2. REFLECTION GROUPS AND COXETER GROUPS

Let V be a f.d. vsp $/\mathbb{R}$. An element $s \in GL(V)$ is a <u>reflection</u> if $H_s := \operatorname{Ker}(s - \mathrm{Id}_V)$ is a hyperplane and $s^2 = 1$ ($\Leftrightarrow$ s has a unique eigenvalue not equal to 1, which has to be -1).

A ref^n s has the form $s(x) = x - \check{\alpha}_s(x)\alpha_s$ where

- $\check{\alpha}_s \in V^*$ is a linear form of kernel H_s
- α_s is an eigenvector of s of eigenvalue -1, s.t. $\check{\alpha}_s(\alpha_s) = 2$.

Let $W \subseteq GL(V)$ be a subgroup, $\mathrm{Ref}(W) := \{ s \in W \mid s \text{ is a reflection} \}$. Then W is a <u>(real) reflection group</u> if $W = \langle \mathrm{Ref}(W) \rangle$. (⚠ depends on V!)

<u>Note</u>: reflections are stable by W-conjugacy

We call α_s (resp. $\check{\alpha}_s$) a <u>root</u> (resp. <u>coroot</u>) attached to s.

We have already seen that $\mathfrak{S}_n$ and D_{2n} are reflection groups.

~~In fact: every (finite) Coxeter group~~

A reflection group is <u>irreducible</u> if V is an irreducible representation of W. If W is finite: V is a direct sum of irred. rep^n (and W is the direct product of the corresp. subgroups).

<u>Aim</u>: show that every finite reflection group is a Coxeter group!

Let $\overline{W} \subseteq GL(V)$, $V = \mathbb{R}^n$, be a finite reflection group.
Let $\mathcal{A}_{\overline{W}} := \{H_s\}_{s \in \overline{W}}$ be the corresponding ($\overline{W}$-invariant) set of reflecting hyperplanes.

<u>Observation</u>: For $H \in \mathcal{A}_{\overline{W}}$, $\exists!$ reflection $s_H \in W$ s.t.
$$H_{s_H} = H.$$

<u>Proof</u>: Let t be a ref. of hyperplane H. Then $t \in C_W(H)$. Since $\overline{W}$ is finite, H has a $C_W(H)$-stable complement, which is a line L. Let $x \in C_W(H)$. Since $\overline{W}$ is finite, and V is real, x acts on L by ± 1. If it acts on 1 then $x = id$. if not then $x = s_H$. Hence $t = s_H$. $\square$

A key lemma to show that FRG are CG is the following characterization of CG:

<u>Lemma 1</u> Let $\overline{W}$ be a group generated by a finite set S of involutions. Let $\{D_s\}_{s \in S}$ be a set of subsets of $\overline{W}$ s.t.

(i) $1 \in D_s$

(ii) $D_s \cap s D_s = \emptyset$

(iii) If for $s, s' \in S$ we have $w \in D_s$, $ws' \notin D_s$, then $ws' = sw$.

Then (✱) (W, S) is a Coxeter system, and $D_s = \{w \in \overline{W} \mid \ell(sw) > \ell(w)\}$

<u>Proof</u>: We show that (W, S) satisfies the exchange condition.

(✱ <u>Exercise 1</u>: Show the converse!)

Assume that $w \notin D_S$, and let $w = s_1 s_2 \cdots s_k$ be a REX of w. Let i be minimal s.t. $s_1 s_2 \cdots s_i \notin D_S$ ($i > 0$ by (i))

$$s_1 \cdots s_{i-1} \in D_S, \quad s_1 \cdots s_{i-1} s_i \notin D_S \overset{(iii)}{\Longrightarrow} s s_1 \cdots s_{i-1} = s_1 \cdots s_{i-1} s_i$$

$$\Longrightarrow \quad sw = s_1 \cdots \hat{s_i} \cdots s_k \text{ , in p'ticular } \ell(sw) < \ell(w)$$

The exchange condition is hence satisfied in this case.
To conclude it suffices to show that if $w \in D_S$, then $\ell(w) < \ell(sw)$. This holds since by (ii) we have $sw \notin D_S$, hence $\ell(w) < \ell(sw)$ by the first case

$\square$.

