathrm{ch} L(x \cdot 0) = \sum_{y \geq x} (-1)^{\ell(x) - \ell(y)} h_{w_0 y, w_0 x}(1) \mathrm{ch} \Delta(y \cdot 0).$$

Infinite dimensional highest weight representations of semi-simple Lie algebras.

Kazhdan-Lusztig character formula (conjectured in 1979):

$$\mathrm{ch} L(x \cdot 0) = \sum_{y \geq x} (-1)^{\ell(x) - \ell(y)} h_{w_0 y, w_0 x}(1) \mathrm{ch} \Delta(y \cdot 0).$$

(A major generalisation of the Weyl character formula.)

The Kazhdan-Lusztig character formula was proved 1981 by Beilinson-Bernstein and Brylinski-Kashiwara using every trick in the book: algebraic differential equations “ D -modules”; the Riemann-Hilbert correspondence (monodromy of differential equations); the theory of perverse sheaves (algebraic topology of singular varieties); Deligne’s theory of weights (arithmetic geometry):

The Kazhdan-Lusztig character formula was proved 1981 by Beilinson-Bernstein and Brylinski-Kashiwara using every trick in the book: algebraic differential equations “ D -modules”; the Riemann-Hilbert correspondence (monodromy of differential equations); the theory of perverse sheaves (algebraic topology of singular varieties); Deligne’s theory of weights (arithmetic geometry):

“The amazing feature of the proof is that it does not try to solve the problem but just keeps translating it in languages of different areas of mathematics (further and further away from the original problem) until it runs into Deligne’s method of weight filtrations which is capable to solve it.

So have a seat; it is going to be a long journey.”

– Joseph Bernstein.

Kazhdan-Lusztig polynomials also play an important role in:

Kazhdan-Lusztig polynomials also play an important role in:

- i) Lusztig's description of the character table of a finite group of Lie type.

Kazhdan-Lusztig polynomials also play an important role in:

- i) Lusztig's description of the character table of a finite group of Lie type.
- ii) The algorithm for the determination of the unitary dual of a semi-simple Lie group by Adams, van Leeuwen, Trapa, Vogan (see work of Schmid and Vilonen.)

Kazhdan-Lusztig polynomials also play an important role in:

- i) Lusztig's description of the character table of a finite group of Lie type.
- ii) The algorithm for the determination of the unitary dual of a semi-simple Lie group by Adams, van Leeuwen, Trapa, Vogan (see work of Schmid and Vilonen.)
- iii) Kazhdan-Lusztig polynomials might end up helping us understand the HOMFLYPT polynomial of a link...

Theorem (Elias-W)

The Kazhdan-Lusztig positivity conjecture holds.

Theorem (Elias-W)

The Kazhdan-Lusztig positivity conjecture holds.

Using results of Soergel we obtain an algebraic proof of the Kazhdan-Lusztig character formula.

The idea (going back to Soergel) is to find a vector space which behaves like the intersection cohomology of a Schubert variety, even if this variety does not exist.

(Much like the coinvariant algebra for a non Weyl group should be regarded as the cohomology of a flag variety, even if no such flag variety exists.)

The idea (going back to Soergel) is to find a vector space which behaves like the intersection cohomology of a Schubert variety, even if this variety does not exist.

(Much like the coinvariant algebra for a non Weyl group should be regarded as the cohomology of a flag variety, even if no such flag variety exists.)

The key property of intersection cohomology is the “decomposition theorem”: the intersection cohomology of a variety is a summand of the cohomology of any resolution.

For Schubert varieties there exist resolutions of singularities (so called Bott-Samelson resolutions) whose cohomology admit elementary algebraic descriptions.

For Schubert varieties there exist resolutions of singularities (so called Bott-Samelson resolutions) whose cohomology admit elementary algebraic descriptions.

For any word (s, t, \dots, u) in S the cohomology of the corresponding Bott-Samelson variety is:

$$BS(s, t, \dots, u) := R \otimes_{R^s} R \otimes_{R^t} \cdots \otimes_{R^u} \mathbb{R}.$$

For Schubert varieties there exist resolutions of singularities (so called Bott-Samelson resolutions) whose cohomology admit elementary algebraic descriptions.

