"Lusztig's exceptional mathematical ability became evident at an early stage of his career at Warwick. . . . His early experience as a mathematician was not without certain difficulties. There was a period during which, for financial reasons, he preferred to live in a tent outside the Mathematics Research Centre houses at Warwick University rather than in the houses themselves." – Roger Carter.

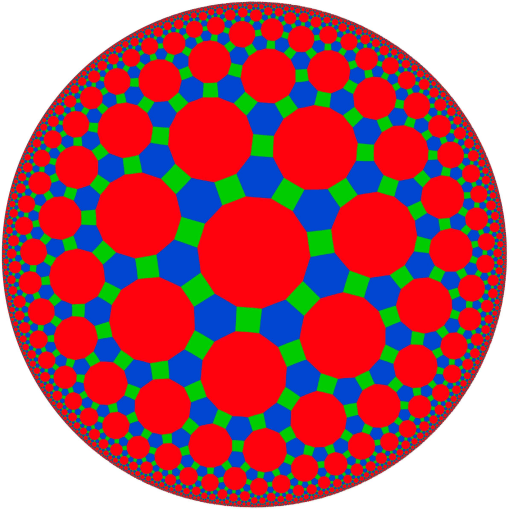
Throughout (W, S) will denote a Coxeter system:

$$\begin{aligned} W &= \langle s \in S \mid s^2 = 1, (st)^{m_{st}} = 1 \rangle \\ &= \langle s \in S \mid s^2 = 1, \underbrace{st\dots}_{m_{st}\text{terms}} = \underbrace{ts\dots}_{m_{st}\text{terms}} \rangle \end{aligned}$$

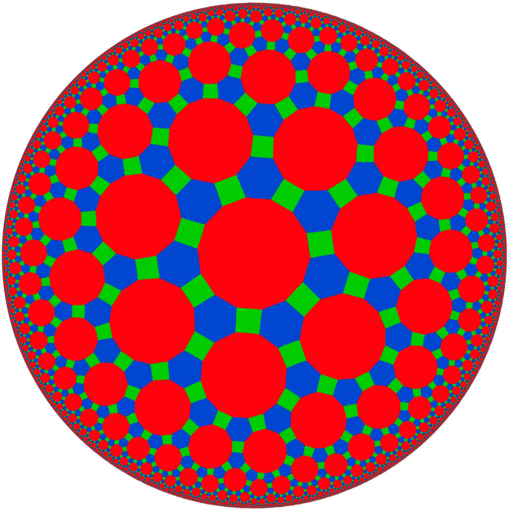
(where $m_{st} \in \{2, 3, \dots, \infty\}$).

For example, we could take W to be a real reflection group...

...or the symmetries of this tesselation of the hyperbolic plane:



...or the symmetries of this tesselation of the hyperbolic plane:



To a Coxeter system (W, S) one may associate a simplicial complex $|(W, S)|$ called the *Coxeter complex* of W.

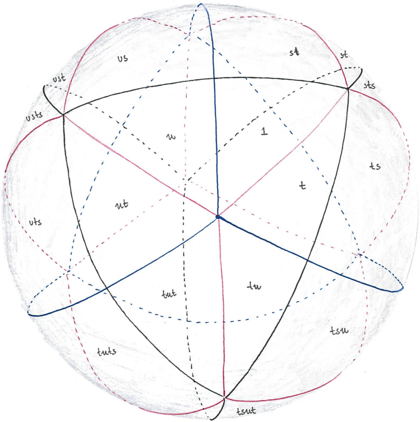
To a Coxeter system (W, S) one may associate a simplicial complex $|(W, S)|$ called the *Coxeter complex* of W.

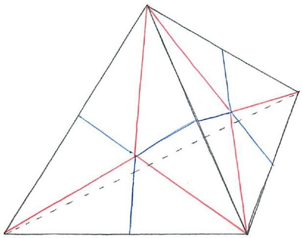
The Coxeter complex is an object of geometric group theory: it allows us to "imagine" (W, S) by providing a nice space on which it acts.

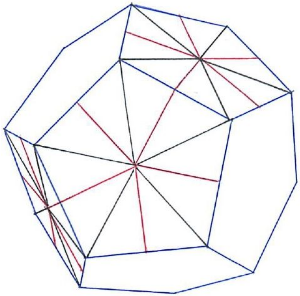
If W is a finite reflection group, then $|(W, S)|$ is a triangulation of the unit sphere in the space where W acts.

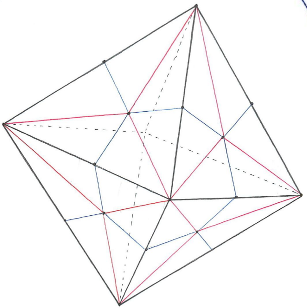
If W is a finite reflection group, then $|(W, S)|$ is a triangulation of the unit sphere in the space where W acts.

For example, $W = S_4$ acting on $\mathbb{R}^3$:





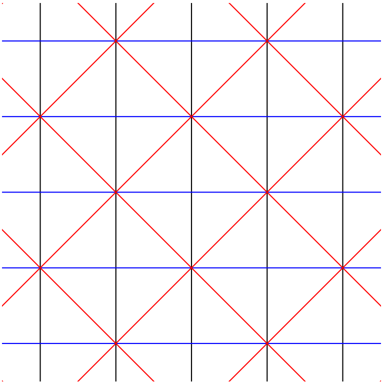




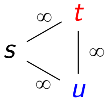
If W is a discrete group of affine transformations of V, then $|(W, S)|$ is a triangulation of V.

If W is a discrete group of affine transformations of V, then $|(W, S)|$ is a triangulation of V.

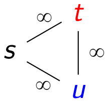
$$s \overset{4}{\text{———}} t \overset{4}{\text{———}} u$$

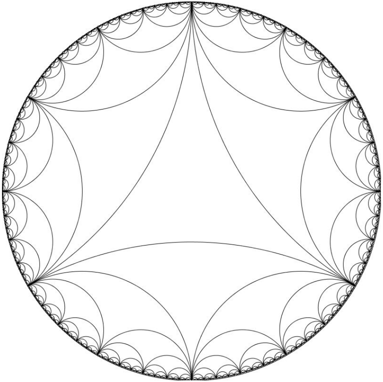


For a hyperbolic reflection group we get a triangulation of hyperbolic space:



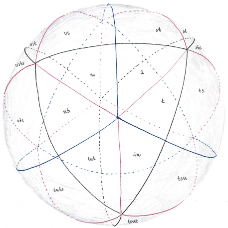
For a hyperbolic reflection group we get a triangulation of hyperbolic space:



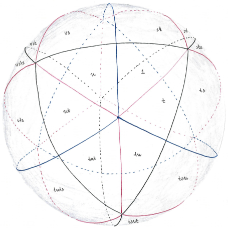


;

Key properties of Coxeter complex:

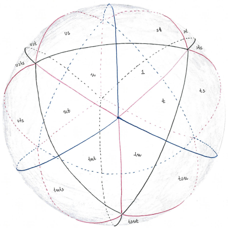


Key properties of Coxeter complex:



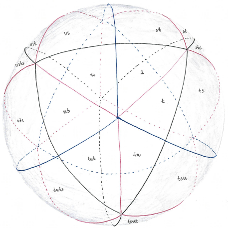
1. Simplices of maximal dimension ("alcoves") are labelled by W.

Key properties of Coxeter complex:



1. Simplices of maximal dimension ("alcoves") are labelled by W.
2. Codimension 1 simplices ("walls") are coloured by S, the simple reflections.

Key properties of Coxeter complex:



1. Simplices of maximal dimension ("alcoves") are labelled by W.
2. Codimension 1 simplices ("walls") are coloured by S, the simple reflections.
3. x and y share an s-coloured wall if and only if $xs = y$.

The Coxeter complex has a (continuous) left action of W.

W also acts on the alcoves of $|(W,S)|$ on the right by

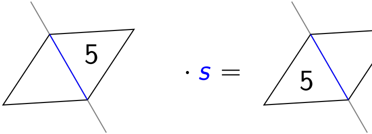
$$\Delta_w \cdot s = \Delta_{ws}.$$

This action is *not* continuous, but is “local”: cross the wall coloured by s.

The Coxeter complex provides a convenient way of visualising the group algebra $\mathbb{Z}W$ of W. Recall that the group algebra $\mathbb{Z}W$ consists of finite formal linear combinations $\sum \lambda_w w$ of elements of W. The product in W induces a multipliction in $\mathbb{Z}W$.

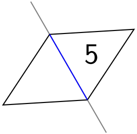
The Coxeter complex provides a convenient way of visualising the group algebra $\mathbb{Z}W$ of W. Recall that the group algebra $\mathbb{Z}W$ consists of finite formal linear combinations $\sum \lambda_w w$ of elements of W. The product in W induces a multipliction in $\mathbb{Z}W$.

Hence we can picture an element of $\mathbb{Z}W$ as the assignment of integers to each alcove, such that only finitely many are non-zero. If we view $\mathbb{Z}W$ as a right module over itself it is easy to picture the action of the elements of S:

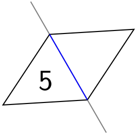


The Coxeter complex provides a convenient way of visualising the group algebra $\mathbb{Z}W$ of W. Recall that the group algebra $\mathbb{Z}W$ consists of finite formal linear combinations $\sum \lambda_w w$ of elements of W. The product in W induces a multipliction in $\mathbb{Z}W$.

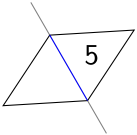
Hence we can picture an element of $\mathbb{Z}W$ as the assignment of integers to each alcove, such that only finitely many are non-zero. If we view $\mathbb{Z}W$ as a right module over itself it is easy to picture the action of the elements of S:



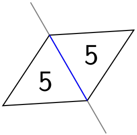
$\cdot\, s =$



Similarly ("s averaging operator")



$\cdot\, (1 + s) =$

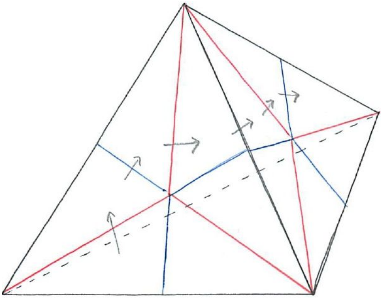


Let $\ell : W \to \mathbb{N}$ denote the length function on W:

$$\ell(w) = \text{length of a minimal expression for } w \text{ in the generators } s$$
$$= \text{number of walls crossed in a minimal path } id \to w \text{ in } |(W,S)|.$$

Let $\ell : W \to \mathbb{N}$ denote the length function on W:

$$\begin{aligned}\ell(w) &= \text{length of a minimal expression for } w \text{ in the generators } s \\ &= \text{number of walls crossed in a minimal path } id \to w \text{ in } |(W,S)|.\end{aligned}$$



The Hecke algebra H is a quantization of $\mathbb{Z}W$. It is an algebra over $\mathbb{Z}[v^{\pm 1}]$ with basis $\{H_x \mid x \in W\}$ parametrised by W. If we write $\underline{H}_s := H_s + vH_{id}$ then the multiplication in H is determined by

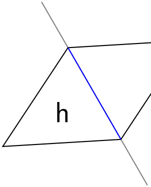
$$H_x \underline{H}_s = \begin{cases} H_{xs} + vH_x & \text{if } \ell(xs) > \ell(x), \\ H_{xs} + v^{-1}H_x & \text{if } \ell(xs) < \ell(x). \end{cases}$$

The Hecke algebra H is a quantization of $\mathbb{Z}W$. It is an algebra over $\mathbb{Z}[v^{\pm 1}]$ with basis $\{H_x \mid x \in W\}$ parametrised by W. If we write $\underline{H}_s := H_s + vH_{id}$ then the multiplication in H is determined by

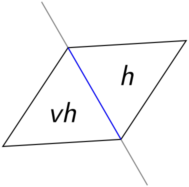
$$H_x\underline{H}_s = \begin{cases} H_{xs} + vH_x & \text{if } \ell(xs) > \ell(x), \\ H_{xs} + v^{-1}H_x & \text{if } \ell(xs) < \ell(x). \end{cases}$$

We can visualise this as follows: ("quantized averaging operator")

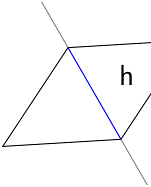
id



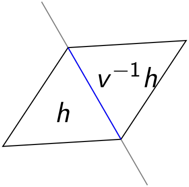
$\cdot \underline{H}_s =$



id



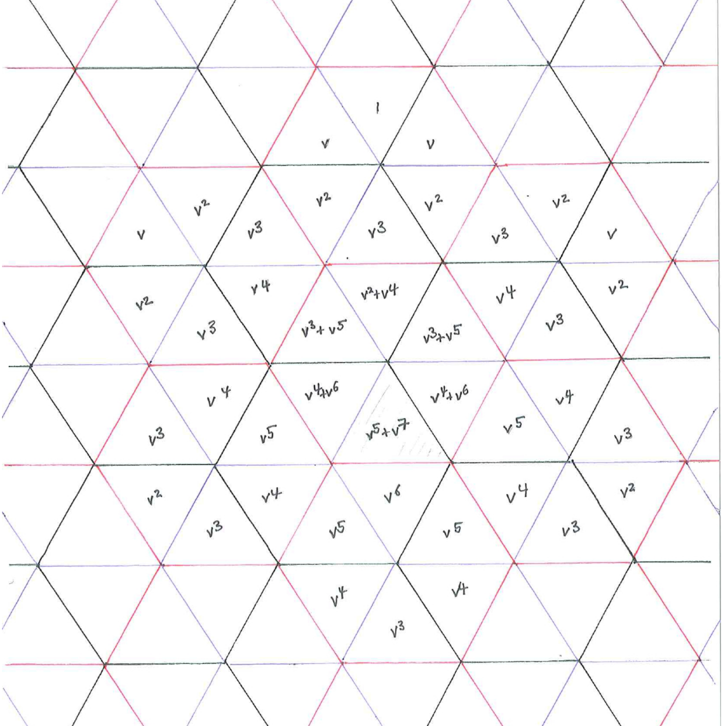
$\cdot \underline{H}_s =$



In 1979 Kazhdan and Lusztig defined a new basis for the Hecke algebra using the combinatorial structure of W. We denote this new basis by $\{\underline{H}_x \mid x \in W\}$. It satisfies

$$\underline{H}_x := H_x + \sum_{\substack{y \in W \\ \ell(y) < \ell(x)}} h_{y,x} H_y$$

with $h_{y,x} \in v\mathbb{Z}[v]$. These polynomials are the *Kazhdan-Lusztig polynomials*.


