Exercise Let (W,S) be a Coxeter system, $I \subseteq S$, $W_I := \langle s \mid s \in I \rangle \subseteq W$. Let $\ell_I : W_I \to \mathbb{Z}_{\geq 0}$ be the length function wrt I

① Show that if $w \in W_I$, then $\ell(w) = \ell_I(w)$.

② Show that (W_I, I) is a Coxeter system

③ Show that if $s_1 s_2 \cdots s_k$ is a REX of w in W, and $w \in W_I$, then $s_i \in I$ $\forall i = 1, \dots, k$.

Solution ① Let $s_1 s_2 \cdots s_k$ be a word for w with $s_i \in I$ $\forall i = 1, \dots, k$. In the lecture we have seen that every expression for an element of a Coxeter group has a subexpression which is a REX (apply the exchange condition repeatedly!). Since W is a Coxeter group, there must be a subword $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ of $s_1 s_2 \cdots s_k$ which is a REX for w in W. But since $s_i \in I$ $\forall i$, this REX has all its letters in I $\Rightarrow$ $\ell(w) \geq \ell_I(w)$.
Since $I \subseteq S$, we have $\ell_I(w) \geq \ell(w)$ $\Rightarrow$ $\ell(w) = \ell_I(w)$

② We show that (W_I, I) satisfies the exchange condition
Let $s_1 s_2 \cdots s_k$ be a REX for $w \in W_I$ (hence with $s_i \in I$ $\forall i$) and $s \in I$ s.t. $\ell_I(sw) \leq \ell_I(w)$. By ① we have that $s_1 s_2 \cdots s_k$ is a REX of w viewed as element of W, and $\ell(sw) = \ell_I(sw)$. We can therefore apply the exchange

condition in $\widetilde{W}$ to conclude that $\exists\, i$ s.t.

$$sw = s_1 s_2 \cdots \hat{s}_i \cdots s_k .$$ Hence (W_I, I)

satisfies the exchange condition $\underset{\text{Theorem 1}}{\Longrightarrow}$ (W_I, I) is a Coxeter system.

③ By ①, there is a REX $s_1' s_2' \cdots s_k'$ of w with $s_i' \in I$ for all i. Matsumoto's lemma implies that $s_1 s_2 \cdots s_k$ and $s_1' s_2' \cdots s_k'$ can be transformed one into the other using braid relations. But a braid move between two words does not change the letters in the word

$$\Rightarrow \quad \{s_1, s_2, \dots, s_k\} = \{s_1', s_2', \dots, s_k'\} \subseteq I$$

$\square$.