For any word (s, t, \dots, u) in S the cohomology of the corresponding Bott-Samelson variety is:

$$BS(s, t, \dots, u) := R \otimes_{R^s} R \otimes_{R^t} \cdots \otimes_{R^u} \mathbb{R}.$$

Theorem (Soergel)

If W is a Weyl group then the intersection cohomology of the Schubert variety $\overline{B \times B / B}$ is the unique largest indecomposable R -module summand of $BS(s, t, \dots, u)$.

For any Coxeter system one can imitate the definition of the action of a Weyl group on $\mathrm{Lie} T$ to define the “geometric representation” \mathfrak{h} of W . Let R denote the regular functions on \mathfrak{h} , a polynomial ring over \mathbb{R} .

For any Coxeter system one can imitate the definition of the action of a Weyl group on $\mathrm{Lie} T$ to define the “geometric representation” \mathfrak{h} of W . Let R denote the regular functions on \mathfrak{h} , a polynomial ring over \mathbb{R} .

Let H_x denote the largest R -module direct summand of $BS(s, t, \dots, u)$ where (s, t, \dots, u) is any reduced expression for x . Soergel shows:

For any Coxeter system one can imitate the definition of the action of a Weyl group on $\mathrm{Lie} T$ to define the “geometric representation” \mathfrak{h} of W . Let R denote the regular functions on \mathfrak{h} , a polynomial ring over \mathbb{R} .

Let H_x denote the largest R -module direct summand of $BS(s, t, \dots, u)$ where (s, t, \dots, u) is any reduced expression for x . Soergel shows:

- 1 H_x is well-defined up to isomorphism,

For any Coxeter system one can imitate the definition of the action of a Weyl group on $\mathrm{Lie} T$ to define the “geometric representation” \mathfrak{h} of W . Let R denote the regular functions on \mathfrak{h} , a polynomial ring over \mathbb{R} .

Let H_x denote the largest R -module direct summand of $BS(s, t, \dots, u)$ where (s, t, \dots, u) is any reduced expression for x . Soergel shows:

- 1 H_x is well-defined up to isomorphism,
- 2 H_x has a filtration $\Gamma_{\leq x}$ indexed by W and its Bruhat order.

For any Coxeter system one can imitate the definition of the action of a Weyl group on $\mathrm{Lie} T$ to define the “geometric representation” \mathfrak{h} of W . Let R denote the regular functions on \mathfrak{h} , a polynomial ring over \mathbb{R} .

Let H_x denote the largest R -module direct summand of $BS(s, t, \dots, u)$ where (s, t, \dots, u) is any reduced expression for x . Soergel shows:

- 1 H_x is well-defined up to isomorphism,
- 2 H_x has a filtration $\Gamma_{\leq x}$ indexed by W and its Bruhat order.

Examples:

- 1 If W is a Weyl group then $H_x = IH^*(\overline{BxB/B}; \mathbb{R})$, the intersection cohomology of a Schubert variety.

For any Coxeter system one can imitate the definition of the action of a Weyl group on $\mathrm{Lie} T$ to define the “geometric representation” \mathfrak{h} of W . Let R denote the regular functions on \mathfrak{h} , a polynomial ring over \mathbb{R} .

Let H_x denote the largest R -module direct summand of $BS(s, t, \dots, u)$ where (s, t, \dots, u) is any reduced expression for x . Soergel shows:

- 1 H_x is well-defined up to isomorphism,
- 2 H_x has a filtration $\Gamma_{\leq x}$ indexed by W and its Bruhat order.

Examples:

- 1 If W is a Weyl group then $H_x = IH^*(\overline{Bx\bar{B}/\bar{B}}; \mathbb{R})$, the intersection cohomology of a Schubert variety.
- 2 If W is finite, with longest element w_0 , then H_{w_0} is the coinvariant algebra.

Conjecture (Soergel)

The graded dimension of

$$\Gamma_{\leq y} H_x / \Gamma_{< y} H_x$$

is given by the Kazhdan-Lusztig polynomial $h_{y,x}$.

Conjecture (Soergel)

The graded dimension of

$$\Gamma_{\leq y} H_x / \Gamma_{< y} H_x$$

is given by the Kazhdan-Lusztig polynomial $h_{y,x}$.

