(3) Classical Braid group (topologically)

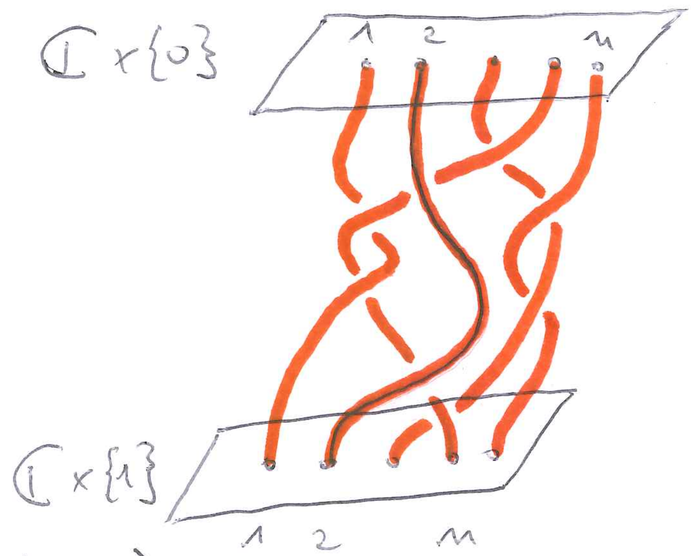
Configuration space of n ordered distinct points in $\mathbb{C}$:

$$M_n := \{ (z_1, z_2, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \ \forall i \neq j \}$$

Define $PB_n := \pi_1(M_n)$ <u>pure braid group on n strands</u>

$$\beta : [0,1] \longrightarrow M_n, \ t \longmapsto (\beta_1(t), \dots, \beta_n(t))$$

represent β by a <u>geometric braid</u>



(homotopy of loops $\Longleftrightarrow$ isotopy of braids)

(product of loops $\Longleftrightarrow$ concatenation of braids)

<u>note</u> starting and ending points have to agree. We want to allow any strand to end at any point.

Consider $\mathrm{Conf}_n(\mathbb{C})$ = conf. space of n <u>unordered</u> distinct points in $\mathbb{C}$ = $\{ \{z_1, \dots, z_n\} \subseteq \mathbb{C}^n \mid z_i \neq z_j \} \cong M_n / \mathfrak{S}_n$

$p: M_n \to \mathrm{Conf}_n(\mathbb{C})$

Define $B_n := \pi_1(\mathrm{Conf}_n(\mathbb{C}))$

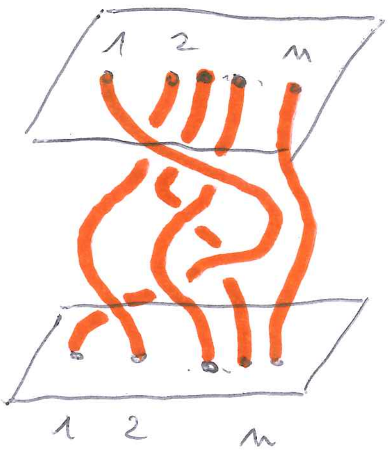
Fix the basepoint $\{(1,0),(2,0),\dots,(n,0)\} = \underset{\overset{\shortparallel}{e}}{((1,0),\dots,(n,0))}$

a loop $\beta: [0,1] \longrightarrow \mathrm{Conf}_n(\mathbb{C})$ with $\beta(0) = e = \beta(1)$

lifts uniquely to a path $\hat{\beta}: [0,1] \longrightarrow M_n$ s.t.

$\hat{\beta}(0) = ((1,0),\dots,(n,0))$

$(\hat{\beta}(1) \in p^{-1}(e))$



$\hat{\beta} = (\beta_1, \beta_2, \dots, \beta_n)$ $\quad \beta_i$ "i-th strand"

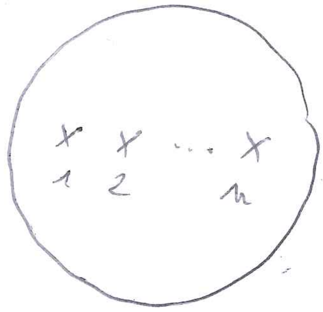
$\beta_i(0) = (i,0)$

$\beta_i(1) \in (k,0)$ for some k

Mapping class groups

Let D be the closed disk, consider n distinct points in D° which for simplicity we denote by $1,2,\dots,n$.

