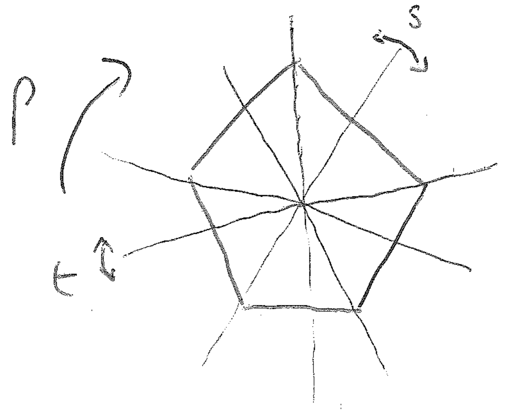
BRAID GROUPS

Basic example of Coxeter group : dihedral groups = "groups of symmetries of regular polygons"

e.g. pentagon



- 5 reflections (along 5 axes)
- one rotation of order 5 (in fact, 4 rotations)

$$D_{10} = \langle \underset{\swarrow \atop \text{rotation}}{\rho}, \underset{\searrow \atop \text{reflection}}{s} \mid \rho^5 = 1 = s^2, \; s\rho s = \rho^{-1} \rangle$$

<u>Observation</u> This group can also be generated by 2 reflections $\quad st = \rho \quad (st)^5 = 1$

$$D_{10} \cong \langle s, t \mid s^2 = 1 = t^2, \; \underbrace{ststs = tstst}_{\text{"braid relation"}} \rangle$$

[More generally : if n is a regular n-gon

$$D_{2n} \cong \langle s, t \mid s^2 = 1 = t^2, \; \underbrace{stst\cdots}_{n \text{ factors}} = \underbrace{tsts\cdots}_{n \text{ factors}} \rangle$$
]

The braid group $\mathcal{B}r(D_{2n})$ is defined by

$$\mathcal{B}r(D_{2n}) = \langle \underline{s}, \underline{t} \mid \underbrace{\underline{s}\,\underline{t}\,\underline{s}\,\underline{t}\cdots}_{n \text{ factors}} = \underbrace{\underline{t}\,\underline{s}\,\underline{t}\,\underline{s}\cdots}_{n \text{ factors}} \rangle$$

Aim
- introduce Coxeter groups (examples include dihedral groups) and their braid groups.
- understanding groups defined by generators and relations can be hard! A tool: (faithful) representations

2 types of representations:
- linear: $G \rightsquigarrow GL(V)$
- categorical: $G \rightsquigarrow$ Auto-equiv. of $\mathcal{C}$ = triangulated category

deduce properties of G. (and important conjectures)

example: Word problem in a gp G defined by generators and relations: is there an algorithm to determine whether two words in the generators represent the same element of G? In $G \subseteq GL(V)$, $(\dim V < \infty)$ the answer is yes! Just compare two products of matrices ...

Coxeter groups: have a canonical linear representation which is faithful (Tits) (next week!)

Braid groups: some of them are known to be linear; they conjecturally admit a faithful categorical representation! (Rouquier)

1 Coxeter groups

Let S be a finite set. Let $(m_{st})_{s,t\in S}$ be a matrix such that

1. $m_{ss} = 1 \quad \forall s \in S$
2. $m_{st} = m_{ts} \in \{2,3,4,\dots\} \cup \{\infty\}$ if $s \neq t$

<u>Coxeter matrix</u>

It defines a <u>Coxeter group</u> W (or a <u>Coxeter system</u> (W,S)) by

$$W = \left\langle s \in S \;\middle|\; (st)^{m_{st}} = 1 \quad \forall s,t \in S \right\rangle$$

$$\|$$

$$\left\langle s \in S \;\middle|\; s^2 = 1 \ \forall s \in S,\ \underbrace{sts\cdots}_{m_{st}} = \underbrace{tst\cdots}_{=m_{ts} \text{ factors}} \right\rangle$$

$(s \neq t)$

<u>braid relations</u>

<u>can be shown</u>:

- generators of W are pairwise distinct
- order of $st = m_{s,t}$

$|S|$ is the <u>rank</u> of W

<u>Remark</u>:

- $m_{st} = m_{ts} = 2$ means $st = ts$, i.e., s and t commute to each other
- Dihedral group of order $2n$ is a Cox. gp with Cox. matrix $\begin{pmatrix} 1 & n \\ n & 1 \end{pmatrix}$

<u>Fundamental example</u>: symmetric groups!

$W = \mathfrak{S}_n$, $S = \{(i,i+1)\}_{i=1,\dots,n-1}$

braid relations! $\begin{cases} s_i s_{i+1} s_i = (i,i+2) = s_{i+1} s_i s_{i+1} & i=1,\dots,n-2 \\ s_i s_j = s_j s_i & |i-j|>1 \end{cases}$

since S generates W we conclude that there is a surjective homomorphism

$$W_n := \left\langle s_1,\dots,s_{n-1} \;\middle|\; \begin{matrix} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \\ i=1,\dots,n-2 \end{matrix}, \begin{matrix} s_i s_j = s_j s_i \\ |i-j|>1 \end{matrix}, \begin{matrix} s_i^2 = 1 \\ \forall i \end{matrix} \right\rangle$$

$$\downarrow$$

$$\mathfrak{S}_n$$

W_n is a Coxeter gp !!