If W is a Weyl group, then Soergel's conjecture follows from the Kazhdan and Lusztig's theorem relating intersection cohomology and Kazhdan-Lusztig polynomials.

Soergel's conjecture obviously implies that Kazhdan-Lusztig polynomials have positive coefficients.

Soergel's conjecture obviously implies that Kazhdan-Lusztig polynomials have positive coefficients.

It is a consequence of work of Soergel from 1990 that his conjecture implies the Kazhdan-Lusztig character formula.

Soergel's conjecture obviously implies that Kazhdan-Lusztig polynomials have positive coefficients.

It is a consequence of work of Soergel from 1990 that his conjecture implies the Kazhdan-Lusztig character formula.

Since then Soergel modules and bimodules have popped up throughout representation theory, and have even been used by Khovanov to construct HOMFLY-PT homology.

A key idea in our proof of Soergel's conjecture is to show that each H_x “looks like the cohomology of a smooth projective variety”.

A key idea in our proof of Soergel's conjecture is to show that each H_x “looks like the cohomology of a smooth projective variety”.

In 2006 de Cataldo and Migliorini gave Hodge theoretic proofs of the decomposition theorem, a deep result about the topology of algebraic maps between algebraic varieties.

The modules $BS(s, t, \dots, u)$ are equipped with an intersection form, using a combinatorial analogue of the fundamental class. In a complicated induction over the length of x we show that this intersection form restricts to a non-degenerate “intersection form” on $H_x \subset BS(s, t, \dots, u)$ and that analogues of the hard Lefschetz theorem and the Hodge-Riemann bilinear relations inductively, following the ideas of de Cataldo and Migliorini.

The modules $BS(s, t, \dots, u)$ are equipped with an intersection form, using a combinatorial analogue of the fundamental class. In a complicated induction over the length of x we show that this intersection form restricts to a non-degenerate “intersection form” on $H_x \subset BS(s, t, \dots, u)$ and that analogues of the hard Lefschetz theorem and the Hodge-Riemann bilinear relations inductively, following the ideas of de Cataldo and Migliorini.

As is the case for de Cataldo and Migliorini, one needs the whole package of statements for the induction to work.

Theorem (Elias-W)

For any $\rho \in V^$ in the interior of the fundamental alcove:*

- 1 *(Hard-Lefschetz theorem) left multiplication by ρ^i gives an isomorphism*

$$(H_x)^{\ell(x)-i} \rightarrow (H_x)^{\ell(x)+i}$$

- 2 *(Hodge-Riemann bilinear relations) The restriction of the form $(\alpha, \beta) := \langle \alpha, \rho^i \beta \rangle$ to the kernel of ρ^{i+1} in $(H_x)^{\ell(x)-i}$ is definite.*

These results are new even for the coinvariant algebra of a finite non-crystallographic reflection group.


Some examples of Betti numbers ...

Some examples of Betti numbers ...

$I_2(5)$: symmetries of the pentagon:

1 2 2 2 2 1



H_3 : symmetries of  :

1 3 5 7 9 11 12 12 12 12 11 9 7 5 3 1

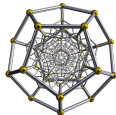
H_3 : symmetries of



:

1 3 5 7 9 11 12 12 12 12 11 9 7 5 3 1

H_4 : symmetries of 120 cell



$\subset \mathbb{R}^4$:

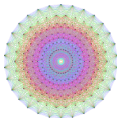
1 4 9 16 25 36 49 64 81 100 121 144 169 196 225 256 289 324 361 396 441 484 529 576 625 676 729 784 841 900 961 1024 1089 1156 1225 1296 1369 1444 1521 1600 1681 1764 1849 1936 2025 2116 2209 2304 2401 2500 2601 2704 2809 2916 3025 3136 3249 3364 3481 3600 3721 3844 3969 4096 4225 4356 4489 4624 4761 4900 5041 5184 5329 5476 5625 5776 5929 6084 6241 6400 6561 6724 6889 7056 7225 7396 7569 7744 7921 8100 8281 8464 8649 8836 9025 9216 9409 9604 9801 10000 10201 10404 10609 10816 11025 11236 11449 11664 11881 12100 12321 12544 12769 12996 13225 13456 13689 13924 14161 14400 14641 14884 15129 15376 15625 15876 16129 16384 16641 16900 17161 17424 17689 17956 18225 18496 18769 19044 19321 19600 19881 20164 20449 20736 21025 21316 21609 21904 22201 22500 22801 23104 23409 23716 24025 24336 24649 24964 25281 25600 25921 26244 26569 26896 27225 27556 27889 28224 28561 28900 29241 29584 29929 30276 30625 30976 31329 31684 32041 32400 32761 33124 33489 33856 34225 34596 34969 35344 35721 36100 36481 36864 37249 37636 38025 38416 38809 39204 39601 40000 40401 40804 41209 41616 42025 42436 42849 43264 43681 44100 44521 44944 45369 45796 46225 46656 47089 47524 47961 48400 48841 49284 49729 50176 50625 51076 51529 51984 52441 52900 53361 53824 54289 54756 55225 55696 56169 56644 57121 57600 58081 58564 59049 59536 60025 60516 61009 61504 62001 62500 63001 63504 64009 64516 65025 65536 66049 66564 67081 67600 68121 68644 69169 69696 70225 70756 71289 71824 72361 72900 73441 73984 74529 75076 75625 76176 76729 77284 77841 78400 78961 79524 80089 80656 81225 81796 82369 82944 83521 84100 84681 85264 85849 86436 87025 87616 88209 88804 89401 90000 90601 91204 91809 92416 93025 93636 94249 94864 95481 96100 96721 97344 97969 98596 99225 99856 100489 101124 101761 102400 103041 103684 104329 104976 105625 106276 106929 107584 108241 108900 109561 110224 110889 111556 112225 112896 113569 114244 114921 115600 116281 116964 117649 118336 119025 119716 120409 121104 121801 122500 123201 123904 124609 125316 126025 126736 127449 128164 128881 129600 130321 131044 131769 132496 133225 133956 134689 135424 136161 136900 137641 138384 139129 139876 140625 141376 142129 142884 143641 144400 145161 145924 146689 147456 148225 148996 149769 150544 151321 152100 152881 153664 154449 155236 156025 156816 157609 158404 159201 160000 160801 161604 162409 163216 164025 164836 165649 166464 167281 168100 168921 169744 170569 171396 172225 173056 173889 174724 175561 176400 177241 178084 178929 179776 180625 181476 182329 183184 184041 184900 185761 186624 187489 188356 189225 190096 190969 191844 192721 193600 194481 195364 196249 197136 198025 198916 199809 200704 201601 202500 203401 204304 205209 206116 207025 207936 208849 209764 210681 211600 212521 213444 214369 215296 216225 217156 218089 219024 219961 220900 221841 222784 223729 224676 225625 226576 227529 228484 229441 230400 231361 232324 233289 234256 235225 236196 237169 238144 239121 240100 241081 242064 243049 244036 245025 246016 247009 248004 249001 250000 251001 252004 253009 254016 255025 256036 257049 258064 259081 260100 261121 262144 263169 264196 265225 266256 267289 268324 269361 270400 271441 272484 273529 274576 275625 276676 277729 278784 279841 280900 281961 283024 284089 285156 286225 287296 288369 289444 290521 291600 292681 293764 294849 295936 297025 298116 299209 300304 301401 302500 303601 304704 305809 306916 308025 309136 310249 311364 312481 313600 314721 315844 316969 318096 319225 320356 321489 322624 323761 324900 326041 327184 328329 329476 330625 331776 332929 334084 335241 336400 337561 338724 339889 341056 342225 343396 344569 345744 346921 348100 349281 350464 351649 352836 354025 355216 356409 357604 358801 360000 361201 362404 363609 364816 366025 367236 368449 369664 370881 372100 373321 374544 375769 376996 378225 379456 380689 381924 383161 384400 385641 386884 388129 389376 390625 391876 393129 394384 395641 396900 398161 