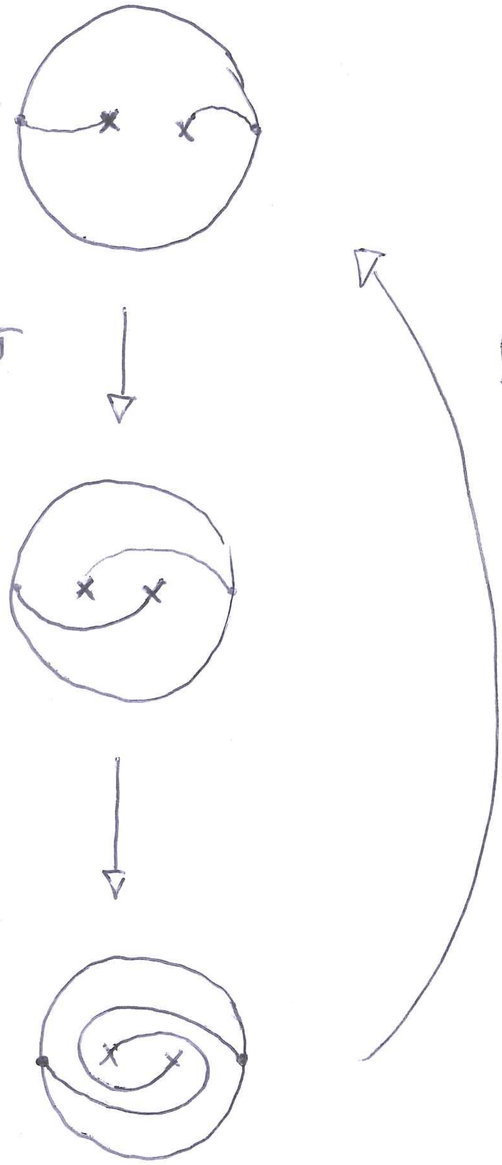
Let $\mathrm{Homeo}^+(D_n) = \begin{pmatrix}\text{orientation preserving}\end{pmatrix}$ homeo's $D_n := D \setminus \{1,2,\dots,n\} \to D_n$ fixing ∂D pointwise.



Define $MCG(D_n) := \mathrm{Homeo}^+(D_n) / \text{isotopy} = \pi_0(\mathrm{Homeo}^+(D_n))$

⚠ The condition "fixing the boundary pointwise" is essential!

"half twist" σ



σ^2

no isotopy between σ^2 and id

~~but there is an isotopy f_t s.t.~~

~~preserving~~

but there is an isotopy between σ^2 and id in the space of Homeo's preserving the boundary.

(**Exercise** Show that $MCG(D_n) \not\cong MCG(\mathbb{C} \setminus \{1,2,\dots,n\})$)

In fact one can show that $MCG(D_n) \cong B_n$

Idea: let $[f] \in MCG(D_n)$. Then f extends uniquely to an homeo $\hat{f}$ of D (s.t. $\hat{f}(\{1,2,\dots,n\}) = \{1,2,\dots,n\}$)

"Alexander's trick": every homeo $\hat{f}$ of D which fixes ∂D is isotopic to id_D. Hence consider an isotopy $\{f_t : D \to D\}_{t\in[0,1]}$, $f_0 = id_D$, $f_1 = \hat{f}$

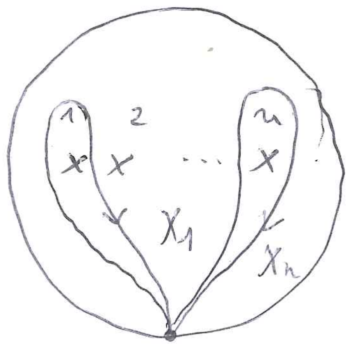
Define a loop $\ell_{[f]} : [0,1] \longrightarrow Conf_n(\mathbb{C})$
$$t \longmapsto f_t(\{1,2,\dots,n\})$$

Action of B_n on $Free_n$

$$B_n \cong MCG(D \setminus \{1,2,\dots,n\}) \circlearrowright \pi_1(D\setminus\{1,2,\dots,n\}) \cong Free_n$$

(fix a base point $d \in \partial D$)