<u>Exercise</u> Show that it is an isomorphism! (<u>hint</u>: show by induction that $|W_n| \leq n!$ To this end, consider the subgroup P_n of W_n generated by $s_1,\dots,s_{n-2}$ and show that there are at most n left cosets of P_n in W_n)

<u>Exercise</u> Let $V = \bigoplus_{i=1}^{n} \mathbb{R} e_i$.

1. Show that $\mathfrak{S}_n$ acts on V via $w(e_i) = e_{w(i)}$.
   Is this representation faithful?
2. Check that s_i acts by a reflection, i.e., an element of $GL(V)$ of order two, fixing a hyperplane.

<u>CONCLUSION</u> $(\mathfrak{S}_n, S)$ — is a Coxeter system (will be an easier characterization later!)
— can be realized as a group generated by reflections.

<u>Natural question</u>: is the word problem in a Coxeter group solvable?

The <u>length</u> $\ell(w)$ of $w \in W$ is the minimal $k \geq 0$ s.t. $w = s_1 s_2 \cdots s_k$, $s_i \in S$. In that case $s_1 s_2 \cdots s_k$ is a <u>reduced expression</u> of w. (short: redex)

We wish to characterize Coxeter systems in terms of reduced expressions of their elements.

Let $R = \{ wsw^{-1} \mid s \in S, w \in W \}$ ("<u>reflections</u>")

<u>Theorem 1</u> Let W be a group generated by a finite set S of involutions. TFAE

① (W, S) is a Coxeter system.

② (Exchange lemma) If $s_1 s_2 \cdots s_k$ is a redex for $w \in W$ and $s \in S$ is s.t. $\ell(sw) \leq \ell(w)$, $\exists i$ s.t. $sw = s_1 s_2 \cdots \hat{s}_i \cdots s_k$

omission of letter s_i

③ $\exists$ a function $N : W \longrightarrow \mathcal{P}(R)$ s.t.

(a) $N(s) = \{s\}$ $\forall s \in S$

(b) $\forall x, y \in W$, $N(xy) = N(x) \dotplus x N(y) x^{-1}$

symmetric difference

④ (Matsumoto's lemma) W satisfies $\ell(sw) \neq \ell(w)$ for $s \in S$, $w \in W$, and any two redex of an elt $w \in W$ can be transformed one into the other using a sequence of <u>braid relations</u> ("braid graph of an element is connected!")

$\underbrace{st\cdots}_{o(st) \text{ factors}} = \underbrace{ts\cdots}_{o(st) \text{ fact.}}$

A few comments before the proof !

- ③ may look strange. Later we will see a geometric interpretation of the function N. In fact ③ can be extremely useful to check that a given group is a Coxeter group !

- Example of braid graph : $(1,4) \in \mathfrak{S}_4$

(Word pblem)

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & s_1s_3s_2s_1s_3 \arrow[dr, no head] & & \\
s_3s_2s_1s_2s_3 \arrow[r, no head] & s_3s_1s_2s_1s_3 \arrow[ur, no head] \arrow[dr, no head] & & s_1s_3s_2s_3s_1 \arrow[r, no head] & s_1s_2s_3s_2s_1 \\
 & & s_3s_1s_2s_3s_1 \arrow[ur, no head] & &
\end{tikzcd}$$

④ allows one to get all the redex of an elt starting from a given one. Hence: makes it possible to say if two redex represent the same element of W!

It is possible to inductively get a redex of $w \in W$ starting from any expression $s_1 s_2 \cdots s_k$ of w, as follows: assume that we already know that $s_i s_{i+1} \cdots s_k$ is reduced and we want to know if $s_{i-1} s_i s_{i+1} \cdots s_k$ is reduced or not.

if $s_{i-1} s_i s_{i+1} \cdots s_k$ is not reduced, then by ② $\exists\, \ell$ s.t.

$$s_i s_{i+1} \cdots \hat{s}_\ell \cdots s_k = s_{i-1} s_i s_{i+1} \cdots s_k$$

$$\Longrightarrow \quad s_{i-1} s_i \cdots \hat{s}_\ell \cdots s_k = s_i s_{i+1} \cdots s_k \Longrightarrow \underline{s_i s_{i+1} \cdots s_k}$$

$s_{i-1}^2 = 1$ has a redex beginning by s_{i-1}

So to know if $s_{i-1} s_i \cdots s_k$ is reduced or not, compute the braid graph of $s_i s_{i+1} \cdots s_k$

- if one finds a redex beginning by s_{i-1} in the graph, then $s_{i-1}s_i \cdots s_k$ is not reduced
  ($\overset{\parallel}{s_i \cdots \hat{s}_\ell \cdots s_k}$ which is reduced)

- if one finds no redex beginning by s_{i-1}, then $s_{i-1}s_i \cdots s_k$ is reduced.