1
v v
v^2 v^2 v^2 v^2
v v^3 v^3 v^3 v
v^2 v^4 v^2+v^4 v^4 v^2
v^3 v^3+v^5 v^3+v^5 v^3
v^4 v^4+v^6 v^4+v^6 v^4
v^3 v^5 v^5+v^7 v^5 v^3
v^2 v^4 v^6 v^4 v^2
v^3 v^5 v^5 v^3
v^4 v^4
v^3

The definition is inductive. The first few Kazhdan-Lusztig basis elements are easily defined:

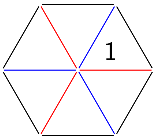
$$\underline{H}_{id} := H_{id}, \quad \underline{H}_s := H_s + vH_{id} \quad \text{for } s \in S.$$

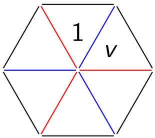
Now the work begins. Suppose that we have calculated $\underline{H}_y$ for all y with $\ell(y) \leqslant \ell(x)$. Choose $s \in S$ with $\ell(xs) > \ell(x)$ and write

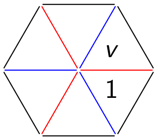
$$\underline{H}_x \underline{H}_s = H_{xs} + \sum_{\ell(y) < \ell(xs)} g_y H_y.$$

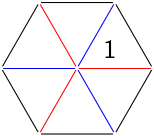
The formula for the action of $\underline{H}_s$ shows that $g_y \in \mathbb{Z}[v]$ for all $y < \ell(xs)$. If all $g_y \in v\mathbb{Z}[v]$ then $\underline{H}_{xs} := \underline{H}_x \underline{H}_s$. Otherwise we set

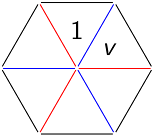
$$\underline{H}_{xs} = \underline{H}_x \underline{H}_s - \sum_{\substack{y \\ \ell(y) < \ell(x)}} g_y(0) \underline{H}_y.$$

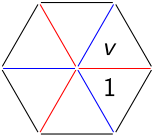
$\underline{H}_{id} =$ 

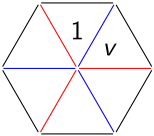
$\underline{H}_{t} =$ 

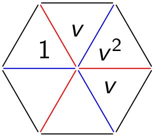
$\underline{H}_{s} =$ 

$\underline{H}_{id} =$ 

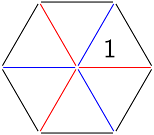
$\underline{H}_{t} =$ 

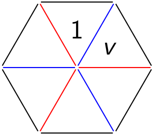
$\underline{H}_{s} =$ 

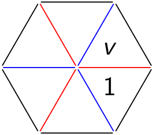
$\underline{H}_{t}\underline{H}_{s} =$ 

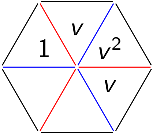
$\cdot \underline{H}_{s} =$ 

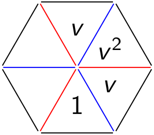
$= \underline{H}_{ts}$

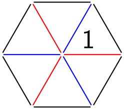
$\underline{H}_{id} =$ 

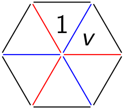
$\underline{H}_{t} =$ 

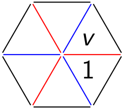
$\underline{H}_{s} =$ 

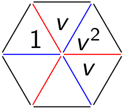
$\underline{H}_{ts} =$ 

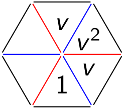
$\underline{H}_{st} =$ 

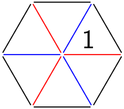
$\underline{H}_{id} =$ 

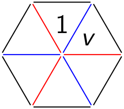
$\underline{H}_{t} =$ 

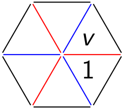
$\underline{H}_{s} =$ 

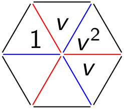
$\underline{H}_{ts} =$ 

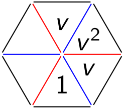
$\underline{H}_{st} =$ 

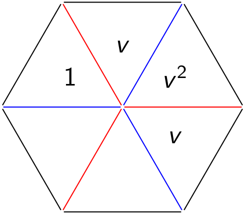
$\underline{H}_{id} =$ 

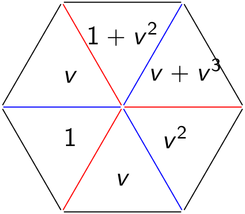
$\underline{H}_{t} =$ 

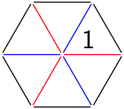
$\underline{H}_{s} =$ 

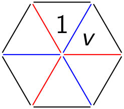
$\underline{H}_{ts} =$ 

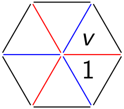
$\underline{H}_{st} =$ 

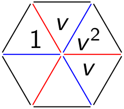
$\underline{H}_{ts}\underline{H}_{t} =$ 

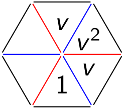
$\cdot\, \underline{H}_{t} =$ 

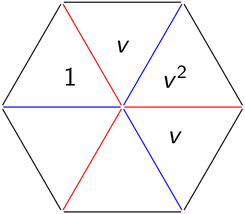
$\underline{H}_{id} =$ 

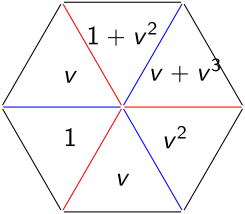
$\underline{H}_{t} =$ 

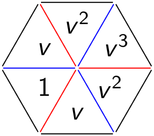
$\underline{H}_{s} =$ 

$\underline{H}_{ts} =$ 

$\underline{H}_{st} =$ 

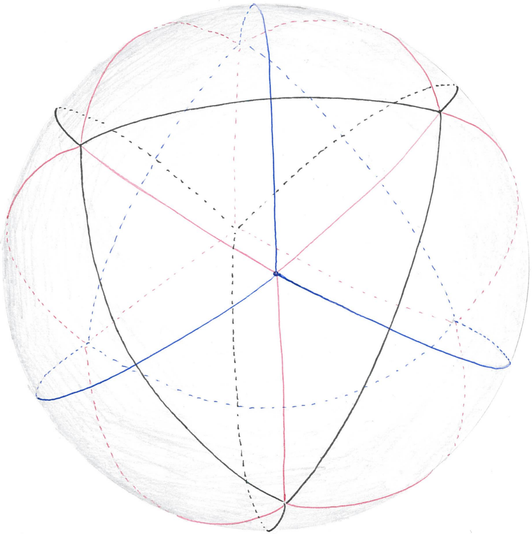
$\underline{H}_{ts}\underline{H}_{t} =$ 

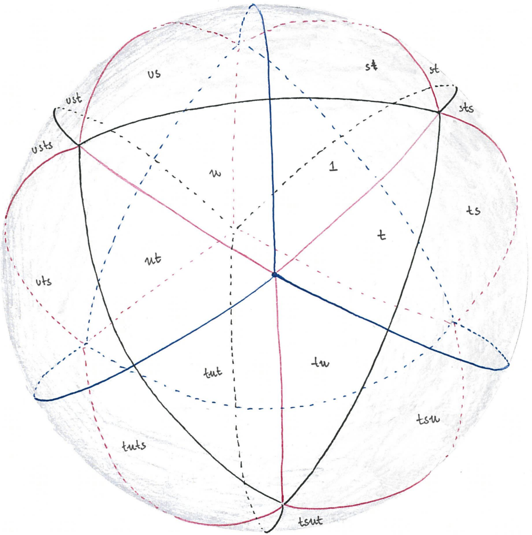
$\cdot\, \underline{H}_{t} =$ 

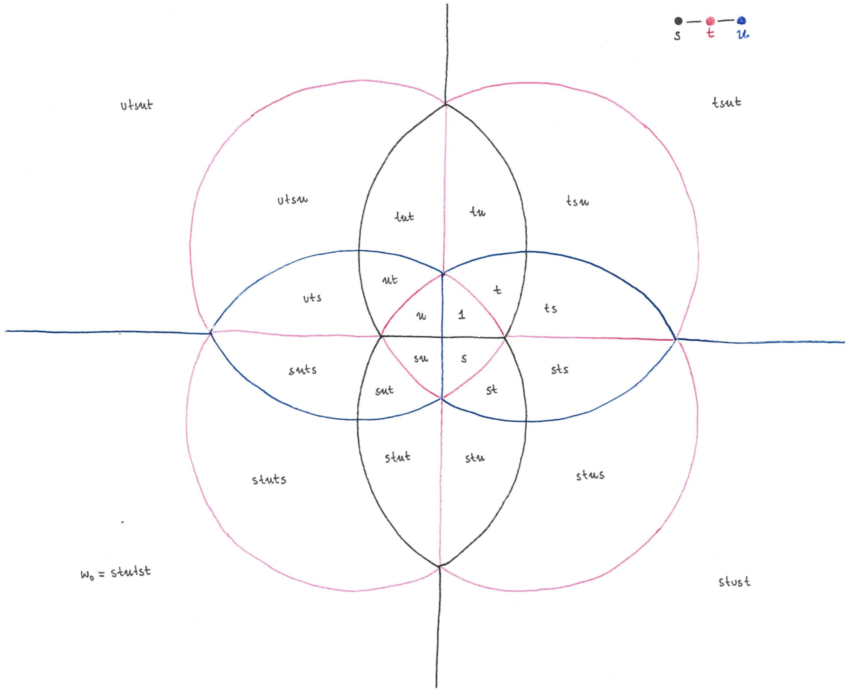
Hence: $\underline{H}_{tst} = \underline{H}_{ts}\underline{H}_{t} - \underline{H}_{t} =$ 

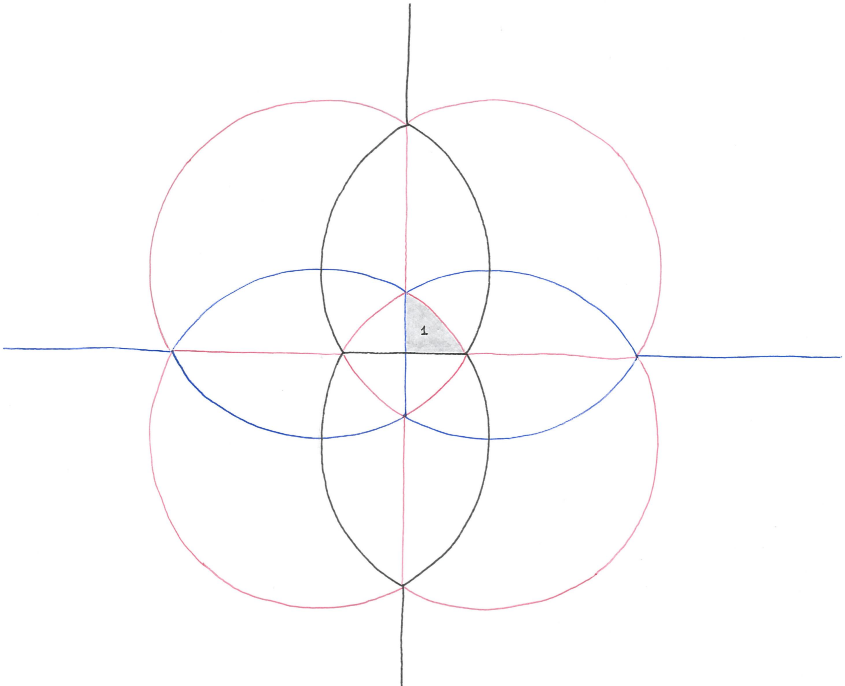
For dihedral groups (rank 2) we always have $h_{y,x} = v^{\ell(x)-\ell(y)}$ (Kazhdan-Lusztig basis elements are *smooth*.)

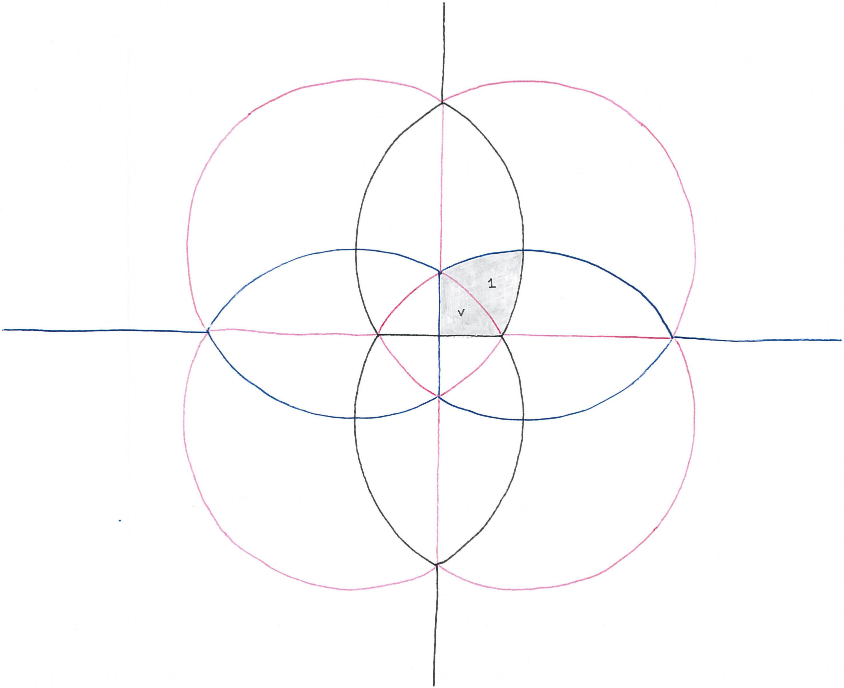
However in higher rank the situation quickly becomes more interesting...