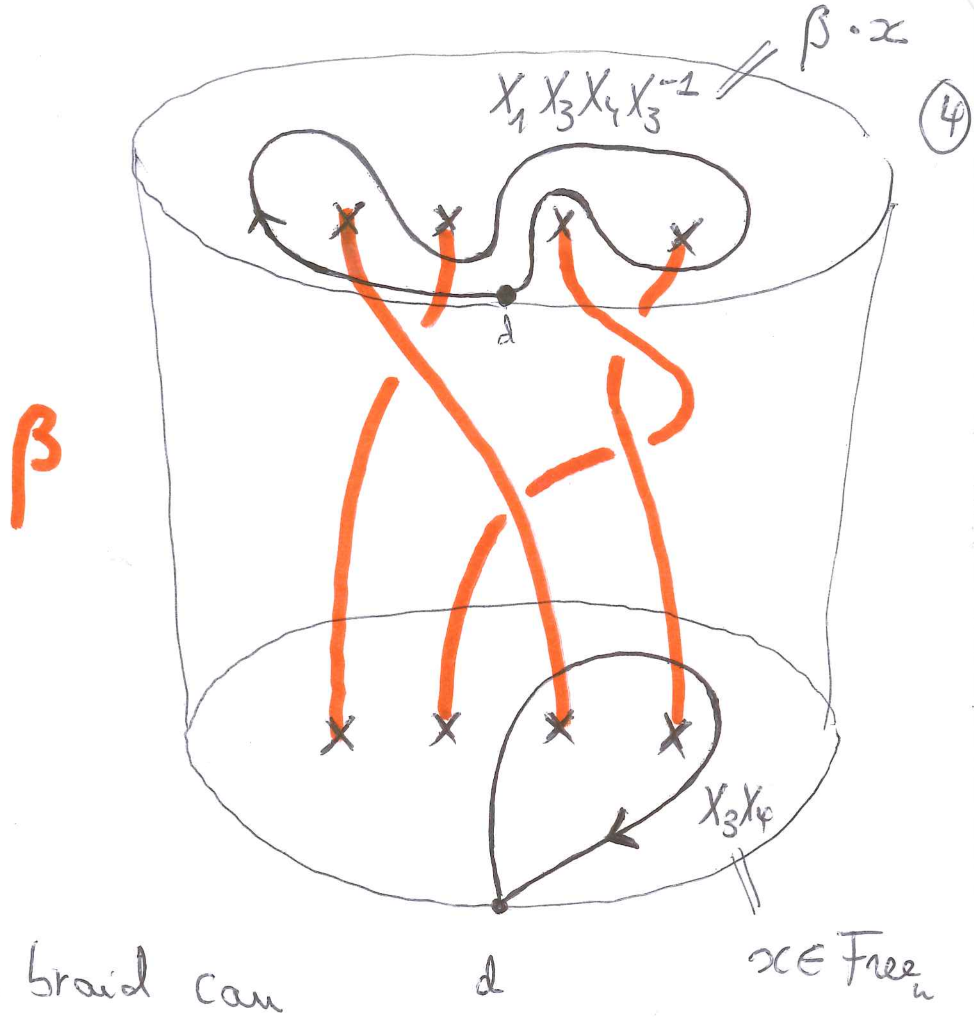
The action is defined in terms of geometric braids as follows



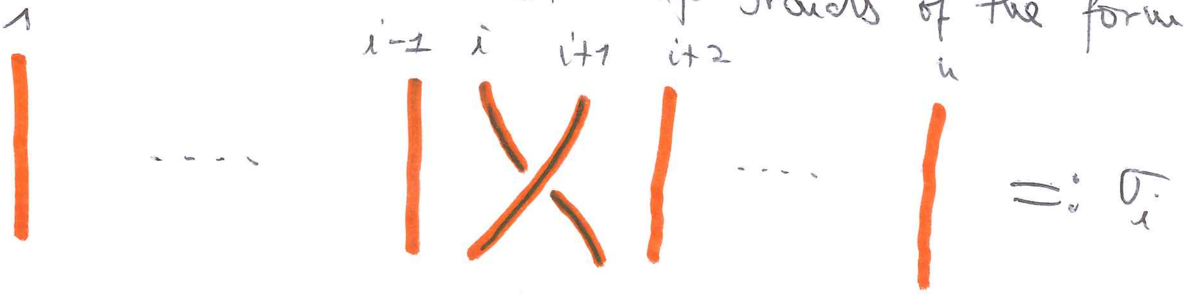
Exercise 1: Show Alexander's trick!

