Last time:

$A = A_n$ zig-zag algebra $\quad Q = \begin{tikzcd} 1 \arrow[r, shift left] & 2 \arrow[l, shift left] \arrow[r, shift left] & 3 \arrow[l, shift left] \arrow[r, shift left] & 4 \arrow[l, shift left] \arrow[r, shift left] & 5 \arrow[l, shift left] \end{tikzcd}$ $\quad A = A_n = kQ \Big/ \begin{array}{l} (i|i+1|i+2) = (i+2|i+1|i) = 0 \\ (i|i+1|i) = (i|i-1|i) \end{array}$

(also $(1|2|1|2) = (2|1|2|1) = 0$ if $n=2$).

~~As~~ $kQ$ and $A_n$ are graded by path length, $A$ is finite-dimensional.

$A_n\text{-mod}^{\mathbb{Z}}$ : graded left modules, finitely generated.

$A_n\text{-proj}^{\mathbb{Z}}$ : projective left modules, finitely generated

$A_n\text{-bim}^{\mathbb{Z}}$ : graded $A_n$-bimodules, f.g. as left and right $A_n$-modules.

$K^b(A_n\text{-mod}^{\mathbb{Z}})$ :

$K^b(A_n\text{-proj}^{\mathbb{Z}})$ :

$\Big\}$ their homotopy categories.

$K^b(A_n\text{-bim}^{\mathbb{Z}}) \curvearrowright K^b(A_n\text{-mod}^{\mathbb{Z}})$

$A_n\text{-bim}^{\mathbb{Z}} \curvearrowright A_n\text{-mod}^{\mathbb{Z}}$

$(B, M) \mapsto B \otimes_{A_n} M$

Last time: $A_n\text{-bim}^{\mathbb{Z}}$ can be thought of as certain endofunctors $A_n\text{-mod} \to A_n\text{-mod}$.

Grothendieck groups:

$\mathcal{C}$ is abelian: $[\mathcal{C}]$ is the ~~free~~ abelian group w/ generators $[c]$ for each $c \in \mathcal{C}$ and relations $[c] = [c'] + [c'']$ whenever there exists an exact ~~triangle~~ sequence $0 \to c' \to c \to c'' \to 0$.

Example: Show that $[\text{f.d. vect}] = \mathbb{Z}$, $[\text{vect}/k] = 0$.

$\mathcal{A}$ ~~$\mathcal{C}$~~ additive: $[\mathcal{A}]_{\oplus}$ is the abelian group w/ generators $[a]$ $c \in \mathcal{C}$ and relations $[a] = [a'] + [a'']$ whenever $a \cong a' \oplus a''$.

Example: If $\mathcal{A}$ is the category of projective modules over a finite-dimensional algebra then $[\mathcal{A}_A]_{\oplus}$ is free w/ basis $\{[P_\bullet]\}$ where $P$ runs over isoclasses of indecomposable projective modules.

In our example $[A_n\text{-proj}]^{\mathbb{Z}} = \bigoplus_{i=1}^{n} \mathbb{Z} A_n(i)$.

<u>$\mathcal{T}$ is triangulated:</u> $[\mathcal{T}]_\Delta$ is the free abelian group w/ generators

$$[t] \quad t\in\mathcal{T}$$

and relations

$$[t]=[t']+[t'']$$

whenever there exists a distinguished triangle $t'\to t\to t''\xrightarrow{+1}$ in $\mathcal{T}$.

<u>Important consequence:</u> $t\xrightarrow{id} t\to 0\xrightarrow{+1}$ is distinguished

$$\Downarrow$$

$t\to 0\to t(1)\xrightarrow{+1}$ is distinguished $\Rightarrow$ $[t(1)]=-[t]$ in $[\mathcal{T}]_\Delta$.

$\boxed{\text{Exact / additive / triangulated functors induce maps on the Grothendieck group.}}$

<u>Gradings:</u> Often one encounters categories equipped w/ a "shift of grading functor" $(1)$. This allows us to view $[\mathcal{C}]$, $[\mathcal{A}]_\oplus$ and $[\mathcal{T}]_\Delta$ as $\mathbb{Z}[q^{\pm1}]$-modules via $q\cdot[M]:=[M(-1)]$. "$q$ shifts grading up by 1"

$$[\mathrm{Vect}^{\mathbb{Z}}]=\mathbb{Z}[q,q^{-1}]$$

<u>Example:</u> $$[A_n\text{-proj}^{\mathbb{Z}}]=\bigoplus \mathbb{Z}[q^{\pm1}][Ae_i].$$