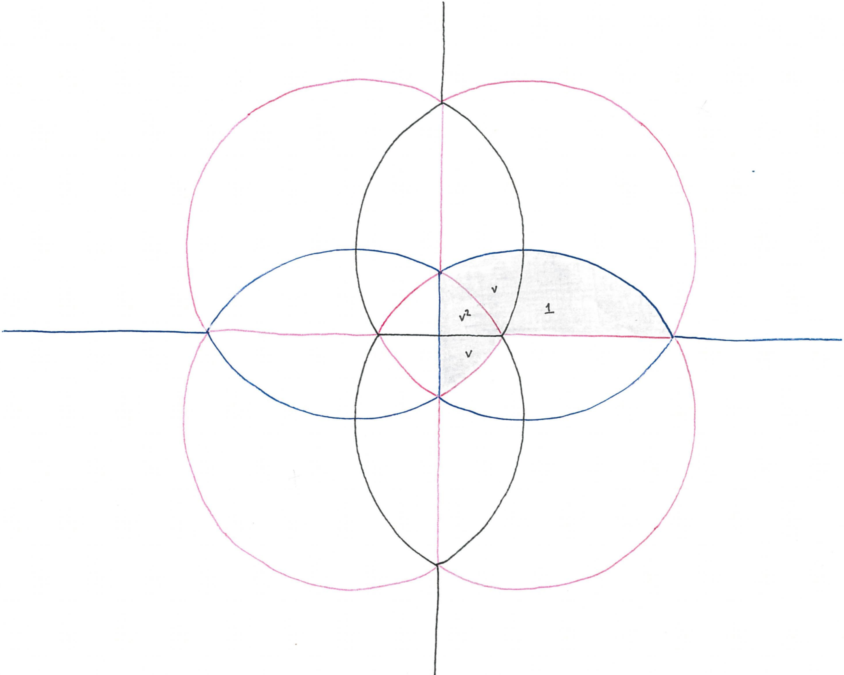


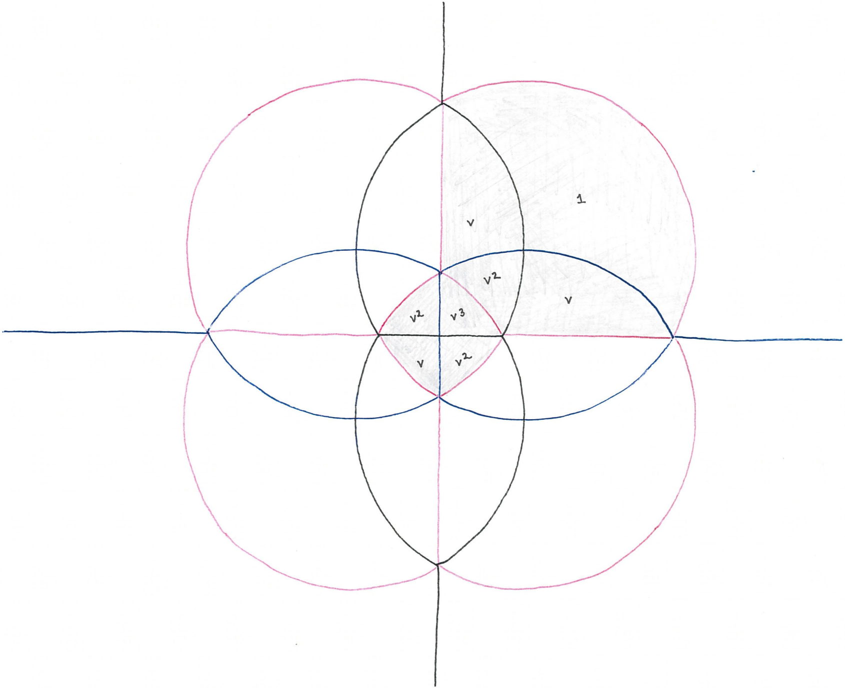

us
st
st
ust
sts
usts
u
1
ts
t
ut
uts
tut
tu
tsu
tuts
tsut

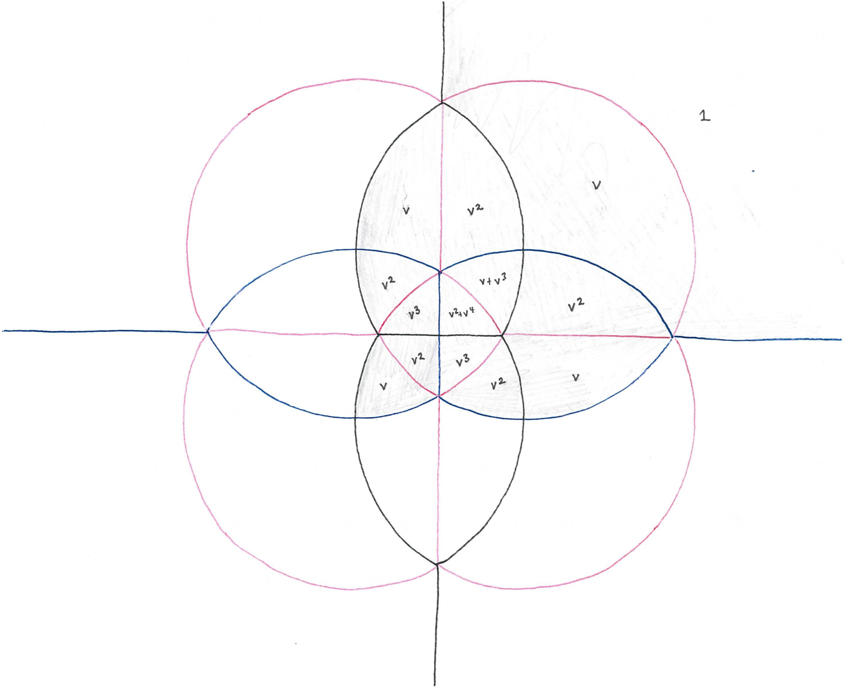

s t u
utsut
tsut
utsu
tut
tu
tsu
ut
t
uts
u
1
ts
su
s
suts
sts
sut
st
stut
stu
stuts
stus
w0 = stutst
stust

