Exercise Prove ③ ⇒ ②, ② ⇒ ④ and ④ ⇒ ① from the main Theorem of lecture 1.

$\boxed{③ \Rightarrow ②}$ Assume that $|N(w)| < k$, where $s_1 s_2 \cdots s_k$ is a REX of $w \in W$. Then $\exists\ i<j$ s.t.

$$s_1 s_2 \cdots s_i s_{i-1} \cdots s_1 = s_1 s_2 \cdots s_j s_{j-1} \cdots s_1$$

$$\Longrightarrow \quad s_{i+1} s_{i+2} \cdots s_j s_{j-1} \cdots s_i = 1 \underset{\substack{s^2=1 \\ \forall e}}{\Longrightarrow} \underbrace{s_i s_{i+1} \cdots s_j}_{\substack{j-i+1 \\ \text{letters}}} = \underbrace{s_{i+1} s_{i+2} \cdots s_{j-1}}_{\substack{j-i-1 \\ \text{letters}}}$$

this contradicts the fact that $s_1 s_2 \cdots s_k$ is reduced.

Hence $N(w) = \{s_1\} \dot\cup \{s_1 s_2 s_1\} \dot\cup \dots \dot\cup \{s_1 s_2 \cdots s_k s_{k-1} \cdots s_1\}$

and $|N(w)| = k = \ell(w)$.

We have $N(sw) = \{s\} \dotplus sN(w)s \Longrightarrow \underset{\ell(sw)}{\underset{\shortparallel}{|N(sw)|}} = \underset{\ell(w)\pm 1}{\underset{\shortparallel}{|N(w)| \pm 1}}$

Hence $\ell(sw) \neq \ell(w)$, and $\ell(sw) \leq \ell(w) \Rightarrow \ell(sw) < \ell(w)$.

If $\ell(sw) < \ell(w)$, then $s \in sN(w)s \Longrightarrow \exists\ i$ s.t.

$$s = s_1 s_2 \cdots s_i s_{i-1} \cdots s_1 \Longrightarrow s s_1 s_2 \cdots s_i = s_1 s_2 \cdots s_{i-1}$$

In particular, $s s_1 s_2 \cdots s_k = s_1 \cdots \hat{s_i} \cdots s_k$, which is what we wanted to show.

$\boxed{② \Rightarrow ④}$ ② implies that $\ell(sw) \neq \ell(w)$

Let $f : S^* \to \Pi$ be as in ④. We show by induction on $\ell(w)$ that f is constant on reduced expressions. Let $s_1 s_2 \cdots s_k$, ~~be~~

$s_1's_2'\cdots s_k'$ be two REX of w which differ by f, we will find a contradiction. ② $\Rightarrow \exists\, i$ s.t. $s_1's_1s_2\cdots s_k = s_1s_2\cdots\hat{s_i}\cdots s_k$.

If $i<k$, then by induction then since $s_1s_2\cdots s_{k-1} = s_1's_1\cdots\hat{s_i}\cdots s_{k-1}$ by induction we have $f(s_1s_2\cdots s_{k-1}) = f(s_1's_1\cdots\hat{s_i}\cdots s_{k-1})$, hence $f(s_1s_2\cdots s_k) = f(s_1's_1\cdots\hat{s_i}\cdots s_k)$ since f is a morphism of monoids. Similarly, $f(s_2'\cdots s_k') = f(s_1's_2\cdots\hat{s_i}\cdots s_k)$ by induction, implying that $f(s_1's_2'\cdots s_k') = f(s_1's_1\cdots\hat{s_i}\cdots s_k) = f(s_1s_2\cdots s_k)$, contradicting our assumption. Hence we can assume that $i=k$ and $s_1's_1\cdots s_{k-1}$ is a REX s.t. $f(s_1's_1\cdots s_{k-1}) \neq f(s_1s_2\cdots s_k)$.

Arguing in the same way, but starting from the REX $s_1's_1\cdots s_{k-1}$ and $s_1s_2\cdots s_k$, we get that $s_1s_1's_1\cdots s_{k-2}$ and $s_1's_1\cdots s_{k-1}$ are two REX for w s.t. $f(s_1s_1's_1\cdots s_{k-2}) \neq f(s_1's_1\cdots s_{k-1})$.

We go on until we get two REX (which differ by f) $s_1s_1's_1\cdots$ and $s_1's_1s_1'\cdots$ of w with all letters in $\{s_1, s_1'\}$. Since the expressions are reduced, we cannot have $k > o(s_1s_1')$, and $k < o(s_1s_1')$ would not be possible since $(s_1s_1')^k = 1$. Hence $k = o(s_1s_1')$, meaning that our REX define a braid relation, hence that

$$f(s_1s_1's_1\cdots) = f(s_1's_1s_1'\cdots),$$ a contradiction $\square$.

$\boxed{④ \Rightarrow ①}$ We define $g: W \to G$ by $g(w) = f(s_1s_2\cdots s_k)$ where $s_1s_2\cdots s_k$ is a REX of w. The map is well-defined by ④. To show that f factors as $S^* \to W \xrightarrow{g} G$, we have to show that for any expression $s_1s_2\cdots s_k$ of $w \in W$ (not necessarily reduced), we have $f(s_1s_2\cdots s_k) = g(w)$.

This follows, by induction on $\ell(w)$, if we can prove that

$$f(s)\,g(w) = g(sw) \quad \forall s \in S,\ v \in \widetilde{W}. \qquad (*)$$

If $\ell(sw) > \ell(w)$ then $(*)$ is an immediate consequence of the definition of g. If $\ell(sw) < \ell(w)$, then by def. of g we have $f(s)\,g(sw) = g(w)$, and using $f(s)^2 = 1$ we rewrite it as $g(sw) = f(s)\,g(w)$. The case where $\ell(sw) = \ell(w)$ cannot appear, by assumption.

$\square$