$\Longrightarrow$ can inductively get a redex. To summarize:

- starting from a word in S, we can find a redex of the elt of $\overline{W}$ which it represents
- having a redex of $w \in \overline{W}$, we can find all redex of w by computing the braid graph.

$\Longrightarrow$ THE WORD PROBLEM IN A COXETER GROUP IS SOLVABLE

__Proof of the Theorem__ ① $\Rightarrow$ ③ Define $N : \underset{\text{words in } S}{S^*} \longrightarrow \mathcal{P}(R)$

by $N(s_1 s_2 \cdots s_k) = \{s_1\} \dot{+} \{s_1 s_2 s_1\} \dot{+} \cdots \dot{+} \{s_1 s_2 \cdots s_{k-1} s_k s_{k-1} \cdots s_2 s_1\}$

~~using the fact that symmetric difference~~ it suffices to check that $N(ss) = \emptyset$ and $N(\underbrace{sts\cdots}_{m_{st}}) = N(\underbrace{tst\cdots}_{m_{st}})$ since it is clear that $N(xy) = N(x) \dot{+} xN(y)x^{-1}$. But

⊛ $N(ss) = \{s\} \dot{+} \{\overset{\overset{s}{\parallel}}{sss}\} = \emptyset$

⊛ $N(\underbrace{st\cdots}_{m_{st}}) = \{s, sts, \ldots\ldots, tst, t\}$ $\quad$ ($t \overset{?}{=} s$)

We leave the rest of the proof as an <u>exercise</u>:

3 $\Rightarrow$ 2 : (i) First show that if $\overset{w=}{s_1 s_2 \cdots s_k}$ is reduced, then $|N(w)| = k$. (hence $|N(w)| = \ell(w)$!)

(ii) Then show that $\ell(sw) \leq \ell(w) \Rightarrow \ell(sw) < \ell(w)$

(iii) Conclude

2 $\Rightarrow$ 4 Note that it suffices to show that:

> If M is a monoid and $f: S^* \longrightarrow M$ is s.t. $f(\underbrace{st\cdots}_{m_{st}}) = f(\underbrace{ts\cdots}_{m_{ts}})$ $\forall\, s,t \in S$, then f is constant on reduced expressions of elts in W.

Prove it by induction on $\ell(w)$: take two redex $s_1 s_2 \cdots s_k$, $s_1' s_2' \cdots s_k'$ of w which differ by f. Exch. cd^n $\Rightarrow \exists\, i$ s.t.

$$s_1' s_1 \cdots s_k = s_1 \cdots \hat{s}_i \cdots s_k$$

show that one can assume that $i = k$

$\Rightarrow$ $s_1' s_1 \cdots s_{k-1}$ is a redex of w s.t.

$$f(s_1' s_1 \cdots s_{k-1}) \neq f(s_1 \cdots s_k)$$

repeat the procedure until you get two redex ~~starting~~ of the form ~~of~~ $\underbrace{s_1 s_1' s_1 \cdots}_{m}$ and $\underbrace{s_1' s_1 s_1' \cdots}_{m}$. Conclude.

4 $\Rightarrow$ 1 Suffices to show:

> If G is a group and $f: S^* \longrightarrow G$ a morphism of monoids s.t. $f(\underbrace{st\cdots}_{m_{st}}) = f(\underbrace{ts\cdots}_{m_{ts}})$, then f factors through a morphism $f: W \longrightarrow G$

Show it by defining $g: W \to G$ by $g(w) = f(s_1 s_2 \cdots s_k)$, where $s_1 s_2 \cdots s_k$ is a REX of w.

$\square$

<u>Exercise</u> (i) Show that $\mathfrak{S}_n$ is a Coxeter group by showing that $N(w) := \{ (i,j) \mid i<j,\ w^{-1}(i) > w^{-1}(j) \}$ satisfies the assumptions of Thm 1 (3)

(ii) Show that there is a unique el^t of maximal length in $\mathfrak{S}_n$.

<u>Exercise</u> Let (W, S) be a Coxeter system. Let $I \subseteq S$. Let $W_I := \langle s \mid s \in I \rangle \subseteq W$. Let $\ell_I : W_I \to \mathbb{Z}_{\geq 0}$ be the length function wrt to I.

(1) Show that if $w \in W_I$, then $\ell(w) = \ell(w_I)$

(2) Show that (W_I, I) is a Coxeter system. It is called a <u>standard parabolic subgroup</u> of W.

(3) Show that if $w \in W_I$ and $s_1 s_2 \cdots s_k$ is a REX of w in W, then $s_i \in I$ $\forall i = 1, \dots, k$