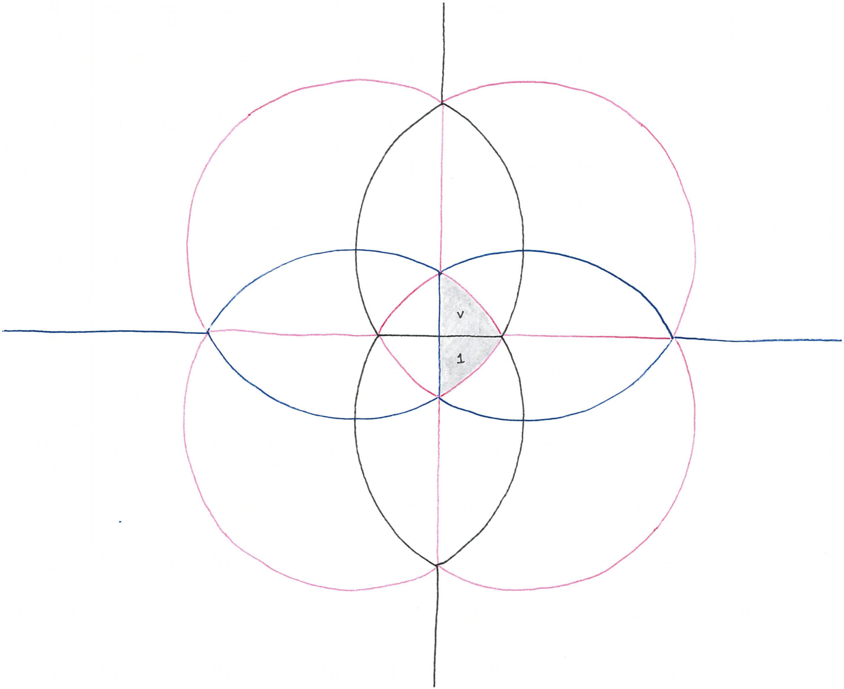

1

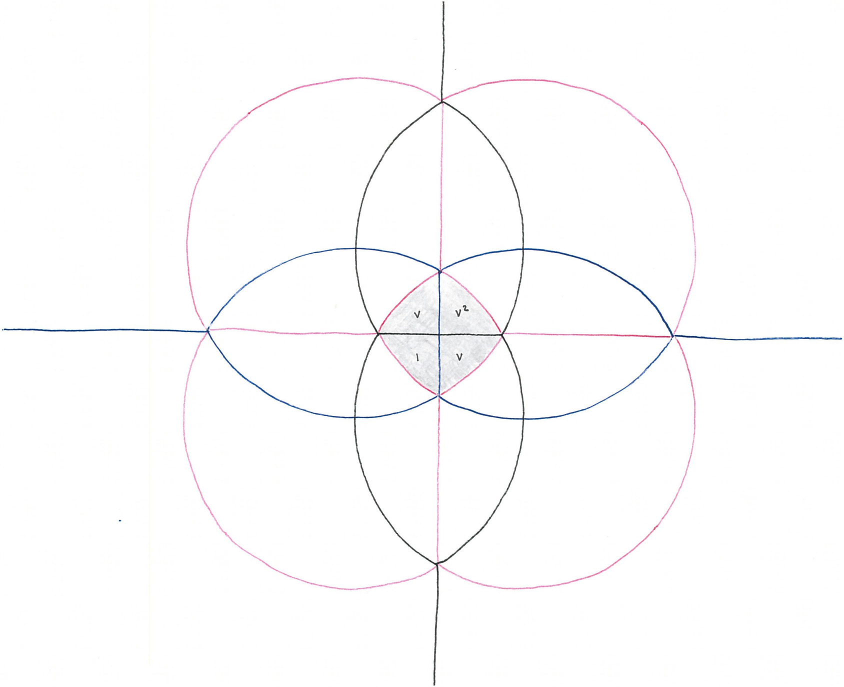

1
v

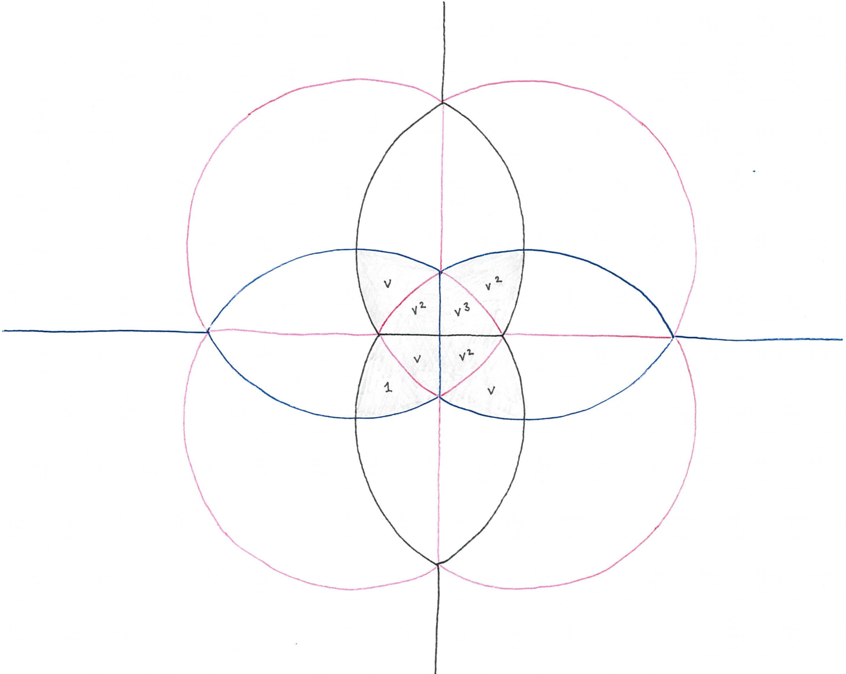

v
v²
1
v

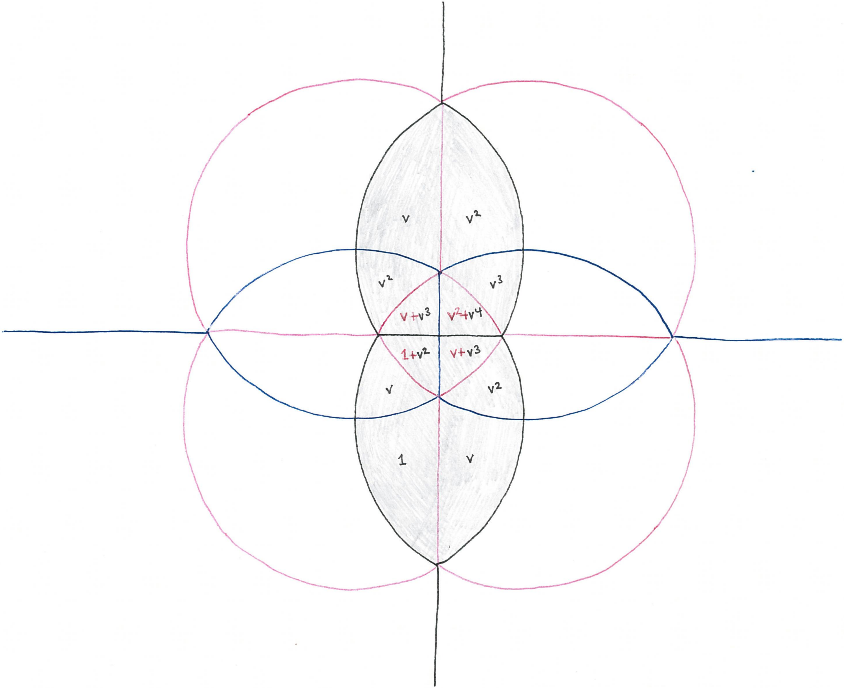

1
v
v2
v
v2
v3
v
v2

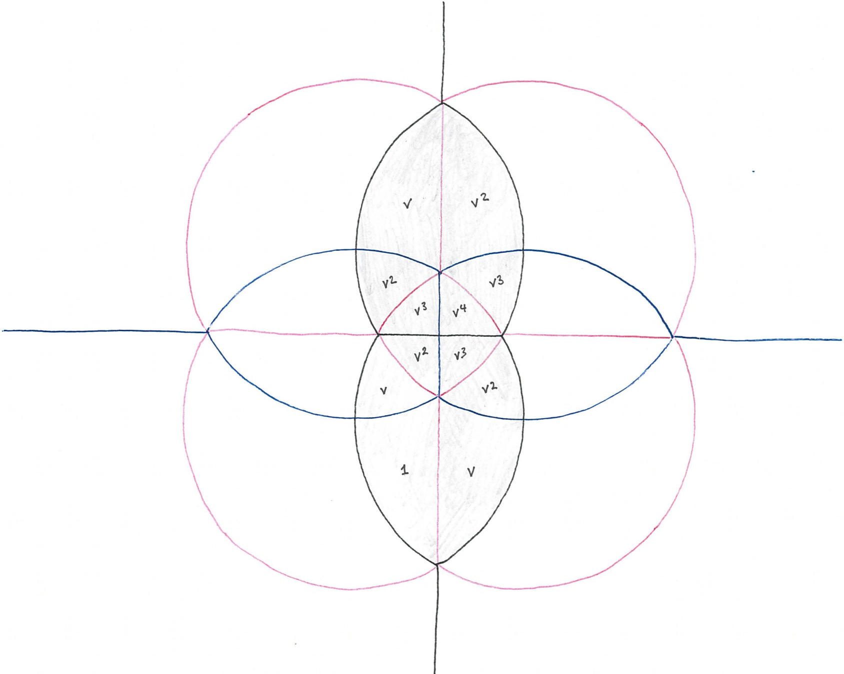

1
v
v^2
v
v^2
$v+v^3$
v^3
v^2+v^4
v^2
v
v^2
v^3
v^2
v

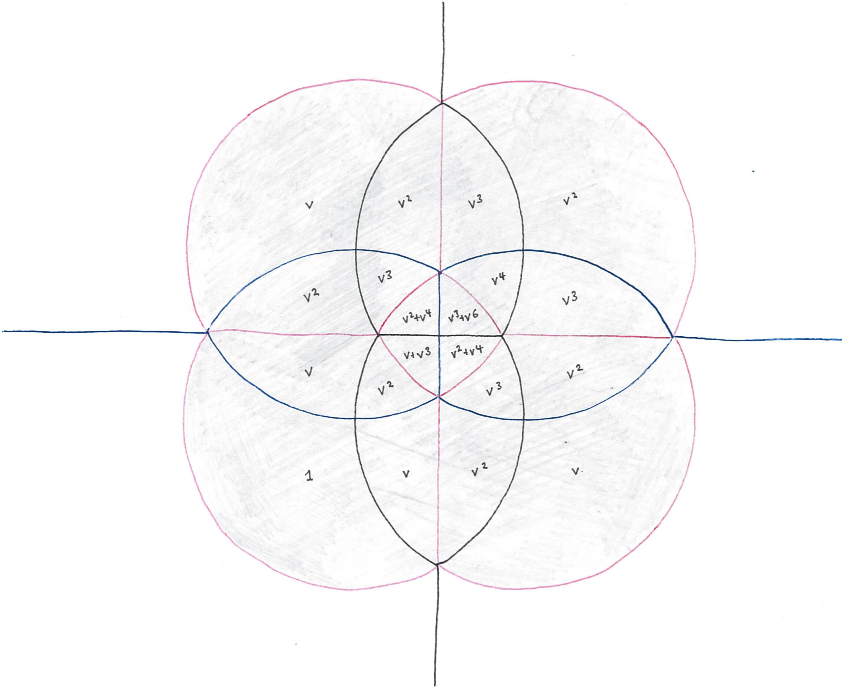

v
1


v
v²
1
v

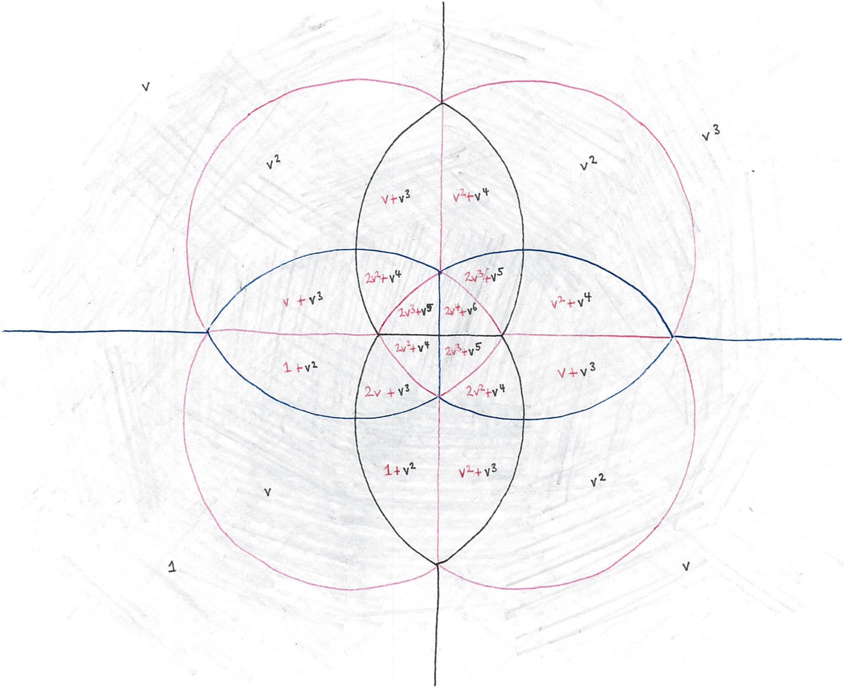

v
v2
v2
v3
v
v2
1
v

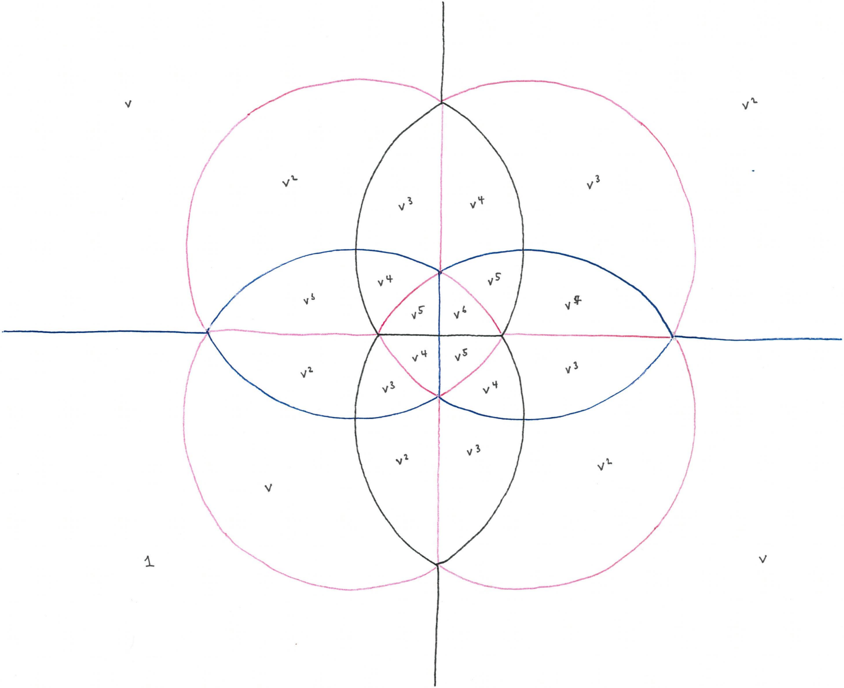

v
v2
v2
v3
v+v3
v2+v4
1+v2
v+v3
v
v2
1
v


v
v2
v2
v3
v3
v4
v2
v3
v
v2
1
v

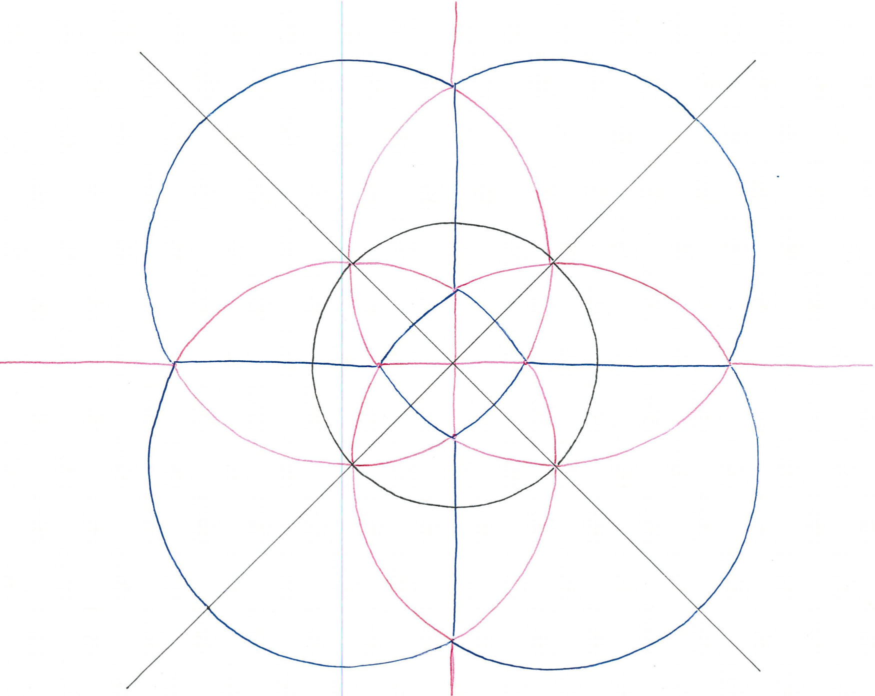


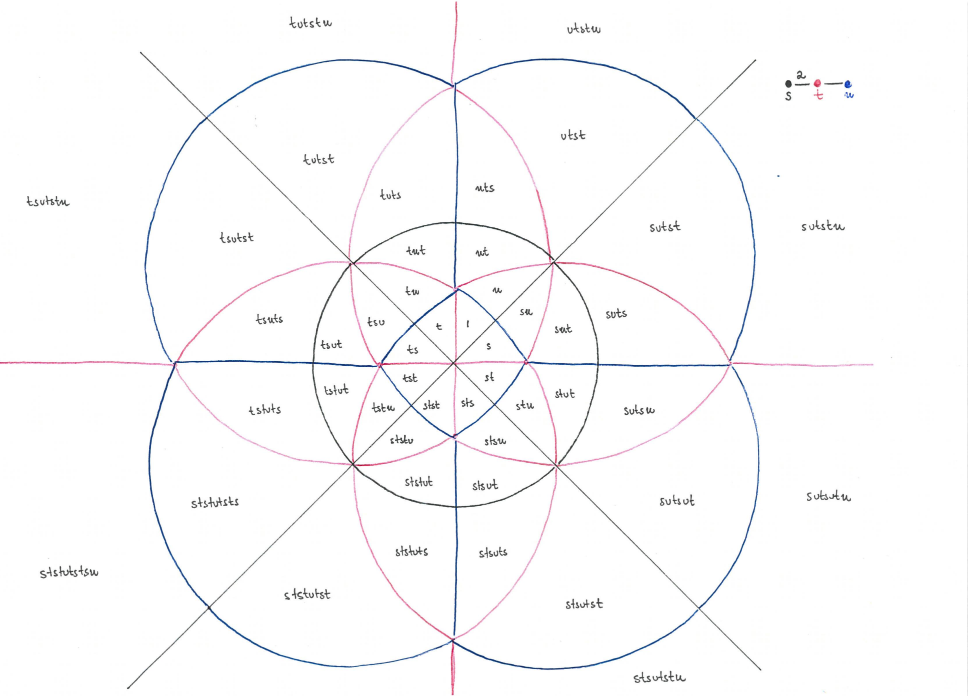
v
v²
v3
v²
v3
v4
v²
v3
v²+v4
v³+v6
v
v+v3
v²+v4
v²
v2
v3
v²
1
v
v²
v.

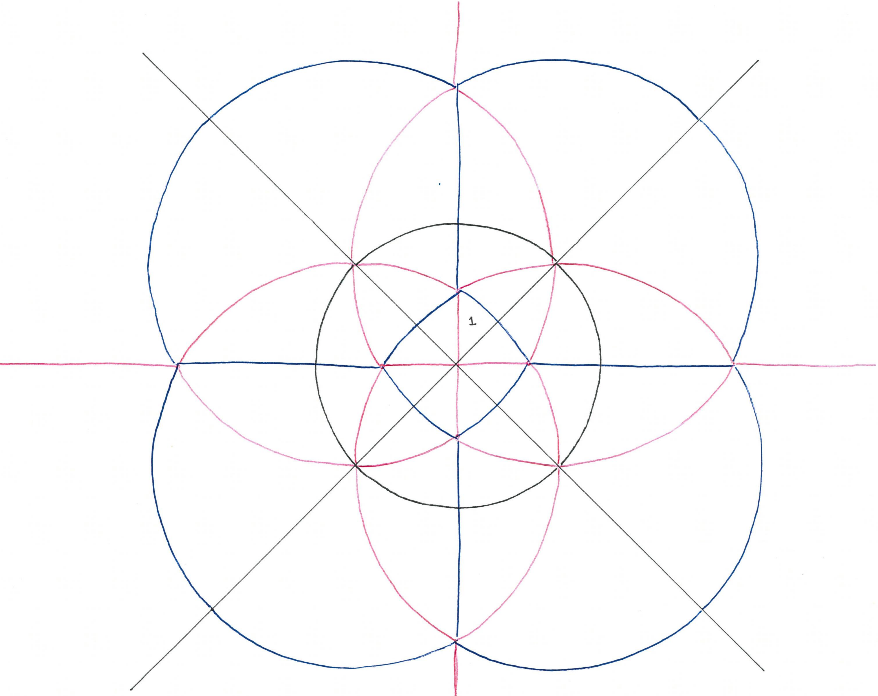

v
v^2
v^3
v^2
$v+v^3$
v^2+v^4
$2v^2+v^4$
$2v^3+v^5$
$v+v^3$
$2v^3+v^5$
$2v^4+v^6$
v^2+v^4
$1+v^2$
$2v^2+v^4$
$2v^3+v^5$
$v+v^3$
$2v+v^3$
$2v^2+v^4$
$1+v^2$
v^2+v^3
v
v^2
1
v