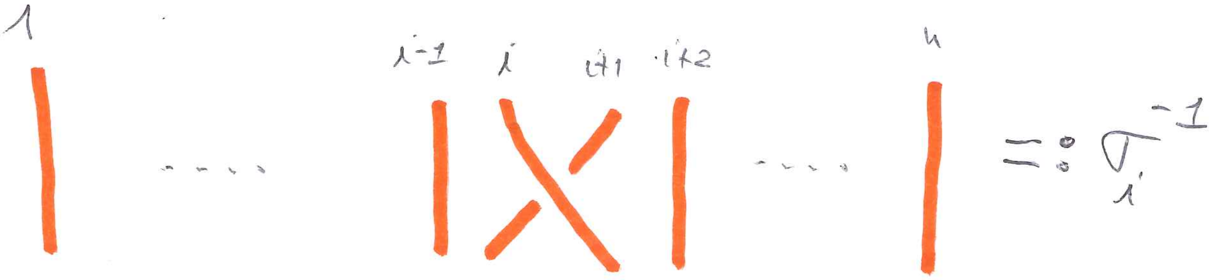
NB

⊛ the action is well-defined (it only depends on the isotopy class of β); it is clear that it defines a group action.



⊛ Every geometric braid can be written as a concatenation of braids of the form





$i = 1, \ldots, n$ $\Longrightarrow$ $\sigma_1, \ldots, \sigma_{n-1}$ and their inverses generate B_n

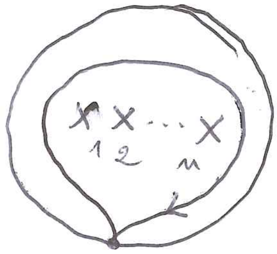
We describe the action of the generators σ_i on the generators $X_1, X_2, \ldots, X_n$ of $Free_n$.

Exercise 2 Show that

$$\sigma_i(X_k) = \begin{cases} X_{k+1} & \text{if } k=i \\ X_k^{-1}X_{k-1}X_k & \text{if } k=i+1 \\ X_k & \text{othw.} \end{cases}$$

$$\sigma_i^{-1}(X_k) = \begin{cases} X_k X_{k+1} X_k^{-1} & \text{if } k=i \\ X_{k-1} & \text{if } k=i+1 \\ X_k & \text{othw.} \end{cases} \qquad (*)$$

Remark



$= X_1X_2\cdots X_n$ is fixed by every $\beta\in B_n$.

A useful way to write the above action is

$$\sigma_i\cdot(X_1,X_2,\dots,X_i,X_{i+1},\dots,X_n) = (X_1,\dots,X_{i-1},X_{i+1},X_{i+1}^{-1}X_iX_{i+1},\dots,X_n)$$

$$\sigma_i^{-1}\cdot(X_1,\dots,X_i,X_{i+1},\dots,X_n) = (X_1,\dots,X_{i-1},X_iX_{i+1}X_i^{-1},X_i,\dots,X_n)$$

"Hurwitz action" ; the remarks above suggests the following definition

Braid automorphisms of B_n

Def^n We say that an automorphism φ of Free_n is a braid automorphism if (i) $\exists\,\mu\in\mathfrak{S}_n$ s.t. $\varphi(X_k)$ is conjugate to $X_{\mu(k)}$ in Free_n for all $k=1,2,\dots,n$

(ii) $\varphi(X_1X_2\cdots X_n) = X_1X_2\cdots X_n$

Write $Br(Free_n)$ for the set of braid automorphisms.

<u>Exercise 3</u> Write φ_β for the braid automorphism induced by $\beta \in B_n$. Show that the map $B_n \longrightarrow Br(Free_n) \subseteq Free_n$, $\beta \mapsto \varphi_\beta$, is surjective (i.e., every braid automorphism is induced by a braid).

(<u>Hint:</u> Let $\varphi \in Br(Free_n)$; we have $\varphi(X_k) = A_k X_{\mu(k)} A_k^{-1}$ $\forall k = 1, 2, \dots, n$, where A_k is a word in $X_1^{\pm 1}, \dots, X_n^{\pm 1}$.

(i) Explain why A_k can be chosen s.t. $A_k X_{\mu(k)} A_k^{-1}$ is a reduced word in $Free_n$.