<u>Chamber geometry</u> Let A be a finite collection of hyperplanes. The connected components of $V - \bigcup_{H \in A} H$ are called the <u>chambers</u> (or <u>alcoves</u>) of the arrangement A. The <u>walls</u> of a chamber C are those $H \in A$ s.t. $H \cap \overline{C}$ contains a nonempty open set of H.

<u>Exercise 2</u> (1) Let C be a chamber. Let $H \in A$. Show that H is a wall of C iff $\exists$ a point in $H \cap \overline{C}$ which lies on no other hyperplane of A.

(2) Deduce the following property: if $x \in C$, $y \notin C$, then the line segment $[x, y]$ intersects at least one wall nontrivially.

(3) Prove that $C = \bigcap_{\substack{H \in A \\ H \text{ is a wall of } C}} H_C^+$, where H_C^+ is the open H-halfspace containing C.

<u>Theorem</u> Let W be a finite reflection group in a finite-dimensional vector space V. Let C be a chamber with set of walls $\mathcal{H}$, let $S := \{ s_H \mid H \in \mathcal{H} \}$. Then (W, S) is a Coxeter system.

<u>Proof</u> Let $W' := \langle s, s \in S \rangle$. We claim that for any $x \in V$, $\exists$ an element of the W'-orbit of x in $\overline{C}$. To this end we need the following lemma (left as exercise) <u>Exercise 3</u>:

<u>Lemma</u> Let $G \leq GL(V)$ be a finite subgroup, where V is a f.d. real vector space. There exists a G-invariant scalar product on V.

(in particular, reflections in G are orthogonal ref^n wrt this scalar product, i.e., the (-1)-eigenspace of s is equal to $H_s^\perp$)

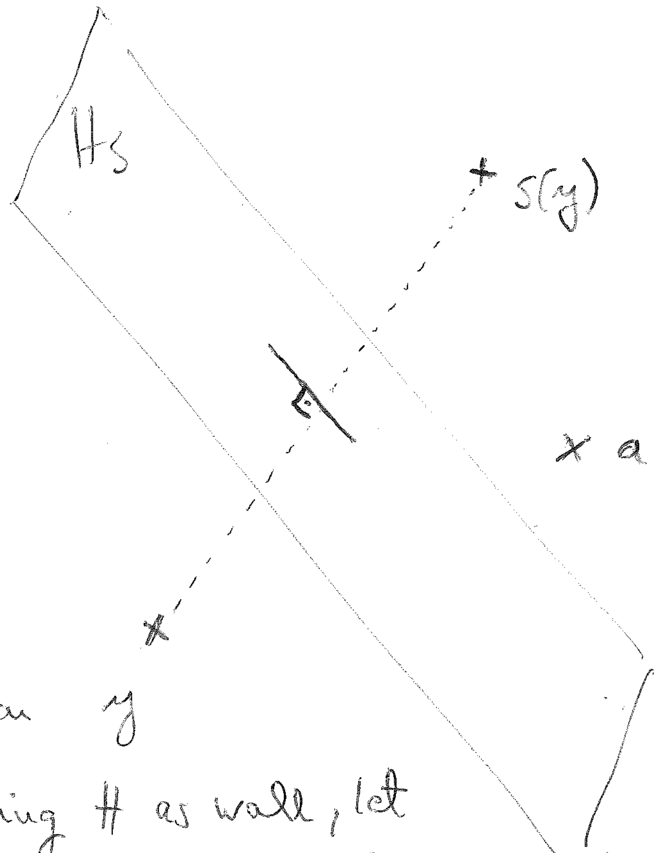
To show the claim, choose $a \in C$ and let y be in the W'-orbit of x s.t. y is at minimal distance of a. If $y \notin \overline{C}$, $\exists$ a hyperplane $H_s \in \mathcal{H}$ separating a and y, hence $s_{H_s}(y)$ is closer to a than y (reflections are orthogonal!)

it follows that any chamber is in the W'-orbit of C:

$\exists\, w \in W' : w(\bar{C'}) \cap \bar{C} \neq \emptyset$

$\Rightarrow w(C') = C.$

(in particular, the action on chambers is transitive)



Let $s_H \in W$ be any reflection

Let C' be a chamber having H as wall, let $w \in W'$ s.t. $w(C') = C$. Since w acts as an homeo, $w(H)$ is a wall of C. By ~~Lemma~~ the observation above, we have that $s_{w(H)} = w s_H w^{-1} \in S \Rightarrow s_H \in W'$. Hence W' contains every reflection in $W \Rightarrow W' = W$.