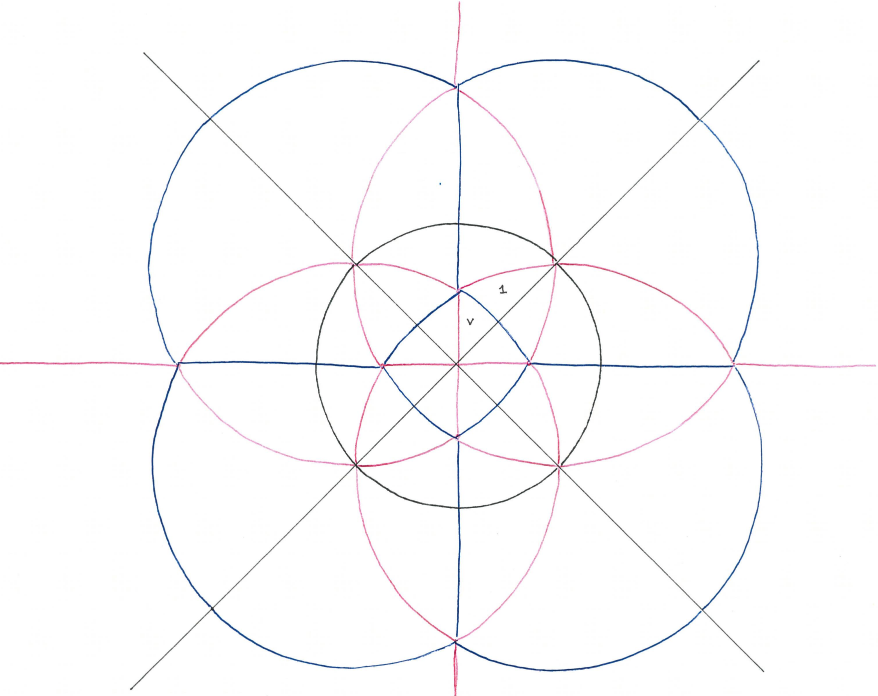


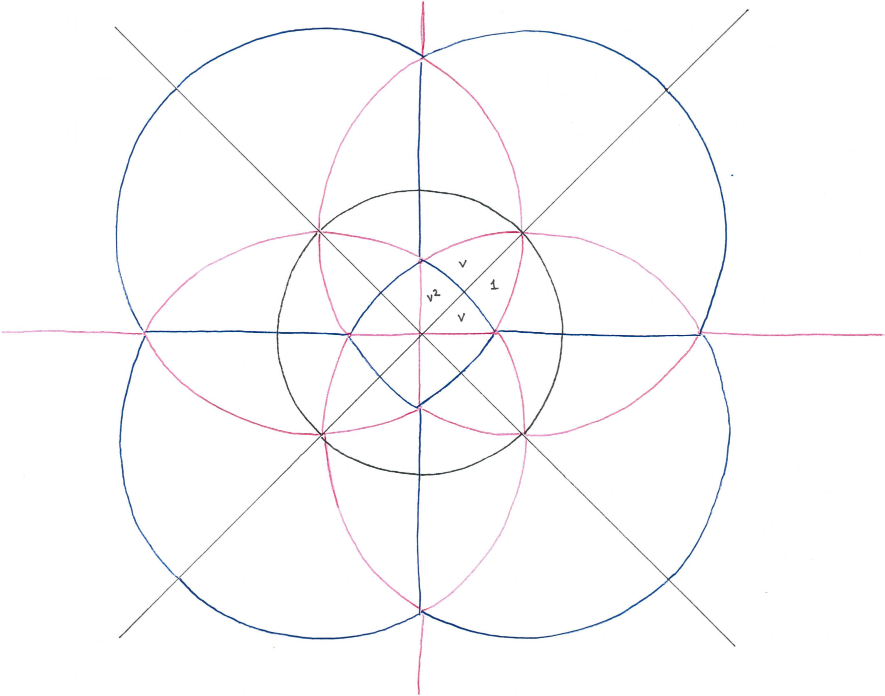
v
v^2
v^2
v^3
v^4
v^3
v^4
v^5
v^5
v^5
v^6
v^4
v^2
v^4
v^5
v^3
v^3
v^4
v^2
v^3
v^2
v
1
v

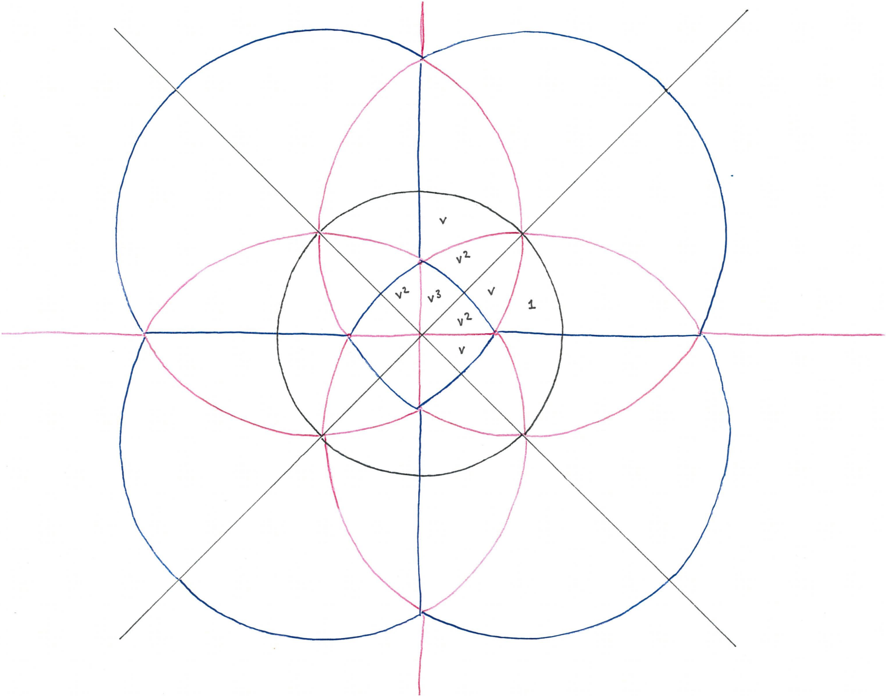



s 2 t u
tutstu
utstu
tsutstu
sutstu
tutst
utst
tuts
uts
tsutst
sutst
tut
ut
tu
u
tsu
su
t
1
sut
tsuts
suts
tsut
ts
s
tst
st
tstut
stut
tstuts
tstu
stst
sts
stu
sutsu
ststu
stsu
ststut
stsut
sutsut
sutsutu
ststutsts
ststuts
stsuts
ststutstsu
ststutst
stsutst
stsutstu