Finally, last time we considered

$$U_i:=Ae_i\otimes_k e_iA(1)\in A\text{-bim}^{\mathbb{Z}}.$$

and the complexes

$$F_i:\quad 0\to U_i(-1)\to \overset{*}{A}\to 0$$

$$F_i':\quad 0\to A\to U_i(1)\to 0$$

<u>Lemma:</u> ① $F_i\otimes_A F_i'\cong F_i'\otimes_A F_i\cong \overset{*}{A}$

② $F_i\otimes_A F_{i+1}\otimes_A F_i\cong F_{i+1}\otimes_A F_i\otimes_A F_{i+1}$ $\forall i$ ③ $F_i\otimes_A F_j\cong F_j\otimes_A F_i$ if $|i-j|>1$.

Recall how $U_i$ acts on $P_j$.

$$U_i \otimes_A P_j = \begin{cases} P_i(1) \oplus P_i(-1) & \text{if } i = j \\ P_i & \text{if } |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the tensoring w/ the complexes $F_i$ preserves $K^b(A\text{-proj}^{\mathbb{Z}})$. Let us calculate how $F_i$ acts on the Grothendieck group. Page 8 here!

$$F_i \qquad Ae_i \otimes e_i A \xrightarrow{(*)} A$$

Good normalisation

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$$F_1 : \quad \begin{aligned} P_1 &\longmapsto P_1(-2) \xrightarrow{(*)} 0 \\ P_2 &\longmapsto P_1(-1) \longrightarrow P_2 \\ P_3 &\longmapsto 0 \longrightarrow P_3 \end{aligned} \qquad \begin{pmatrix} -q^2 & -q & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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$$F_2 : \quad \begin{aligned} P_1 &\longmapsto P_2(-1) \xrightarrow{(*)} P_1 \\ P_2 &\longmapsto P_2(-2) \longrightarrow 0 \\ P_3 &\longmapsto P_2(-1) \longrightarrow P_3 \end{aligned} \qquad \begin{pmatrix} 1 & 0 & 0 \\ -q & -q^2 & -q \\ 0 & 0 & 1 \end{pmatrix}$$

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$$F_3 : \quad \begin{aligned} P_1 &\longmapsto \cancel{P_3} \quad\; P_1 \\ P_2 &\longmapsto P_3(-1) \xrightarrow{(*)} P_2 \\ P_3 &\longmapsto P_3(-2) \longrightarrow 0 \end{aligned} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -q & -q^2 \end{pmatrix}$$

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<u>Exercise:</u> Show that ~~conj~~ this representation is equivalent to the Burau representation after $q \mapsto q^2$.

( <u>Hint:</u> Conjugate by $\begin{pmatrix} q & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{-1} \end{pmatrix}$. )

## Minimal complexes:

$M$ is indecomposable if $M \cong M_1 \oplus M_2$
$$\Downarrow$$
$$M_1 = 0 \text{ or } M_2 = 0$$

Suppose $A$ is a f.gl. k-algebra.

Then objects in finitely generated $A$-modules have the following properties:

① ~~Ma~~ Any $M \in A$-mod can be written $M = M_1 \oplus \dots \oplus M_k$ $(*)$ w/ $M_i$ indecomposable.

② $M \in A$-mod is indecomposable $\iff$ $\mathrm{End}(M)$ is local.

③ Summands/iso. and multiplicities in $(*)$ are well-defined ("Krull-Schmidt theorem").

In fact ① + ② ⟹ ③. An additive category $\mathcal{A}$ is Krull-Schmidt if it satisfies ① and ②.

E.g. $\mathbb{Z}$-mod, even-dimensional vector spaces fails ②

Suppose $\mathcal{A}$ is an additive category. The complex

$$0 \longrightarrow M \longrightarrow M \longrightarrow 0 \qquad (*)$$

is isomorphic to $0$ in $K(\mathcal{A})$ (its identity is null-homotopic).

A contractible summand of a complex $N^\circ$ is a summand isomorphic to $(*)$.

