Solution to exercise 6

(1) By Matsumoto's property, every two REX of $w \in \widetilde{W}$ are related by a sequence of braid moves. But there are no braid relations $\Rightarrow$ the braid graph of every element is a singleton.

(2) Let $w \in \widetilde{W}$ with $\ell(w)$ odd. Let $s_1 s_2 \cdots s_{2k+1}$ be its redex. Then either $\{s_i = s$ for all even i and $s_i = t$ for all odd $i\}$ or $\{s_i = t$ for even i and $s_i = s$ for odd $i\}$

$$\Rightarrow \quad w = s_1 s_2 \cdots\cdots s_k \, s_{k+1} \, s_{k+2} \cdots s_{2k+1}$$

but by the observation above we have

$$s_1 s_2 \cdots s_k = s_{2k+1} s_{2k} \cdots s_{k+2} = (s_{k+2} s_{k+3} \cdots s_{2k+1})^{-1}$$

$$= \quad w = (s_1 s_2 \cdots s_k) \, s_{k+1} \, (s_1 s_2 \cdots s_k)^{-1} \in R.$$

Conversely, let $w = xqx^{-1}$, $q \in \{s,t\}$

Note: $\det(\sigma_s) = -1 = \det(\sigma_t)$ since σ_s and σ_t are reflections. Hence $\det(\sigma_w) = -1$.

If $w = s_1 s_2 \cdots s_{2k}$, $s_i \in \{s,t\}$, then $\det(\sigma_w) = 1$, contradiction. Hence $\ell(w)$ is odd.

(3) We have $B(e_s, e_t) = -1$

$$B(e_s, e_s) = 1 = B(e_t, e_t).$$

$\Rightarrow$ $B(e_s, e_s + e_t) = 0 = B(e_t, e_s + e_t)$

$\Rightarrow$ $H_s = H_t = \mathbb{R}(e_s + e_t). =: H$

$\Rightarrow$ $s(H_s) = s(H_t) = \mathbb{R}(e_s + e_t) = t(H_t) = t(H_s)$

$\Rightarrow$ $w(H) = H$ $\forall$ $w \in W$, hence every reflection has H as reflecting hyperplane.

(4) let $\alpha_n = n e_s + (n+1) e_t \quad , n \in \mathbb{Z}$

$\beta_n = (n+1) e_s + n e_t \quad , n \in \mathbb{Z}$

we have $\sigma_s(\alpha_n) = -n e_s + (n+1)(e_t + 2e_s) = (n+1) e_t + (n+2) e_s$

$= \beta_{n+1}$

$\sigma_t(\alpha_n) = n e_s + (n-1) e_t = \beta_{n-1}$

and exchanging the roles of s and t:

$\sigma_s(\beta_n) = \alpha_{n-1} \qquad \sigma_t(\beta_n) = \alpha_{n+1}$

Since $\alpha_0 = e_t$, $\beta_0 = e_s$, we get

$\Phi = \{\alpha_n, n \in \mathbb{Z}\} \cup \{\beta_n, n \in \mathbb{Z}\}$.

(5) $B(e_s, e_s + e_t) = 0 = B(e_t, e_s + e_t)$. Since $\{e_s, e_t\}$ is a basis of V it follows that $B(v, e_s + e_t) = 0 \ \forall v \in V$

$\Rightarrow$ B is degenerate

(6) $e_s + e_t$ is fixed by s and t $\Rightarrow L = \mathbb{R}(e_s + e_t)$ is fixed by W $\Rightarrow$ V is not irreducible