We show the Thm by applying Lemma 1 in the following setting: for $s \in S$, let

$$\boxed{D_s := \{ w \in W \mid C \text{ and } w(C) \text{ are on the same side of } H_s \}}$$

(i) and (ii) in Lemma 1 are trivial. To show: if $w \in D_s$, $ws' \notin D_s$, then $ws' = sw$. By assumpⁿ we have that $w(C)$ and $ws'(C)$ are on different sides of $H_s \Rightarrow C$ and $s'(C)$ are on different sides of $w^{-1}(H_s)$. But $H_{s'}$ is the only wall separating C from $s'(C) \Rightarrow s' = w^{-1}sw \Rightarrow ws' = sw$.

$\square$

Exercise 4 Let $w \in W$ be an element with REX $s_1 s_2 \cdots s_k$.
Show that the hyperplanes separating C and $w(C)$ are given by $H_1, \; s_1 H_2, \; s_1 s_2 H_3, \ldots, s_1 s_2 \cdots s_{k-1} H_k$
(hint: Consider the sequence of chambers $C, s_1 C, s_1 s_2 C, \ldots, s_1 s_2 \cdots s_k C$)
Deduce that $N(w) = \{ t \in T \mid H_t \text{ separates } C \text{ and } w(C) \}$
(in particular, $\ell(w) = \# \{ H \in \mathcal{A}_W \mid H \text{ separates } C \text{ and } w(C) \}$)

$\Rightarrow$ $\boxed{\text{ACTION OF } W \text{ on THE CHAMBERS IS FREE}}$

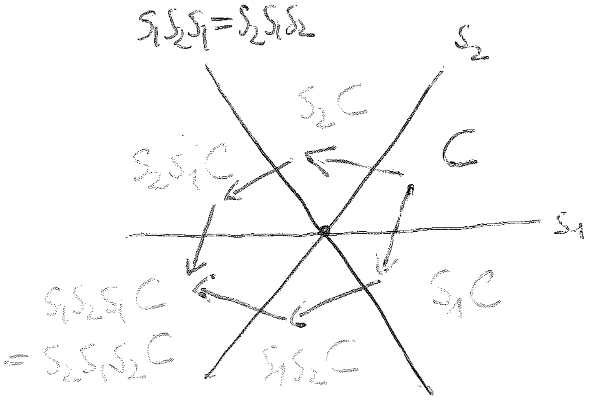
(It can be shown that $m_{s_H s_{H'}} = |\{ H'' \in \mathcal{A}_W \mid H'' \supseteq H \cap H' \}|$)

Example (type A_2)

$W = \mathfrak{S}_3$

$H_{s_2} \cap H_{s_1} = 0$, 3 hyperplanes

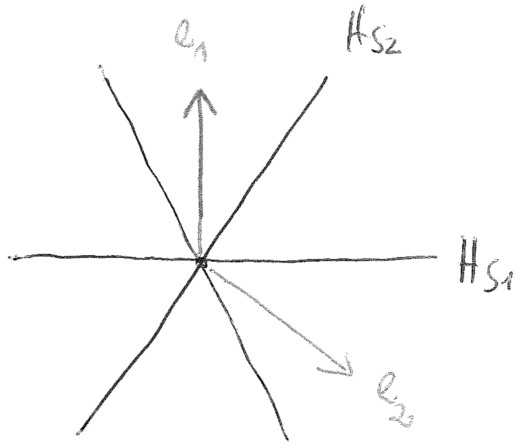
$(m_{s_1 s_2} = 3)$



Hence: FINITE REFLECTION GROUPS ARE COXETER GROUPS

Question: Can we realize every (finite) Coxeter group as a reflection group?