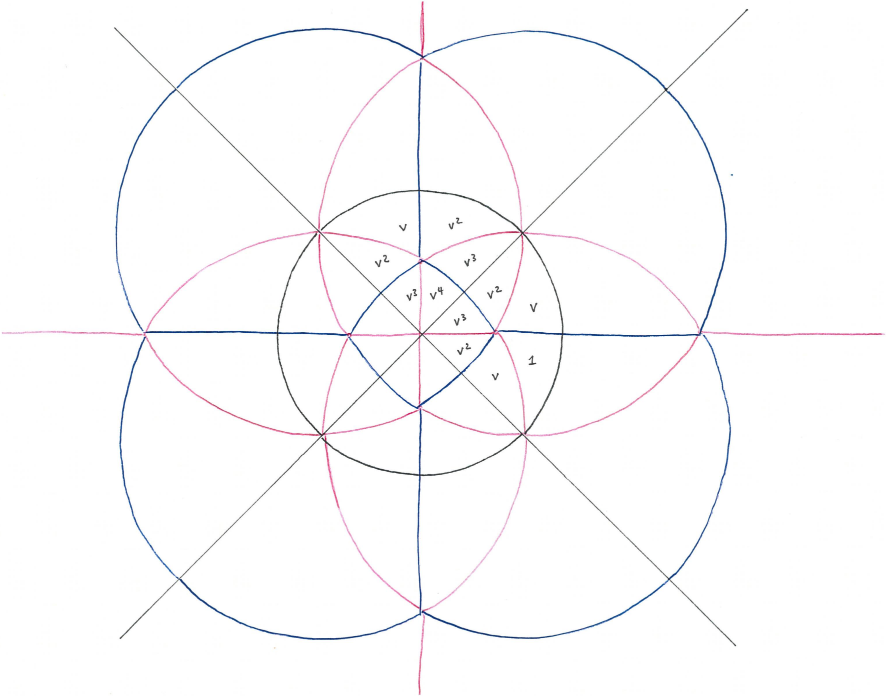

1


1
v

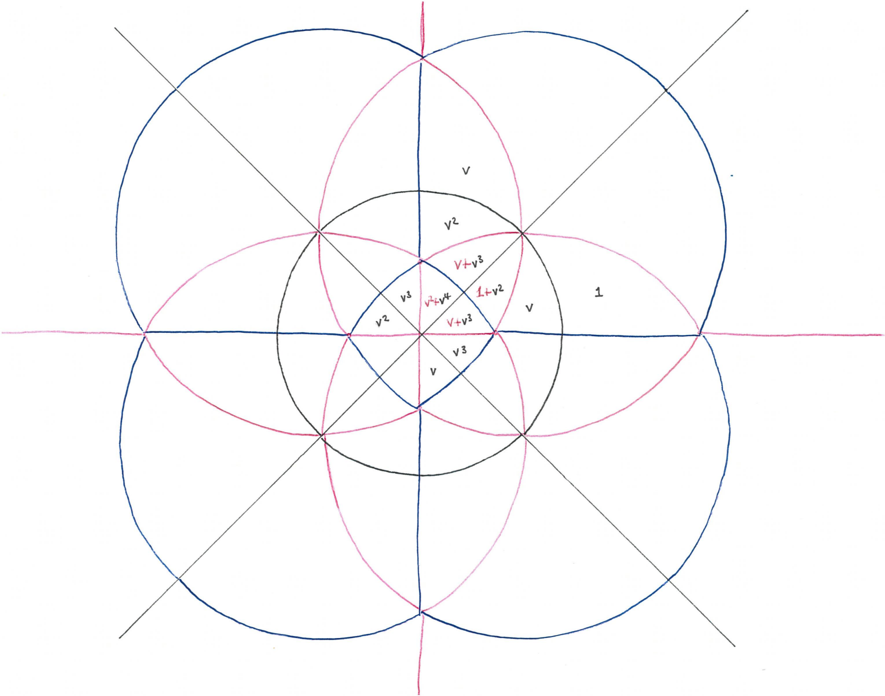

v
1
v2
v



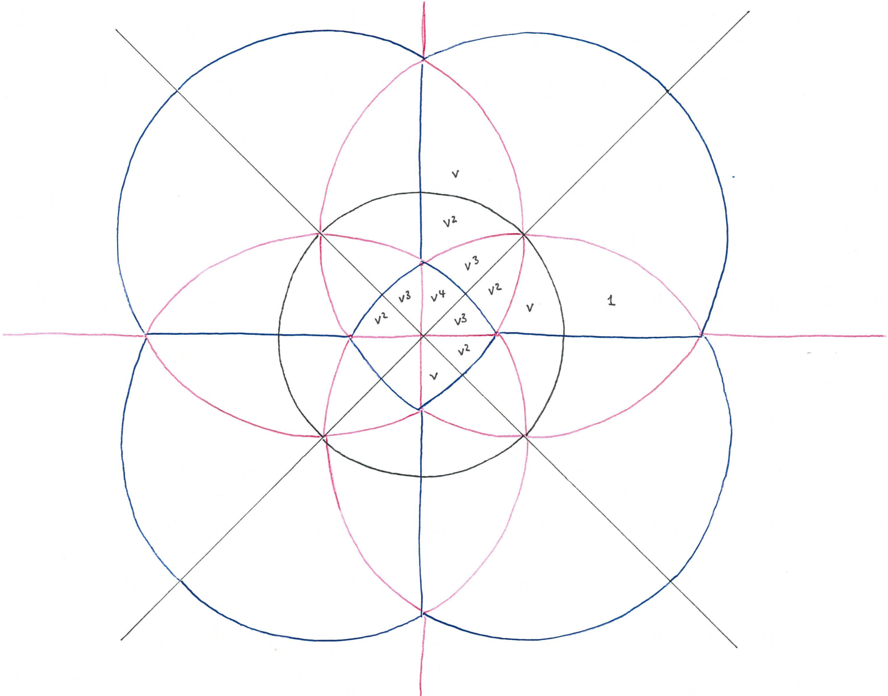
v
v2
v2
v3
v
1
v2
v

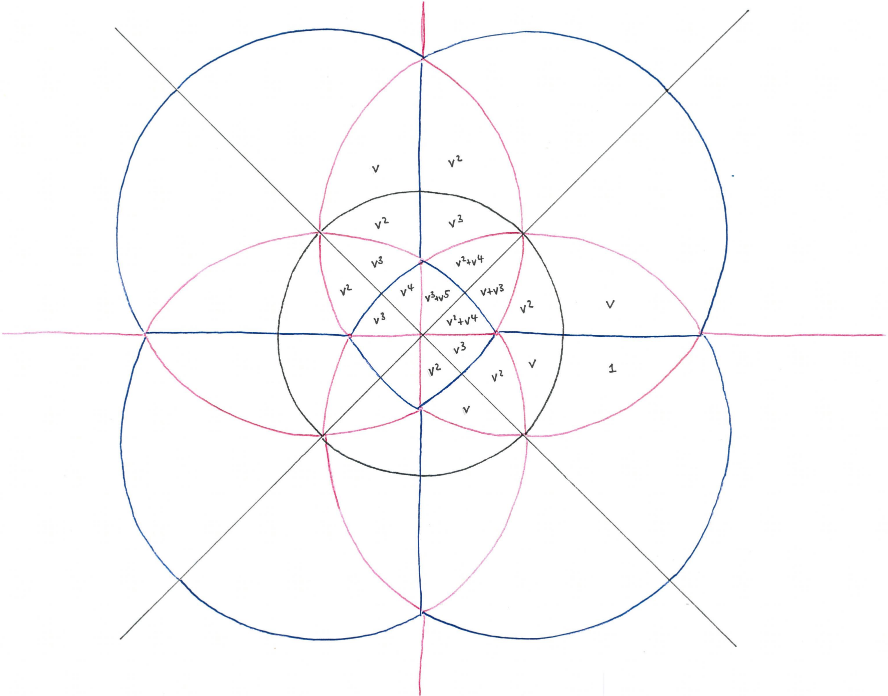


v
v2
v2
v3
v3
v4
v2
v
v3
v2
1
v

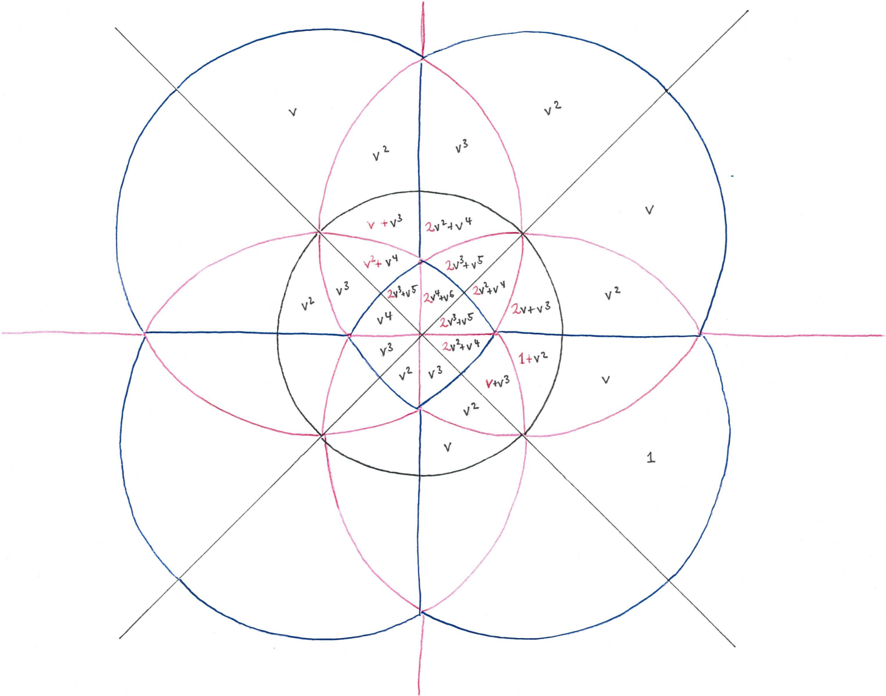


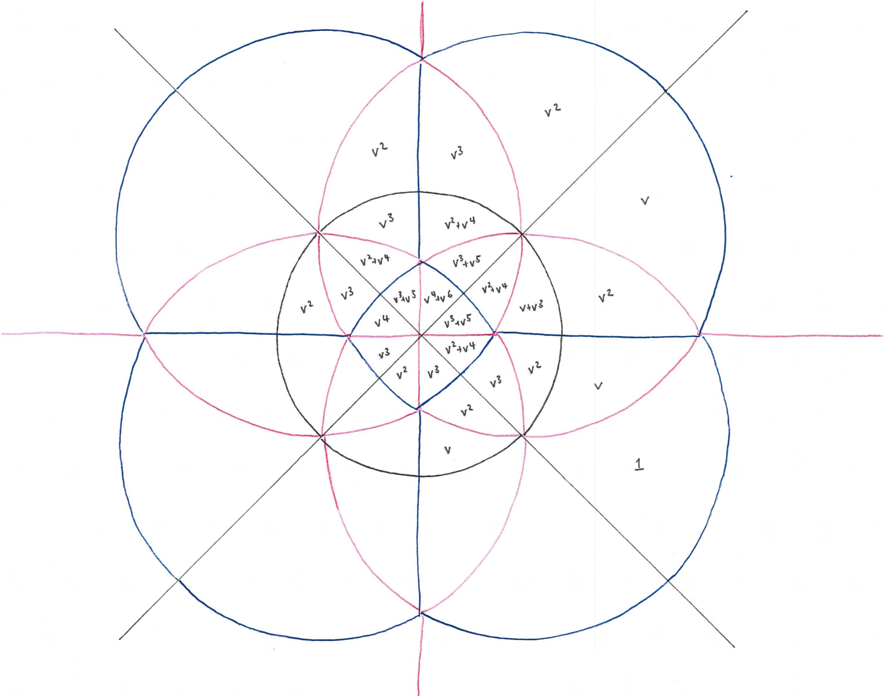
v
v^2
$v+v^3$
v^3
v^2+v^4
$1+v^2$
1
v
v^2
$v+v^3$
v^3
v

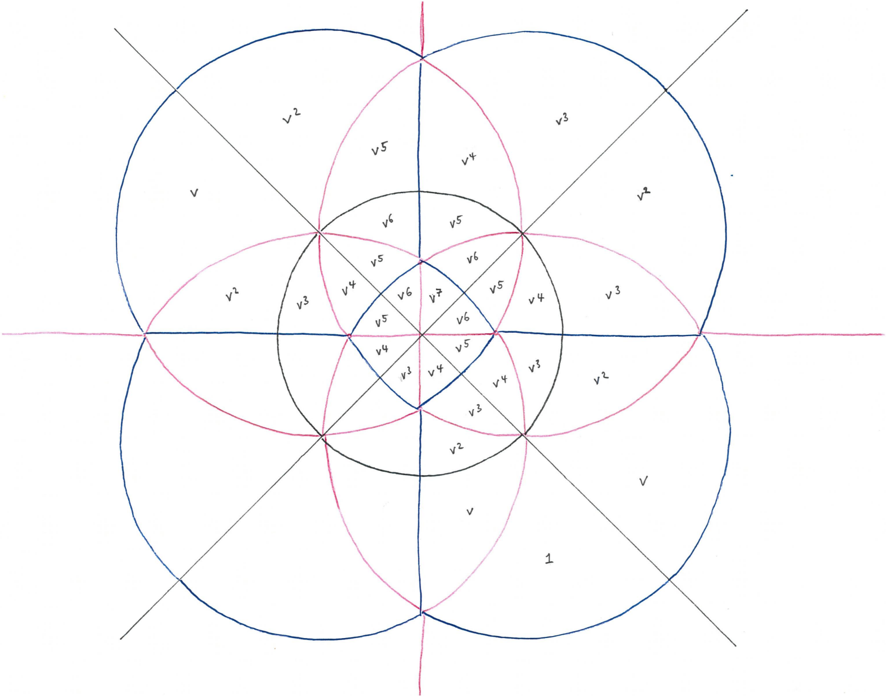

v
v2
v3
v3
v4
v2
v2
v
1
v3
v2
v

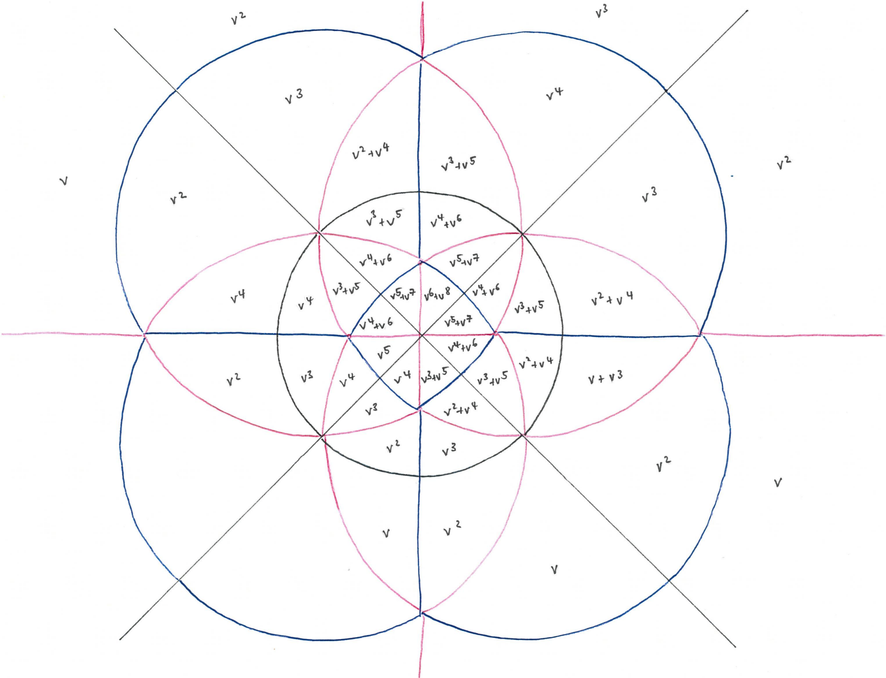


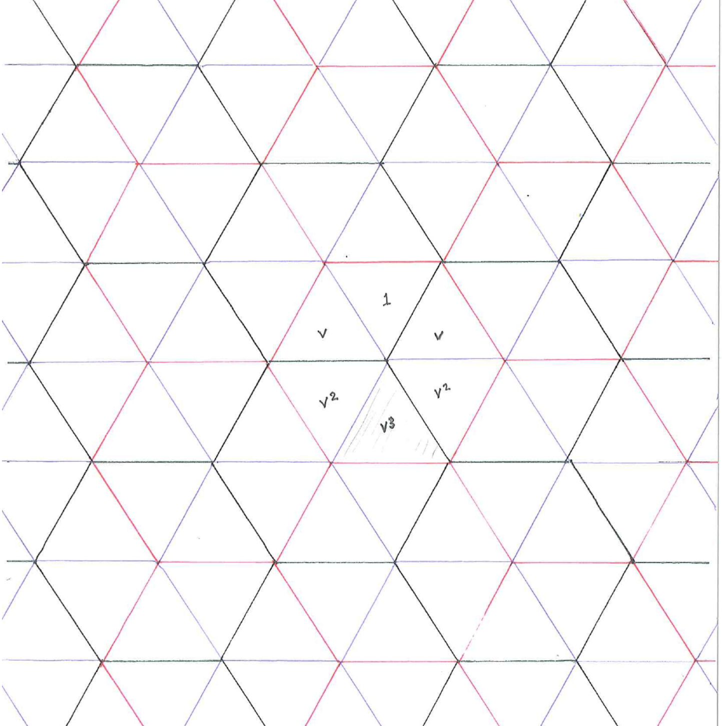
v
v2
v2
v3
v3
v2+v4
v2
v4
v3+v5
v+v3
v2
v
v3
v2+v4
v3
v2
v2
v
1
v

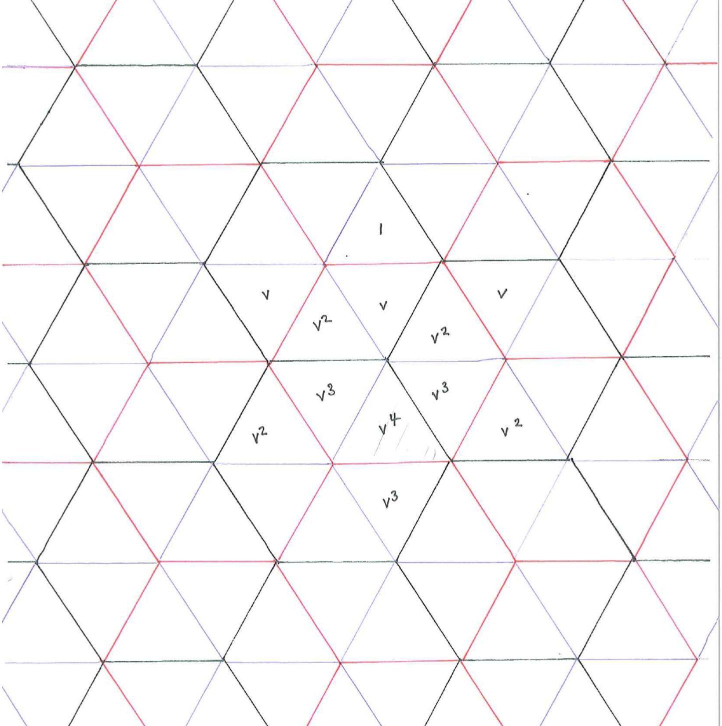

v
v2
v2
v3
v
v + v3
2v2+v4
v2+ v4
2v3+v5
v2
v3
2v3+v5
2v4+v6
2v2+v4
2v+v3
v2
v4
2v3+v5
v3
2v2+v4
1+v2
v2
v3
v+v3
v
v2
v
1

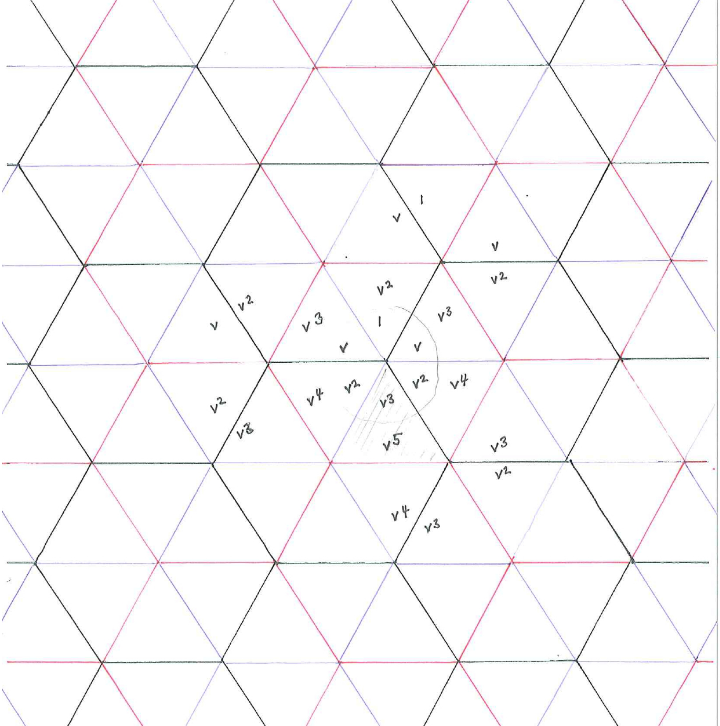

v2
v3
v2
v
v3
v2+v4
v2+v4
v3+v5
v2
v3
v3+v5
v4+v6
v3+v4
v+v3
v2
v4
v3+v5
v3
v2+v4
v2
v3
v3
v2
v
v2
v
1