399424 400689 401956 403225 404496 405769 407044 408321 409600 410881 412164 413449 414736 416025 417316 418609 419904 421201 422500 423801 425104 426409 427716 429025 430336 431649 432964 434281 435600 436921 438244 439569 440896 442225 443556 444889 446224 447561 448900 450241 451584 452929 454276 455625 456976 458329 459684 461041 462400 463761 465124 466489 467856 469225 470596 471969 473344 474721 476100 477481 478864 480249 481636 483025 484416 485809 487204 488601 490000 491401 492804 494209 495616 497025 498436 499849 501264 502681 504100 505521 506944 508369 509796 511225 512656 514089 515524 516961 518400 519841 521284 522729 524176 525625 527076 528529 529984 531441 532896 534356 535816 537281 538744 540209 541676 543149 544624 546101 547584 549069 550556 552045 553536 555029 556524 558021 559520 561021 562524 564029 565536 567045 568556 570069 571584 573101 574624 576149 577676 579205 580736 582269 583804 585341 586880 588421 589964 591509 593056 594605 596156 597709 599264 600821 602380 603941 605504 607069 608636 610205 611776 613349 614924 616501 618080 619661 621244 622829 624416 626005 627596 629189 630784 632381 633980 635581 637184 638789 640396 642005 643616 645229 646844 648461 650080 651701 653324 654949 656576 658205 659836 661469 663104 664741 666380 668021 669664 671309 672956 674605 676256 677909 679564 681221 682880 684541 686204 687869 689536 691205 692876 694549 696224 697901 699580 701261 702944 704629 706316 708005 709696 711389 713084 714781 716480 718181 719884 721589 723296 725005 726716 728429 730144 731861 733580 735301 737024 738749 740476 742205 743936 745669 747404 749141 750880 752621 754364 756109 757856 759605 761356 763109 764864 766621 768380 770141 771904 773669 775436 777205 778976 780749 782524 784301 786080 787861 789644 791429 793216 795005 796796 798589 800384 802181 803980 805781 807584 809389 811196 812996 814801 816604 818409 820216 822025 823836 825649 827464 829281 831100 832921 834744 836569 838396 840225 842056 843889 845724 847561 849400 851241 853084 854929 856776 858625 860476 862329 864184 866041 867896 869756 871616 873479 875344 877211 879080 880951 882824 884701 886580 888461 890344 892229 894116 896005 897896 899789 901684 903581 905480 907381 909284 911189 913096 915005 916916 918829 920744 922661 924580 926501 928424 930349 932276 934205 936136 938069 940004 941941 943880 945821 947764 949709 951656 953605 955556 957509 959464 961421 963380 965341 967304 969269 971236 973205 975176 977149 979124 981101 983080 985061 987044 989029 991016 993005 994996 996989 998984 1000981 1002980 1004981 1006984 1008989 1010996 1013005 1015016 1017029 1019044 1021061 1023080 1025101 1027124 1029149 1031176 1033205 1035236 1037269 1039304 1041341 1043380 1045421 1047464 1049509 1051556 1053605 1055656 1057709 1059764 1061821 1063880 1065941 1068004 1070069 1072136 1074205 1076276 1078349 1080424 1082501 1084580 1086661 1088744 1090829 1092916 1095005 1097096 1099189 1101284 1103381 1105480 1107581 1109684 1111789 1113896 1116005 1118116 1120229 1122344 1124461 1126580 1128701 1130824 1132949 1135076 1137205 1139336 1141469 1143604 1145741 1147880 1149921 1152064 1154209 1156356 1158505 1160656 1162809 1164964 1167121 1169280 1171441 1173604 1175769 1177936 1180105 1182276 1184449 1186624 1188801 1190980 1193161 1195344 1197529 1199716 1201905 1204096 1206289 1208484 1210681 1212880 1215081 1217284 1219489 1221696 1223905 1226116 1228329 1230544 1232761 1234980 1237201 1239424 1241649 1243876 1246105 1248336 1250569 1252804 1255041 1257280 1259521 1261764 1264009 1266256 1268505 1270756 1273009 1275264 1277521 1279780 1282041 1284304 1286569 1288836 1291105 1293376 1295649 1297924 1300201 1302480 1304761 1307044 1309329 1311616 1313905 1316196 1318489 1320784 1323081 1325380 1327681 1329984 1332289 1334596 1336905 1339216 1341529 1343844 1346161 1348480 1350801 1353124 1355449 1357776 1360105 1362436 1364769 1367104 1369441 1371780 1374121 1376464 1378809 1381156 1383505 1385856 