(ii) Show that $\exists\, j \in \{1, 2, \dots, n-1\}$ ~~and a word A in $X_1^{\pm 1}, \dots, X_n^{\pm 1}$~~ s.t. one of the following holds

Ⓐ $A_j = A_{j+1} X_{\mu(j+1)} A$ for some word A in $X_1^{\pm 1}, \dots, X_n^{\pm 1}$

Ⓑ $A_{j+1} = A_j X_{\mu(j)}^{-1} A$ " "

(iii) Show that if Ⓐ (resp. Ⓑ) holds, then $\varphi\sigma_j$ (resp. $\varphi\sigma_j^{-1}$) has shorter length than φ; here the length of φ is the sum of the lengths of the words $A_k X_{\mu(k)} A_k^{-1} = \varphi(X_k)$.

(iv) Conclude)

<u>Word problem in B_n: Artin's solution</u>

<u>Question</u>: Determine if two braids (or rather, two words in the generators $\sigma_i^{\pm 1}$) are the same,

i.e., represent the same element of B_n

NB 1 : the word problem in Free_n is trivial: given a word in $X_1^{\pm 1}, \dots, X_n^{\pm 1}$, perform cancellations $X_j X_j^{-1}$ or $X_j^{-1} X_j$ until you can't go on. This gives the unique REX of the group element.

Theorem (Artin 1925) The group homomorphism

$$\varphi: B_n \longrightarrow Br(\mathrm{Free}_n) \subseteq \mathrm{Aut}(F_n), \ \beta \mapsto \varphi_\beta$$

is an isomorphism.

NB it implies that the word problem in B_n is solvable. Let $\beta, \beta' \in B_n$ be two braids. Choose words in $\sigma_1^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}$ to represent them. For all $k = 1, \dots, n$, compare

$$\varphi_\beta(X_k) \quad \text{and} \quad \varphi_{\beta'}(X_k) \qquad \text{using } (*)$$

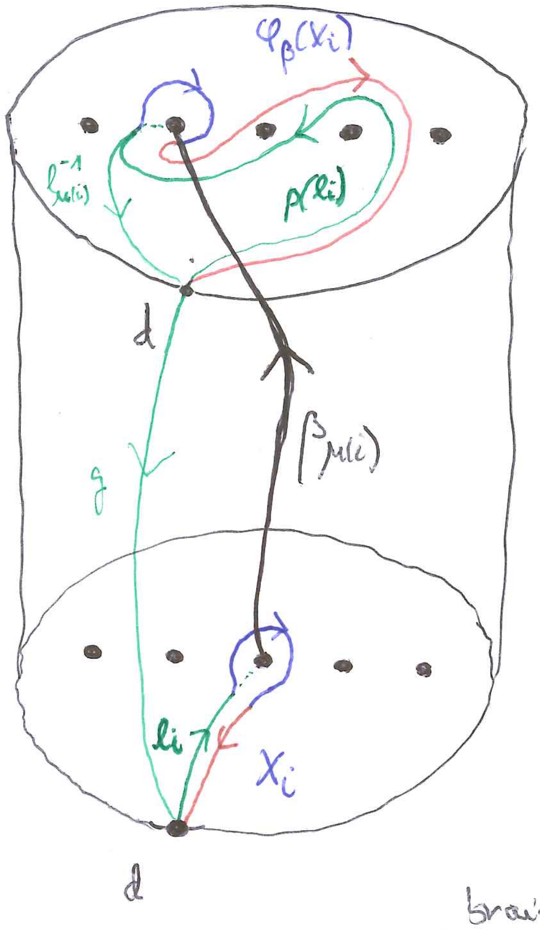
and reducing the obtained words. If they coincide $\forall k$, then φ_β and $\varphi_{\beta'}$ have the same action on the generators of Free_n $\Longrightarrow \varphi_\beta = \varphi_{\beta'} \xRightarrow[\text{Artin's Thm}]{} \beta = \beta'$!!!

The idea of Artin's proof: $\varphi_\beta(X_i) = A_i X_{\mu(i)} A_i^{-1}$

① show that A_i determines the isotopy ~~typ~~ class of the strand of β starting at $\mu(i)$ inside the cylinder with the other strands removed (for a fixed braid diagram for β)

(2) Show by induction on the number of strings that $\varphi_\beta = \varphi_{\beta'} \Rightarrow \beta = \beta'$.