<u>Exercise:</u> (Gaussian elimination)

Suppose $M \in K(\mathcal{A})$ has the form

$$\begin{matrix} C \\ \oplus \\ \widetilde{M}^i \end{matrix} \xrightarrow{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}} \begin{matrix} C' \\ \oplus \\ \widetilde{M}^{i+1} \end{matrix}$$

with $\alpha$ an iso.

$$\cdots \longrightarrow M^i \longrightarrow M^{i+1} \longrightarrow \cdots$$

Then $M$ is isomorphic to a complex of the form

$$\cdots \longrightarrow M^{i-1} \longrightarrow \begin{matrix} C \\ \oplus \\ \widetilde{M}^i \end{matrix} \xrightarrow{\begin{pmatrix} \mathrm{id} & 0 \\ 0 & \gamma \end{pmatrix}} \begin{matrix} C \\ \oplus \\ \widetilde{M}^{i+1} \end{matrix} \longrightarrow M^{i+2} \longrightarrow \cdots$$

in particular, $M$ contains a contractible summand.

Def A <u>minimal complex</u> is a complex without contractible summands.

<u>Lemma:</u> If $\mathcal{A}$ is Krull-Schmidt then any complex $M \in K(\mathcal{A})$ contains a summand $M_{min} \subset M$ s.t.

① $M_{min} \hookrightarrow M$ is an isomorphism ② $M_{min}$ is minimal.

Moreover, any two minimal complexes are isomorphic as complexes.

There is much beauty in the action of $\mathcal{B}_n$ on $K^b(A\text{-proj}^{\mathbb{Z}})$.

Consider: (*)

$$X_m := \quad 0 \longrightarrow \underset{\deg = -m}{P_1(-2m+1)} \longrightarrow P_1(-2m+3) \longrightarrow \dots \longrightarrow P_1(-1) \longrightarrow P_2 \longrightarrow 0$$

<u>Exercise:</u> $F_1^m(P_2) \cong X_m$.

<u>Sketch:</u> ① Show that $U_1 X_m = P_1(-2m)[m] \;\; = \text{"}0 \to \underset{\deg -m}{P_1(-2m)} \to 0\text{"}$

② We have a distinguished triangle of bimodules:

$$A \longrightarrow F_1 \longrightarrow U_1(-1)[1] \xrightarrow{+1}$$

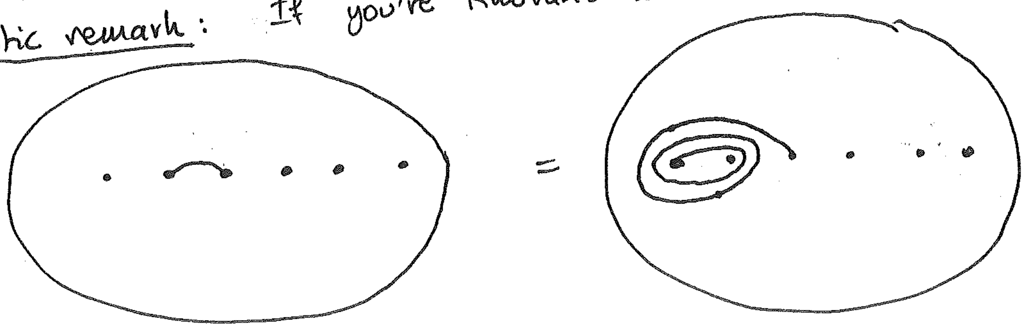
$\rightsquigarrow$ a d.t. of projective A-modules:

$$A \otimes_A X_m \to F_1 \otimes_A X_m \to U_1 \otimes_A \overset{U_1(-1)[1]}{\cancel{P_1(-2m)[1]}} \xrightarrow{+1}$$

$$X_m \longrightarrow F_1^m \otimes_A X_m \longrightarrow P_1(-2m-1)[m+1] \xrightarrow{+1}$$

Now argue that $F_1 \otimes_A X_m \cong X_{m+1}$ and use induction.

<u>Cryptic remark:</u> If you're Khovanov and Seidel this reminds you of:

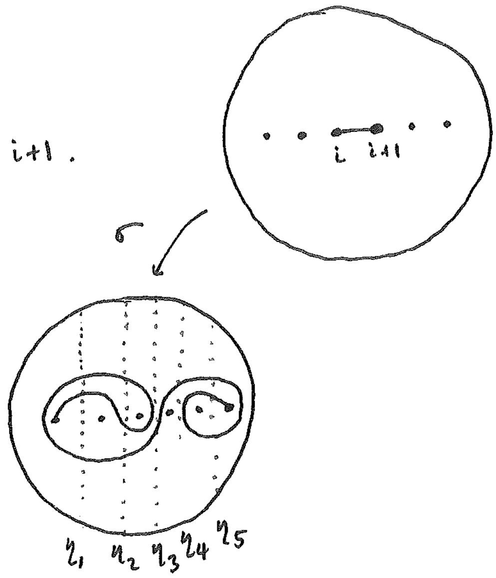
$\sigma_1^m$ 

Khovanov and Seidel's recipe: $\sigma \in B_{n+1}$

Our goal is to understand the minimal complex $F_\sigma(P_i) \in K^b(A_n\text{-proj}^{\mathbb{Z}})$.