1388209 1390564 1392921 1395280 1397641 1400004 1402369 1404736 1407105 1409476 1411849 1414224 1416601 1418980 1421361 1423744 1426129 1428516 1430905 1433296 1435689 1438084 1440481 1442880 1445281 1447684 1450089 1452496 1454905 1457316 1459729 1462144 1464561 1466980 1469401 1471824 1474249 1476676 1479105 1481536 1483969 1486404 1488841 1491280 1493721 1496164 1498609 1501056 1503505 1505956 1508409 1510864 1513321 1515780 1518241 1520704 1523169 1525636 1528105 1530576 1533049 1535524 1537996 1540471 1542948 1545425 1547904 1550385 1552868 1555353 1557840 1560329 1562820 1565313 1567808 1570305 1572804 1575305 1577808 1580313 1582820 1585329 1587840 1590353 1592868 1595385 1597904 1600425 1602948 1605473 1607996 1610521 1613048 1615577 1618108 1620641 1623176 1625713 1628252 1630793 1633336 1635881 1638428 1640977 1643528 1646081 1648636 1651193 1653752 1656313 1658876 1661441 1664008 1666577 1669148 1671721 1674296 1676873 1679452 1682033 1684616 1687201 1689788 1692377 1694968 1697561 1700156 1702753 1705352 1707953 1710556 1713161 1715768 1718377 1720988 1723596 1726207 1728820 1731435 1734052 1736671 1739292 1741915 1744540 1747167 1749796 1752427 1755060 1757695 1760332 1762971 1765612 1768255 1770900 1773547 1776196 1778847 1781500 1784155 1786812 1789471 1792132 1794795 1797460 1800127 1802796 1805467 1808140 1810815 1813492 1816171 1818852 1821535 1824220 1826907 1829596 1832287 1834980 1837675 1840372 1843071 1845772 1848475 1851180 1853887 1856596 1859307 1862020 1864735 1867452 1870171 1872892 1875615 1878340 1881067 1883796 1886527 1889260 1891995 1894732 1897471 1900212 1902955 1905700 1908447 1911196 1913947 1916696 1919448 1922201 1924956 1927713 1930472 1933233 1935996 1938761 1941528 1944297 1947068 1949841 1952616 1955393 1958172 1960953 1963736 1966521 1969308 1972097 1974888 1977681 1980476 1983273 1986072 1988873 1991676 1994481 1997288 2000097 2002908 2005721 2008536 2011353 2014172 2016993 2019816 2022641 2025468 2028297 2031128 2033961 2036796 2039633 2042472 2045313 2048156 2051001 2053848 2056697 2059548 2062401 2065256 2068113 2070972 2073833 2076696 2079561 2082428 2085297 2088168 2091041 2093916 2096793 2099672 2102553 2105436 2108321 2111208 2114097 2116988 2119881 2122776 2125673 2128572 2131473 2134376 2137281 2140188 2143097 2146008 2148921 2151836 2154753 2157672 2160593 2163516 2166441 2169368 2172297 2175228 2178161 2181096 2184033 2186972 2189913 2192856 2195801 2198748 2201697 2204648 2207601 2210556 2213513 2216472 2219433 2222396 2225361 2228328 2231297 2234268 2237241 2240216 2243193 2246172 2249153 2252136 2255121 2258108 2261097 2264088 2267081 2270076 2273073 2276072 2279073 2282076 2285081 2288088 2291097 2294108 2297121 2300136 2303153 2306172 2309193 2312216 2315241 2318268 2321297 2324328 2327361 2330396 2333433 2336472 2339513 2342556 2345601 2348648 2351697 2354748 2357801 2360856 2363913 2366972 2370033 2373096 2376161 2379228 2382297 2385368 2388441 2391516 2394593 2397672 2400753 2403836 2406921 2410008 2413097 2416188 2419281 2422376 2425473 2428572 2431673 2434776 2437881 2440988 2444097 2447208 2450321 2453436 2456553 2459672 2462793 2465916 2469041 2472168 2475297 2478428 2481561 2484696 2487833 2490972 2494113 2497256 2500401 2503548 2506697 2509848 2512996 2516147 2519301 2522456 2525613 2528772 2531933 2535096 2538261 2541428 2544597 2547768 2550941 2554116 2557293 2560472 2563653 2566836 2569921 2573008 2576097 2579188 2582281 2585376 2588473 2591572 2594673 2597776 2600881 2603988 2607097 2610208 2613321 2616436 2619543 2622652 2625763 2628876 2631991 2635108 2638227 2641348 2644471 2647596 2650723 2653852 2656983 2660116 2663251 2666388 2669527 2672668 2675811 2678956 2682103 2685252 2688403 2691556 2694711 2697868 2701027 2704188 2707351 2710516 2713683 2716852 2720023 2723196 2726371 2729548 2732727 2735908 2739091 2742276 2745463 2748652 2751843 2755036 2758231 2761428 2764627 2767828 2771031 2774236 2777443 2780652 2783863 2787076 2790291 2793508 2796727