s_1, s_2 orthogonal wrt standard scalar product on $\mathbb{R}^2$



Choosing roots $e_1 \perp H_{s_1}$, $e_2 \perp H_{s_2}$ s.t. $(e_1, e_1) = 1 = (e_2, e_2)$, and defining $e_i^\vee : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as $e_i^\vee(v) = 2(v, e_i)$, we have

$$s_i(v) = v - e_i^\vee(v)\, e_i .$$

Let (W, S) be any (not nec. finite) Coxeter system.
Let $V = \bigoplus_{s \in S} \mathbb{R} e_s$. Define a symmetric bilinear form $B : V \times V \longrightarrow \mathbb{R}$ by $B(e_s, e_t) = -\cos\left(\frac{\pi}{m_{s,t}}\right)$

(wh. $\frac{\pi}{m_{s,t}} = 0$ if $m_{s,t} = \infty$).

<u>Proposition</u> (<u>Tits representation</u>) The map $s \longmapsto \{\sigma_s : v \longmapsto v - 2B(v, e_s)e_s\}$ (a reflection!) defines an action of W on V, for which $B(-,-)$ is an invariant bilinear form.

<u>Proof</u> The fact that $B(-,-)$ is preserved by every $s \in S$ (hence by every $w \in W$) is a straightforward computation. To conclude

<u>Exercise 5</u>: Show that for $s, s' \in S$, the order of $\sigma_s \sigma_{s'}$ is equal to $m_{s,s'}$. To this end, distinguish the cases where $B(e_s, e_{s'}) = -1$ and $B(e_s, e_{s'}) \neq -1$; in this last case, show that $\sigma_s \sigma_{s'}$ is

a rotation by $\frac{2\pi}{m_{s,s'}}$

□

To conclude that every Coxeter group is a reflection group, one has to show that the Tits representation is faithful

<u>Theorem</u> The representation $\{ s \longmapsto \sigma_s \}$ is faithful

<u>About the proof</u> One needs a notion of "chambers", s.t. if $w \neq 1$, then $\sigma_w(C) \cap C = \emptyset$. The correct setting to do this is not V, because in general when W is infinite the reflecting hyperplanes do not distinguish the reflections. Hence one replaces V by the contragredient representation V^* ($(sf^*)(v) = f^*(sv) \quad \forall s \in S$). Choosing the dual basis $\{e_s^*\}_{s\in S}$ to $\{e_s\}_{s \in S}$ and setting

$$C := \{ x^* \in V^* \mid x^*(e_s) > 0 \quad \forall s \in S \}$$

it can be shown that if $w \neq 1$, then $\sigma_w(C) \cap C = \emptyset$ }.

□

<u>Classification of finite Coxeter groups</u>

Let (W,S) be a Coxeter group. The <u>Coxeter graph</u> Γ_W has vertices labelled by the elements of S. There is an edge between s and s' iff $m_{s,s'} > 2$. The edge is labelled by $m_{s,s'}$ except if $m_{s,s'} = 3$ (no label).

<u>Example</u> If W is the symmetric group $\mathfrak{S}_{n+1}$ ("type A_n").

s_1 — s_2 — s_3 — . . . — s_n

A Coxeter group (W,S) is <u>irreducible</u> if Γ_W is connected ($\iff$ $\nexists$ nontrivial partition $S = S_1 \sqcup S_2$ s.t. $W = \langle s, s \in S_1 \rangle \times \langle s, s \in S_2 \rangle$)

(standard parabolic subgps)

$\boxed{\text{What is the relation with irreducibility of reflection groups?}}$

<u>Lemma</u> W is irreducible $\Rightarrow$ Γ_W is connected

<u>Proof</u> Let $\Gamma = \Gamma_1 \dot\cup \Gamma_2 \dot\cup \dots \dot\cup \Gamma_k$. Let $V_i := \bigoplus_{s \in \Gamma_i} \mathbb{R} e_s$. If $s, s' \in \Gamma_i$, then $s'(e_s) \in V_i$ since it is a linear combin. of e_s and $e_{s'}$. If $s \in \Gamma_i$, $s' \notin \Gamma_i$, then $m_{s,s'} = 2$