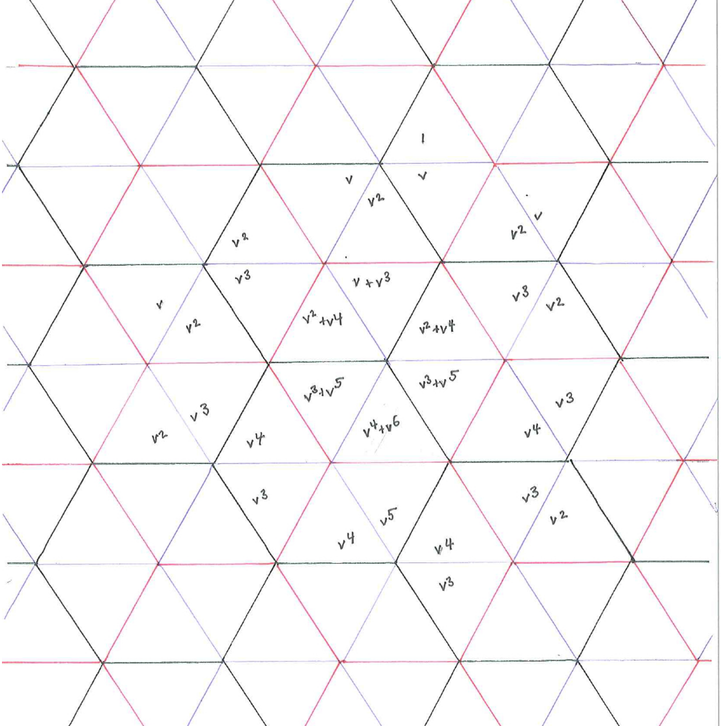

v2
v3
v5
v4
v
v2
v6
v5
v5
v6
v2
v3
v4
v6
v7
v5
v4
v3
v3
v5
v6
v4
v5
v3
v4
v4
v3
v2
v3
v2
v
v
1

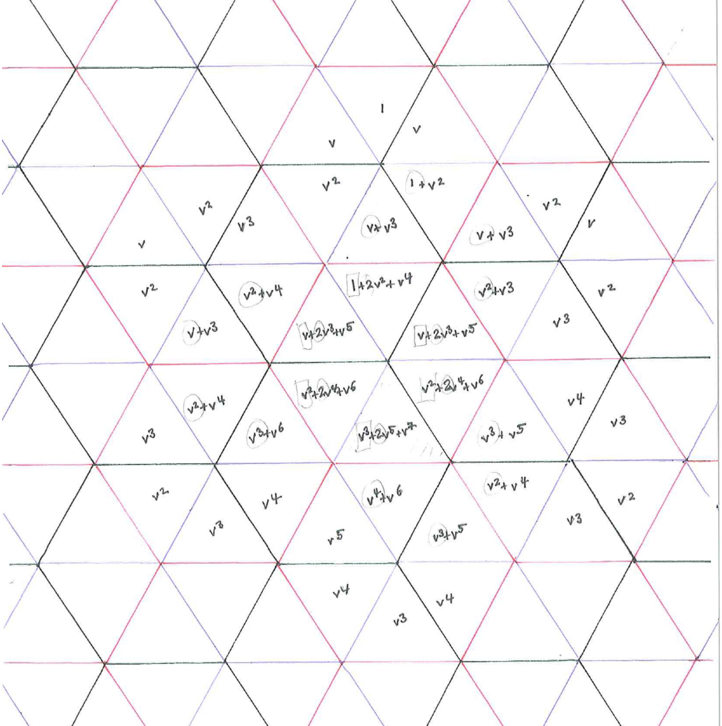

v^2
v^3
v
v^2
$v3$
$v4$
v^2+v^4
v^3+v^5
v^2
v^3
v^3+v^5
v^4+v^6
v^4+v^6
v^5+v^7
$v4$
$v4$
v^3+v^5
v^5+v^7
v^6+v^8
v^4+v^6
v^3+v^5
v^2+v4
v^4+v6
v^5+v^7
$v5$
v^4+v^6
v^2
$v3$
$v4$
$v4$
v^3+v^5
v^3+v^5
v^2+v^4
$v+v3$
$v3$
v^2+v^4
v^2
$v3$
v^2
v
v
v^2
v

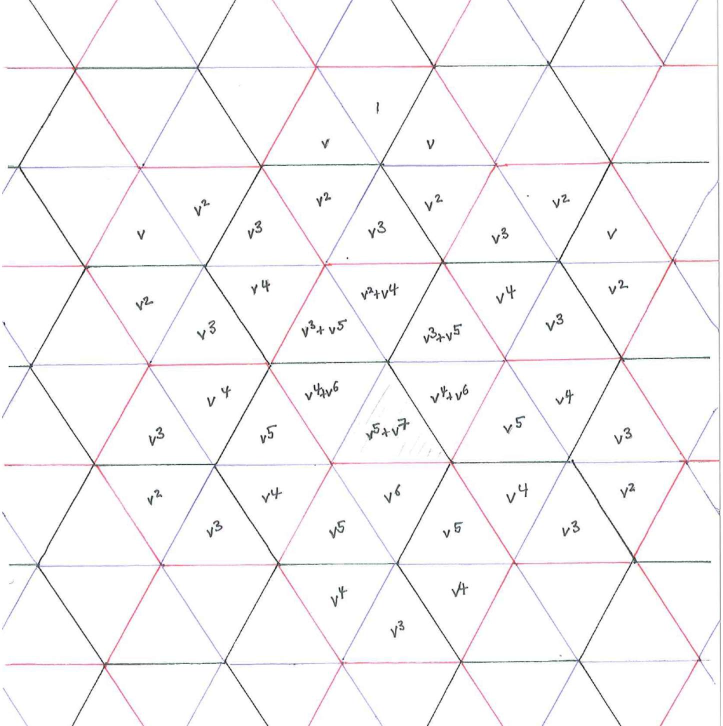

1
v
v
v2
v2
v3

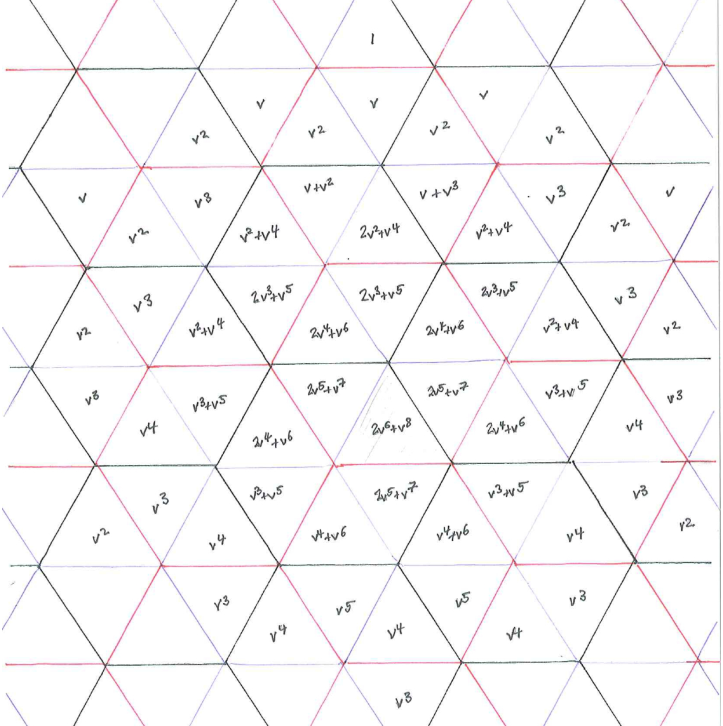

1
v
v^2
v
v^2
v
v^3
v^3
v^2
v^4
v^2
v^3


1
v
v
v2
v2
v2
v
v3
v2
1
v3
v
v
v2
v4
v2
v2
v4
v3
v3
v5
v3
v2
v4
v3


1
v
v
v2
v2
v
v2
v3
v+v3
v3
v2
v
v2
v2+v4
v2+v4
v3+v5
v3+v5
v3
v3
v2
v4
v4+v6
v4
v3
v3
v5
v2
v4
v4
v3


1
v
v
v²
1+v²
v²
v³
v+v³
v+v³
v²
v
v
v²
v²+v⁴
1+2v²+v⁴
v²+v³
v²
v+v³
v+2v³+v⁵
v+2v³+v⁵
v³
v³+2v⁴+v⁶
v²+2v⁴+v⁶
v⁴
v²+v⁴
v³
v³+v⁶
v³+2v⁵+v⁷
v³+v⁵
v³
v²
v⁴
v⁴+v⁶
v²+v⁴
v²
v³
v⁵
v³+v⁵
v³
v⁴
v⁴
v³


1
v v
v v^2 v^3 v^2 v^3 v^2 v^3 v^2 v
v^2 v^3 v^4 v^3+v^5 v^2+v^4 v^3+v^5 v^4 v^3 v^2
v^3 v^4 v^5 v^4+v^6 v^5+v^7 v^4+v^6 v^5 v^4 v^3
v^2 v^3 v^4 v^5 v^6 v^5 v^4 v^3 v^2
v^4 v^3 v^4


1
v v v
v^2 v^2 v^2 v^2
v v^3 $v+v^2$ $v+v^3$ v^3 v
v^2 v^2+v^4 $2v^2+v^4$ v^2+v^4 v^2
v^3 $2v^3+v^5$ $2v^3+v^5$ $2v^3+v^5$ v^3
v^2 v^2+v^4 $2v^4+v^6$ $2v^4+v^6$ v^2+v^4 v^2
v^3 v^3+v^5 $2v^5+v^7$ $2v^5+v^7$ v^3+v^5 v^3
v^4 $2v^4+v^6$ $2v^6+v^8$ $2v^4+v^6$ v^4
v^3 v^3+v^5 $2v^5+v^7$ v^3+v^5 v^3
v^2 v^4 v^4+v^6 v^4+v^6 v^4 v^2
v^3 v^5 v^5 v^3
v^4 v^4 v^4
v^3

Kazhdan-Lusztig positivity conjecture (1979):

$$h_{x,y} \in \mathbb{Z}_{\geqslant 0}[v]$$

Kazhdan-Lusztig positivity conjecture (1979):

$$h_{x,y} \in \mathbb{Z}_{\geqslant 0}[v]$$

Established for crystallographic W by Kazhdan and Lusztig in 1980, using Deligne's proof of the Weil conjectures.

Crystallographic: $m_{st} \in \{2, 3, 4, 6, \infty\}$.

Why are Kazhdan-Lusztig polynomials hard?

Why are Kazhdan-Lusztig polynomials hard?

Polo's Theorem (1999)

For any $P \in 1 + q\mathbb{Z}_{\geqslant 0}[q]$ there exists an m such that $v^m P(v^{-2})$ occurs as a Kazhdan-Lusztig polynomial in some symmetric group.

Why are Kazhdan-Lusztig polynomials hard?

Polo's Theorem (1999)

For any $P \in 1 + q\mathbb{Z}_{\geqslant 0}[q]$ there exists an m such that $v^m P(v^{-2})$ occurs as a Kazhdan-Lusztig polynomial in some symmetric group.

Roughly: all positive polynomials are Kazhdan-Lusztig polynomials!

The most complicated Kazhdan-Lusztig-Vogan polynomial computed by the *Atlas of Lie groups and Representations* project:

$$152q^{22}+3\,472q^{21}+38\,791q^{20}+293\,021q^{19}+1\,370\,892q^{18}+$$
$$+4\,067\,059q^{17}+7\,964\,012q^{16}+11\,159\,003q^{15}+$$
$$+11\,808\,808q^{14}+9\,859\,915q^{13}+6\,778\,956q^{12}+$$
$$+3\,964\,369q^{11}+2\,015\,441q^{10}+906\,567q^{9}+$$
$$+363\,611q^{8}+129\,820q^{7}+41\,239q^{6}+$$
$$+11\,426q^{5}+2\,677q^{4}+492q^{3}+61q^{2}+3q$$

(This polynomial is associated to the reflection group of type E_8. See `www.liegroups.org`.)

Why are Kazhdan-Lusztig polynomials useful?

Infinite dimensional highest weight representations of semi-simple Lie algebras.

Infinite dimensional highest weight representations of semi-simple Lie algebras.

Kazhdan-Lusztig character formula (conjectured in 1979):

$$\mathrm{ch} L(x \cdot 0) = \sum_{y \geqslant x} (-1)^{\ell(x)-\ell(y)} h_{w_0 y, w_0 x}(1) \mathrm{ch} \Delta(y \cdot 0).$$

Infinite dimensional highest weight representations of semi-simple Lie algebras.

Kazhdan-Lusztig character formula (conjectured in 1979):

$$\mathrm{ch}L(x \cdot 0) = \sum_{y \geqslant x} (-1)^{\ell(x)-\ell(y)} h_{w_0 y, w_0 x}(1) \mathrm{ch}\Delta(y \cdot 0).$$

(A major generalisation of the Weyl character formula.)

The Kazhdan-Lusztig character formula was proved 1981 by Beilinson-Bernstein and Brylinski-Kashiwara using every trick in the book: algebraic differential equations "D-modules"; the Riemann-Hilbert correspondence (monodromy of differential equations); the theory of perverse sheaves (algebraic topology of singular varieties); Deligne's theory of weights (arithmetic geometry):

The Kazhdan-Lusztig character formula was proved 1981 by Beilinson-Bernstein and Brylinski-Kashiwara using every trick in the book: algebraic differential equations "D-modules"; the Riemann-Hilbert correspondence (monodromy of differential equations); the theory of perverse sheaves (algebraic topology of singular varieties); Deligne's theory of weights (arithmetic geometry):

"The amazing feature of the proof is that it does not try to solve the problem but just keeps translating it in languages of different areas of mathematics (further and further away from the original problem) until it runs into Deligne's method of weight filtrations which is capable to solve it.

So have a seat; it is going to be a long journey."

– Joseph Bernstein.

Kazhdan-Lusztig polynomials also play an important role in:

Kazhdan-Lusztig polynomials also play an important role in:

i) Lusztig's description of the character table of a finite group of Lie type.

Kazhdan-Lusztig polynomials also play an important role in:

i) Lusztig's description of the character table of a finite group of Lie type.
ii) The algorithm for the determination of the unitary dual of a semi-simple Lie group by Adams, van Leeuwen, Trapa, Vogan (see work of Schmid and Vilonen.)