1 3 5 7 9 11 12 12 12 12 11 9 7 5 3 1

 $\subset \mathbb{R}^4:$

1 4 9 16 25 36 49 64 81 100 121 144 168 192 216 240 264 288 312 336 359 380 399 416 431 444 455 464 471 476 478 476 471 464 455 444 444 431 416 399 380 359 336 312 288 264 240 216 192 168 144 121 100 81 64 49 36 25 16 9 4 1

 $E_8:$ [illegible]

For non-crystallographic W the spaces H_x are in general not the cohomology of any projective variety. The integral lattice in cohomology is replaced by a lattice over some ring of integers.

For non-crystallographic W the spaces H_x are in general not the cohomology of any projective variety. The integral lattice in cohomology is replaced by a lattice over some ring of integers.

Example: If $m_{st} = 5$ then the role of the integral lattice is replaced by $\mathbb{Z}[\phi]$, where ϕ denotes the golden ratio!

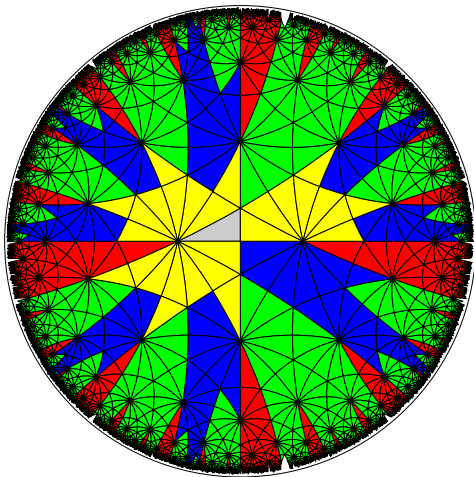
I will finish with two questions:

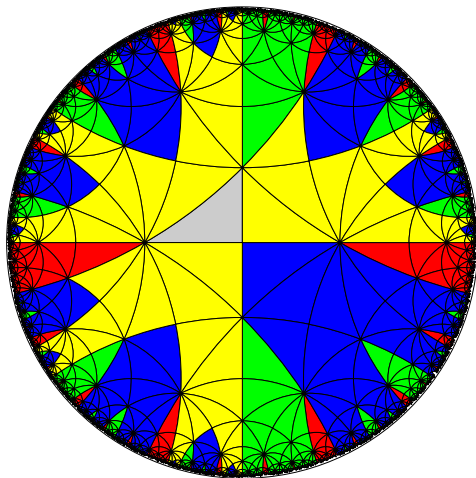
I will finish with two questions:

- i) Is there any geometric interpretation of these spaces? (One can ask a similar question for the intersection cohomology of non-rational polytopes.)

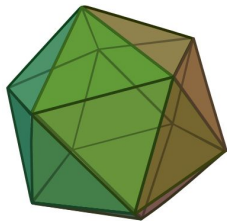
I will finish with two questions:

- i) Is there any geometric interpretation of these spaces? (One can ask a similar question for the intersection cohomology of non-rational polytopes.)
- ii) What does Kazhdan-Lusztig theory mean in the non-crystallographic case?





For more images of two-sided cells in hyperbolic groups see [Paul Gunnell's web page](#).



Thanks for listening!