$\Rightarrow$ $B(e_s, e_{s'}) = -\cos\left(\frac{\pi}{2}\right) = 0$ $\Rightarrow$ $s'(e_s) = e_s$

$\Rightarrow$ each V_i is W-invariant $\square$

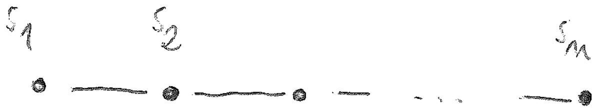
⚠ The converse is false in general (see Exercise 6).

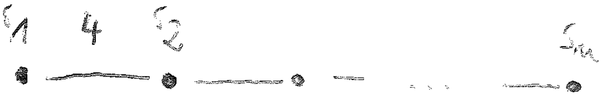
But it holds if W is finite

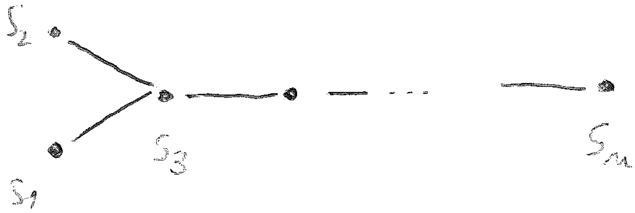
~~(iff W)~~ (it can be shown that $B(-,-)$ is a scalar product iff W is finite)

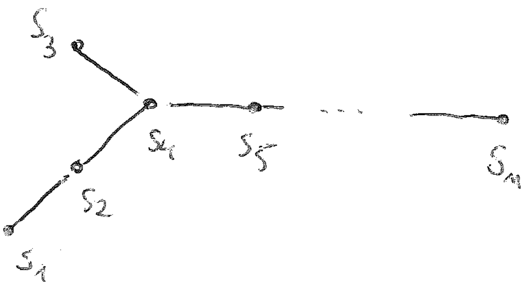
We give the classification of finite Coxeter groups

Theorem Every irreducible finite Coxeter group W has Γ_W belonging to the following list; conversely every Γ in the following list defines an irred. finite Coxeter group.

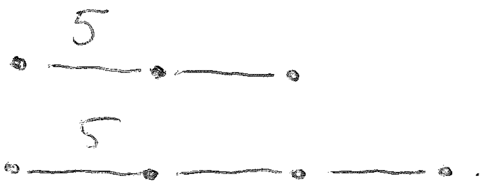
(type A_n)  $(n \geq 1)$

(type B_n / C_n)  $(n \geq 2)$

(type D_n)  $(n \geq 4)$

(type E_6, E_7, E_8)  $(n = 6, 7, 8)$

(type F_4) 

(type H_3, H_4) 

(type $I_2(m)$)  $(m \geq 5)$

Exercise 6 The set $\Phi := \{ w(e_s) \mid w \in W, s \in S \}$ is the set of roots attached to (W, S).

Let $W = \langle s, t \mid s^2 = 1 = t^2 \rangle$ ($m_{st} = \infty$; "infinite dihedral group")

1. Show that every element in $\widetilde{W}$ has a unique reduced expression
2. Show that $R := \bigcup_{v \in W} wSw^{-1}$ is equal to the set $\{ w \in W \mid \ell(w) \text{ is odd} \}$
3. Show that if $t, t' \in R$, then $H_t = H_{t'}$ (reflecting hyperplanes do not distinguish the reflections!)
4. Determine the set Φ of roots (as linear combinations of e_s and e_t)
5. Show that $B(-,-)$ is ~~not a scalar product~~ degenerate.
6. Show that V is not irreducible.

Note Groups which are nonisomorphic as Coxeter groups can be isomorphic as abstract groups

$$W_{I_2(6)} \cong \text{dihedral gp of order } 12 \cong \mathfrak{S}_3 \times \mathfrak{S}_2 \cong W_{A_2} \times W_{A_1}$$