Kazhdan-Lusztig polynomials also play an important role in:

i) Lusztig's description of the character table of a finite group of Lie type.
ii) The algorithm for the determination of the unitary dual of a semi-simple Lie group by Adams, van Leeuwen, Trapa, Vogan (see work of Schmid and Vilonen.)
iii) Kazhdan-Lusztig polynomials might end up helping us understand the HOMFLYPT polynomial of a link...

Theorem (Elias-W)

The Kazhdan-Lusztig positivity conjecture holds.

Theorem (Elias-W)

The Kazhdan-Lusztig positivity conjecture holds.

Using results of Soergel we obtain an algebraic proof of the Kazhdan-Lusztig character formula.

The idea (going back to Soergel) is to find a vector space which behaves like the intersection cohomology of a Schubert variety, even if this variety does not exist.

(Much like the coinvariant algebra for a non Weyl group should be regarded as the cohomology of a flag variety, even if no such flag variety extists.)

The idea (going back to Soergel) is to find a vector space which behaves like the intersection cohomology of a Schubert variety, even if this variety does not exist.

(Much like the coinvariant algebra for a non Weyl group should be regarded as the cohomology of a flag variety, even if no such flag variety extists.)

The key property of intersection cohomology is the "decomposition theorem": the intersection cohomology of a variety is a summand of the cohomology of any resolution.

For Schubert varieties there exist resolutions of singularities (so called Bott-Samelson resolutions) whose cohomology admit elementary algebraic descriptions.

For Schubert varieties there exist resolutions of singularities (so called Bott-Samelson resolutions) whose cohomology admit elementary algebraic descriptions.

For any word $(s, t, \dots, u)$ in S the cohomology of the corresponding Bott-Samelson variety is:

$$BS(s, t, \dots, u) := R \otimes_{R^s} R \otimes_{R^t} \cdots \otimes_{R^u} \mathbb{R}.$$

For Schubert varieties there exist resolutions of singularities (so called Bott-Samelson resolutions) whose cohomology admit elementary algebraic descriptions.

For any word $(s, t, \dots, u)$ in S the cohomology of the corresponding Bott-Samelson variety is:

$$BS(s, t, \dots, u) := R \otimes_{R^s} R \otimes_{R^t} \cdots \otimes_{R^u} \mathbb{R}.$$

Theorem (Soergel)

If W is a Weyl group then the intersection cohomology of the Schubert variety $\overline{BxB/B}$ is the unique largest indecomposable R-module summand of $BS(s, t, \dots, u)$.

For any Coxeter system one can imitate the definition of the action of a Weyl group on $\mathrm{Lie}\, T$ to define the "geometric representation" $\mathfrak{h}$ of W. Let R denote the regular functions on $\mathfrak{h}$, a polynomial ring over $\mathbb{R}$.

For any Coxeter system one can imitate the definition of the action of a Weyl group on $\mathrm{Lie}\, T$ to define the “geometric representation” $\mathfrak{h}$ of W. Let R denote the regular functions on $\mathfrak{h}$, a polynomial ring over $\mathbb{R}$.

Let H_x denote the largest R-module direct summand of $BS(s, t, \ldots, u)$ where $(s, t, \ldots, u)$ is any reduced expression for x. Soergel shows:

For any Coxeter system one can imitate the definition of the action of a Weyl group on $\mathrm{Lie}\, T$ to define the "geometric representation" $\mathfrak{h}$ of W. Let R denote the regular functions on $\mathfrak{h}$, a polynomial ring over $\mathbb{R}$.

Let H_x denote the largest R-module direct summand of $BS(s, t, \dots, u)$ where $(s, t, \dots, u)$ is any reduced expression for x. Soergel shows:

1. H_x is well-defined up to isomorphism,

For any Coxeter system one can imitate the definition of the action of a Weyl group on $\mathrm{Lie}\, T$ to define the "geometric representation" $\mathfrak{h}$ of W. Let R denote the regular functions on $\mathfrak{h}$, a polynomial ring over $\mathbb{R}$.

Let H_x denote the largest R-module direct summand of $BS(s, t, \ldots, u)$ where $(s, t, \ldots, u)$ is any reduced expression for x. Soergel shows:

1. H_x is well-defined up to isomorphism,
2. H_x has a filtration $\Gamma_{\leqslant x}$ indexed by W and its Bruhat order.

For any Coxeter system one can imitate the definition of the action of a Weyl group on $\mathrm{Lie}\, T$ to define the "geometric representation" $\mathfrak{h}$ of W. Let R denote the regular functions on $\mathfrak{h}$, a polynomial ring over $\mathbb{R}$.

Let H_x denote the largest R-module direct summand of $BS(s, t, \ldots, u)$ where $(s, t, \ldots, u)$ is any reduced expression for x. Soergel shows:

1. H_x is well-defined up to isomorphism,
2. H_x has a filtration $\Gamma_{\leqslant x}$ indexed by W and its Bruhat order.

Examples:

1. If W is a Weyl group then $H_x = IH^*(\overline{BxB/B}; \mathbb{R})$, the intersection cohomology of a Schubert variety.

For any Coxeter system one can imitate the definition of the action of a Weyl group on $\mathrm{Lie}\, T$ to define the "geometric representation" $\mathfrak{h}$ of W. Let R denote the regular functions on $\mathfrak{h}$, a polynomial ring over $\mathbb{R}$.

Let H_x denote the largest R-module direct summand of $BS(s, t, \dots, u)$ where $(s, t, \dots, u)$ is any reduced expression for x. Soergel shows:

1. H_x is well-defined up to isomorphism,
2. H_x has a filtration $\Gamma_{\leqslant x}$ indexed by W and its Bruhat order.

Examples:

1. If W is a Weyl group then $H_x = IH^*(\overline{BxB/B}; \mathbb{R})$, the intersection cohomology of a Schubert variety.
2. If W is finite, with longest element w_0, then H_{w_0} is the coinvariant algebra.

Conjecture (Soergel)

The graded dimension of

$$\Gamma_{\leqslant y} H_x / \Gamma_{<y} H_x$$

is given by the Kazhdan-Lusztig polynomial $h_{y,x}$.

Conjecture (Soergel)

The graded dimension of

$$\Gamma_{\leqslant y} H_x / \Gamma_{<y} H_x$$

is given by the Kazhdan-Lusztig polynomial $h_{y,x}$.

If W is a Weyl group, then Soergel's conjecture follows from the Kazhdan and Lusztig's theorem relating intersection cohomology and Kazhdan-Lusztig polynomials.

Soergel's conjecture obviously implies that Kazhdan-Lusztig polynomials have positive coefficients.

Soergel's conjecture obviously implies that Kazhdan-Lusztig polynomials have positive coefficients.

It is a consequence of work of Soergel from 1990 that his conjecture implies the Kazhdan-Lusztig character formula.

Soergel's conjecture obviously implies that Kazhdan-Lusztig polynomials have positive coefficients.

It is a consequence of work of Soergel from 1990 that his conjecture implies the Kazhdan-Lusztig character formula.

Since then Soergel modules and bimodules have popped up throughout representation theory, and have even been used by Khovanov to construct HOMFLY-PT homology.

A key idea in our proof of Soergel's conjecture is to show that each H_x "looks like the cohomology of a smooth projective variety".

A key idea in our proof of Soergel's conjecture is to show that each H_x "looks like the cohomology of a smooth projective variety".

In 2006 de Cataldo and Migliorini gave Hodge theoretic proofs of the decomposition theorem, a deep result about the topology of algebraic maps between algebraic varieties.

The modules $BS(s, t, \ldots, u)$ are equipped with an intersection form, using a combinatorial analogue of the fundamental class. In a complicated induction over the length of x we show that this intersection form restricts to a non-degenerate "intersection form" on $H_x \subset BS(s, t, \ldots, u)$ and that analogues of the hard Lefschetz theorem and the Hodge-Riemann bilinear relations inductively, following the ideas of de Cataldo and Migliorini.

The modules $BS(s, t, \ldots, u)$ are equipped with an intersection form, using a combinatorial analogue of the fundamental class. In a complicated induction over the length of x we show that this intersection form restricts to a non-degenerate "intersection form" on $H_x \subset BS(s, t, \ldots, u)$ and that analogues of the hard Lefschetz theorem and the Hodge-Riemann bilinear relations inductively, following the ideas of de Cataldo and Migliorini.

As is the case for de Cataldo and Migliorini, one needs the whole package of statements for the induction to work.

Theorem (Elias-W)

For any $\rho \in V^*$ in the interior of the fundamental alcove:

1. *(Hard-Lefschetz theorem) left multiplication by ρ^i gives an isomorphism*
$$(H_x)^{\ell(x)-i} \to (H_x)^{\ell(x)+i}$$
2. *(Hodge-Riemann bilinear relations) The restriction of the form $(\alpha, \beta) := \langle \alpha, \rho^i \beta \rangle$ to the kernel of ρ^{i+1} in $(H_x)^{\ell(x)-i}$ is definite.*

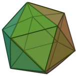
These results are new even for the coinvariant algebra of a finite non-crystallographic reflection group.

Some examples of Betti numbers . . .

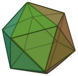
Some examples of Betti numbers ...

$I_2(5)$: symmetries of the pentagon:

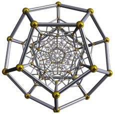
1 2 2 2 2 1

H_3: symmetries of  :

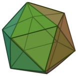
1 3 5 7 9 11 12 12 12 12 11 9 7 5 3 1

H_3: symmetries of  :

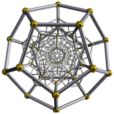
1 3 5 7 9 11 12 12 12 12 11 9 7 5 3 1

H_4: symmetries of 120 cell  $\subset \mathbb{R}^4$:

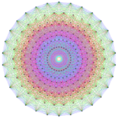
1 4 9 16 25 36 49 64 81 100 121 144 168 192 216 240 264 288 312 336 359 380 399 416 431 444 455 464 471 476 478 476 471 464 455 444 431 416 399 380 359 336 312 288 264 240 216 192 168 144 121 100 81 64 49 36 25 16 9 4 1

H_3: symmetries of  :

1 3 5 7 9 11 12 12 12 12 11 9 7 5 3 1

H_4: symmetries of 120 cell  $\subset \mathbb{R}^4$:

1 4 9 16 25 36 49 64 81 100 121 144 168 192 216 240 264 288 312 336 359 380 399 416 431 444 455 464 471 476 478 476 471 464 455 444 431 416 399 380 359 336 312 288 264 240 216 192 168 144 121 100 81 64 49 36 25 16 9 4 1

E_8: 

1 8 35 112 294 672 1386 2640 4718 8000 12978 20272 30645 45016 64470 90264 123829 166768 220849 287992 370250 469784 588833 729680 894613 1085880 1305640 1555912 1838523 2155056 2506798 2894688 3319268 3780640 4278429 4811752 5379194 5978792 6608029 7263840 7942628 8640288 9352240 10073472 10798593 11521896 12237428 12939064 13620586 14275768 14898464 15482696 16022740 16513208 16949127 17326016 17639957 17887656 18066494 18174568 18210722 18174568 18066494 17887656 17639957 17326016 16949127 16513208 16022740 15482696 14898464 14275768 13620586 12939064 12237428 11521896 10798593 10073472 9352240 8640288 7942628 7263840 6608029 5978792 5379194 4811752 4278429 3780640 3319268 2894688 2506798 2155056 1838523 1555912 1305640 1085880 894613 729680 588833 469784 370250 287992 220849 166768 123829 90264 64470 45016 30645 20272 12978 8000 4718 2640 1386 672 294 112 35 8 1

For non-crystallographic W the spaces H_x are in general not the cohomology of any projective variety. The integral lattice in cohomology is replaced by a lattice over some ring of integers.

For non-crystallographic W the spaces H_x are in general not the cohomology of any projective variety. The integral lattice in cohomology is replaced by a lattice over some ring of integers.

Example: If $m_{st} = 5$ then the role of the integral lattice is replaced by $\mathbb{Z}[\phi]$, where ϕ denotes the golden ratio!

I will finish with two questions:

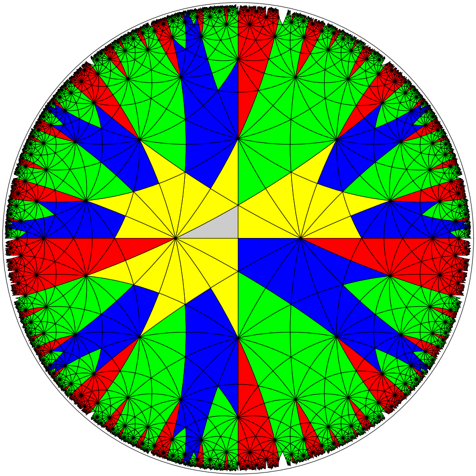
I will finish with two questions:

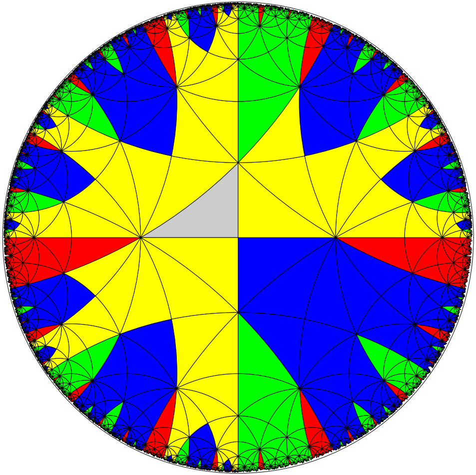
i) Is there any geometric interpretation of these spaces? (One can ask a similar question for the intersection cohomology of non-rational polytopes.)

I will finish with two questions:

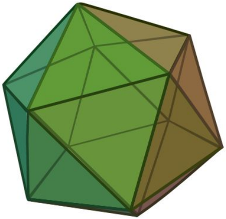
i) Is there any geometric interpretation of these spaces? (One can ask a similar question for the intersection cohomology of non-rational polytopes.)

ii) What does Kazhdan-Lusztig theory mean in the non-crystallographic case?





For more images of two-sided cells in hyperbolic groups see Paul Gunnell's web page.



Thanks for listening!