About (1)



g : vertical path from d in the plan above to d in the plan down.

in the cylinder with the strands β_k for $k \neq \mu(i)$ removed,

A_i is homotopic to $p(l_i)\, l_{\mu(i)}^{-1}$

But $p(l_i)\, l_{\mu(i)}^{-1}$ is the braid projection of the path $g\, l_i\, \beta_{\mu(i)}\, l_{\mu(i)}^{-1}$

$$\Rightarrow \quad A_i \cong g\, l_i\, \beta_{\mu(i)}\, l_{\mu(i)}^{-1} \Rightarrow \beta_{\mu(i)} \cong l_i^{-1} g^{-1} A_i\, l_{\mu(i)}$$

Hence A_i determines the homotopy type of $\beta_{\mu(i)}$ in the cylinder with the other strands removed.

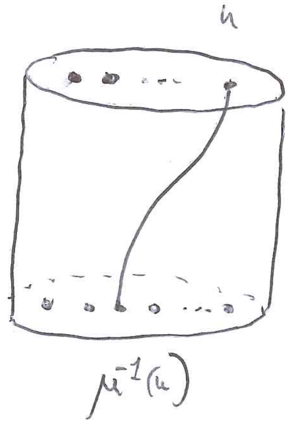
About (2)

If $n=1$ then the claim is true. Let $n>1$.

Let β, β' be two braids on n strands. Suppose that $\varphi_\beta = \varphi_{\beta'}$. Remove the n^{th} string in two braid diagrams for β, β'

$\Longrightarrow$ get two braids $\tilde{\beta}, \tilde{\beta}'$ on $n-1$ strands

$\varphi_{\tilde{\beta}}(X_i)$ is obtained by



- Replacing $X_n^{\pm 1}$ by 1 in
  $\varphi_\beta(X_i) = A_i X_{\mu(i)} A_i^{-1}$ if $i < \mu^{-1}(n)$
- Replacing $X_n^{\pm 1}$ by 1 in $\varphi_\beta(X_{i+1})$ if $i > \mu^{-1}(n)$

$\Longrightarrow \varphi_{\tilde{\beta}}(X_i) = \varphi_{\tilde{\beta}'}(X_i) \quad \forall i.$

Hence $\tilde{\beta}$ and $\tilde{\beta}'$ are isotopic. Extend the isotopy to the whole space and apply it to β $\Longrightarrow$ get a braid $\bar{\beta}$, isotopic to β, s.t. the first $n-1$ strands coincide with those of β'. Take two braid diagrams for $\bar{\beta}$ and β' where the first $n-1$ strands are the same, and remove them. Then by ① the n^{th} strands of β and β' are isotopic in this complement

$\Longrightarrow$ $\bar{\beta}$ and β', hence β and β', are isotopic.

$\square$.

Braid relations

Exercise 4 ① Show that $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \forall i = 1, \dots, n-2$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{if } |i-j| > 1$$

② Show that $\sigma_1, \sigma_1\sigma_2 \cdots \sigma_{n-1}$ generate B_n.

These are the braid relations of $\mathfrak{S}_n$! In fact one can show that every relation in B_n is a consequence of these.

Recall: $\mathfrak{S}_n \circlearrowright V' = \bigoplus_{i=1}^{n} \mathbb{R} e_i$ by permuting the coordinates. For $t = (i,j)$, $H_t = \{x_i = x_j\}$

Note: V' is not irreducible since $\underbrace{\mathbb{R}(e_1+e_2+\dots+e_n)}_{L}$ is fixed by every $w \in \mathfrak{S}_n$.

Exercise 5 Show that $V'/L \cong V$ (Tits representation of $\mathfrak{S}_n$)

Note:
$$V' \otimes \mathbb{C} \Big/ \bigcup_{i<j} H_{i,j} \otimes \mathbb{C} = \{(z_1, \dots, z_n) \mid z_i \in \mathbb{C},\ z_i \neq z_j \text{ if } i \neq j\}$$
$$= M_n$$

since $L \subseteq H_{i,j}$ $\forall i<j$, we have ~~...~~

$$V \otimes \mathbb{C} \Big/ \bigcup_{i<j} \overline{H}_{i,j} \otimes \mathbb{C}$$

(where $\overline{H}_{i,j} = H_{i,j}/L$ is the refl^n hyp. of $t = (i,j)$ in Tits representation)

can be defined for every (finite) Coxeter group

$\wr\wr$ homotopic

$$V' \otimes \mathbb{C} / \bigcup H_{i,j} \otimes \mathbb{C}$$

$\rightsquigarrow$ this allows one to define (pure) braid groups for all finite Coxeter groups.