Do the following:

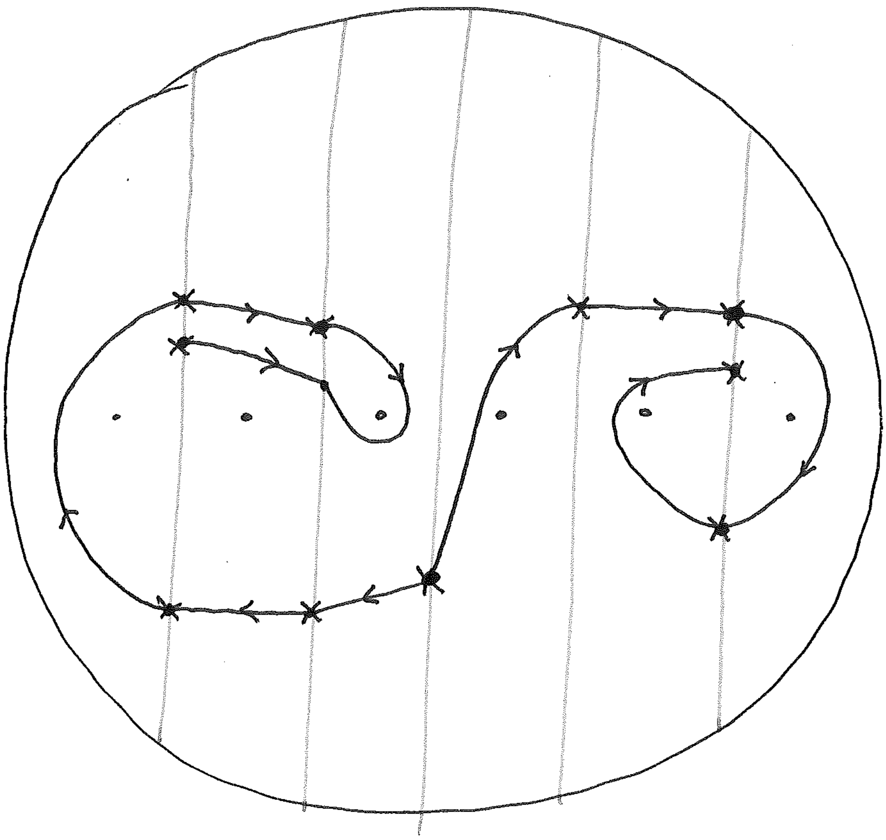
① Consider the curve $f_i$ joining $i$ and $i+1$.




② Act by $\sigma \in MCG(D_n, \partial)$:

(Make sure $\gamma = \sigma(f_i)$ is minimal wrt noodles $\eta_1, \eta_2, \dots, \eta_n$).

③



$$P_1 \to P_2 \leftarrow P_2 \leftarrow P_1 \leftarrow P_1 \leftarrow P_2 \leftarrow P_3 \to P_4 \to P_5 \to P_5 \to P_5$$

④ Make it a complex:

(ignoring shifts)

$$\begin{array}{ccccccccc} & & & & P_1 \oplus P_2 \oplus & \to & P_2 & & \\ & & P_2 \oplus & \to & P_1 \oplus & \to & P_1 \oplus & & \\ P_3 & \to & P_4 & \to & P_5 & \to & P_5 & \to & P_5 \end{array} =: P_{\sigma, i}$$

**Thm:** Up to grading and degree shift, we have

$$\sigma(P_i) \cong F_{\sigma, i}. \quad \text{in} \quad K^b(A\text{-proj}^{\mathbb{Z}}).$$

**Corollary:** The action of $B_n$ on $K^b(A\text{-proj}^{\mathbb{Z}})$ is faithful.

("the categorified braid group action is faithful").

**Proof:** If $F_\sigma \cong F_{id}$ then $F_\sigma(P_i) \cong P_i$ $\forall i$.

Hence $\sigma(f_i)$ is isotopic to $f_i$ for all $i$. Hence $\sigma = w_0^m$ for some $m$.

Hence $\sigma = id$ by considering action on Grothendieck group.