Recall

$$Q = \begin{tikzcd} \underset{1}{\circ} \arrow[r, bend left] & \underset{2}{\circ} \arrow[l, bend left] \arrow[r, bend left] & \circ \arrow[l, bend left] \arrow[r, bend left] & \underset{n}{\circ} \arrow[l, bend left] \end{tikzcd}$$

type A_n zigzag algebra.

$$A = kQ \Big/ \begin{array}{l} (i|i+1|i+2) = (i+2|i+1|i) = 0 \\ (i|i+1|i) = (i|i-1|i) =: X_i \end{array}$$

i

graded by path length

e_i = 'lazy' path at vertex i

$U_i := Ae_i \otimes_k e_i A(1)$ graded A-bimodule

⊛ Tensor products of U_i's satisfy the Temperley-Lieb

⊛ U_i is right projective $\Rightarrow$

$F_i \otimes_R -$ is a well-defined functor

$$K^b(\mathrm{Proj}\ A\text{-}\cancel{\mathrm{bim}}\ \mathrm{mod}^{\mathbb{Z}}) \longrightarrow K^b(\mathrm{Proj}\ A\text{-}\cancel{\mathrm{bim}}\ \mathrm{mod}^{\mathbb{Z}})$$

where $F_i : 0 \longrightarrow U_i(-1) \longrightarrow \overset{*}{A} \longrightarrow 0 \in K^b(A\text{-}\mathrm{bim}^{\mathbb{Z}})$

(*) The F_i are invertible and satisfy the type A_n braid relations.

These definitions can be given starting from any quiver of type ADE, and the same properties hold

Natural question: Do the faithfulness of the categorical action of B_{n+1} extend to types ADE?

Remark: The 'decategorification' of $\mathcal{B}_{n+1} \circlearrowright K^b(\mathrm{Proj}\ A\text{-mod}^{\mathbb{Z}})$, i.e., the induced action on $K_0(\mathrm{Proj}\ A\text{-mod}^{\mathbb{Z}})$, is the Burau representation, which is unfaithful except for $n=3$ and possibly $n=4$, while the categorical action is faithful

"lift non-faithful actions to faithful ones"

Brav & Thomas : homological interpretation of the Garside structure on the braid group Br_W attached to a Coxeter group of type ADE.

Recall Γ = Coxeter diagram of type ADE
with vertices indexed by $1, 2, \dots, n$

braid relations

$$W = \langle s_1, \dots, s_n \mid s_i^2 = 1, \ s_i s_j s_i = s_j s_i s_j \text{ if } i - j, \ s_i s_j = s_j s_i \text{ if } i \not- j \rangle$$

$$Br_W = \langle \sigma_1, \dots, \sigma_n \mid \text{braid relations of } W \rangle$$

$\cup\!\mid$

$$Br_W^+ = \langle \sigma_1, \dots, \sigma_n \mid \text{———} \rangle^+ \quad \sigma \text{ monoid}$$

(Garside monoid)

To show that an action of Br_W is faithful, it suffices to show that the action of Br_W^+ is faithful

Weight structures and t-structures

$\mathcal{A}$ additive category $\rightsquigarrow$ $K^b(\mathcal{A})$ triangulated category

(= bounded homotopy category)

$\mathcal{A}$ abelian category $\rightsquigarrow$ $D^b(\mathcal{A})$ bounded derived category.

(= $K^b(\mathcal{A})$ + localize wrt quasi-isomorphisms)

in general:

$K^b(\mathcal{A})$ has a natural weight structure, while

$D^b(\mathcal{A})$ has a natural t-structure

(in our setting: $K^b(\mathrm{Proj}\ A\text{-mod}^{\mathbb{Z}})$ also has a t-structure).

Weight structure on a triangulated category:

$\mathcal{D}$ triangulated category. A <u>weight structure</u> on $\mathcal{D}$ is the data of two (full) additive subcategories $\mathcal{D}^{\geq 0}$, $\mathcal{D}^{\leq 0}$ s.t.

1. $\mathcal{D}^{\geq 0} \subseteq \mathcal{D}^{\geq 0}[1]$, $\mathcal{D}^{\leq 0}[1] \subseteq \mathcal{D}^{\leq 0}$
2. ("Orthogonality") $\mathrm{Hom}(\mathcal{D}^{\geq 0}, \mathcal{D}^{\leq 0}[1]) = 0$
3. $\forall X \in \mathcal{D}$, $\exists$ distinguished triangle

$$\begin{array}{ccccccc} B & \longrightarrow & X & \longrightarrow & A & \xrightarrow{f} & B[1] \\ \cap & & & & \cap & & \\ \mathcal{D}^{\geq 0}[-1] & & & & \mathcal{D}^{\leq 0} & & \end{array}$$

<u>Standard example</u> $\mathcal{A}$ additive, $K^b(\mathcal{A})$

$K^b(\mathcal{A})^{\leq 0}$ = Complexes which are homotopy equivalent to complexes concentrated in degrees ≤ 0

$K^b(\mathcal{A})^{\geq 0}$ = ———————————————— ≥ 0

1, 2 are clear.

(in derived categories, ② is not satisfied in general! $\mathrm{Hom}_{D^b(A)}(X^\bullet, Y^\bullet)$

$= \mathrm{Hom}_{K^b(A)}(X^\bullet, I^\bullet)$ ← injective resol^n of $Y^\bullet$)

$$
\begin{tikzcd}
\tau_{\geq 1} A^\bullet & A^\bullet & \tau_{\leq} A^\bullet & \tau_{\geq 1} A^\bullet[1] \\
A^2 & A^2 & 0 & {} \\
A^1 \arrow[u] \arrow[r] & A^1 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] \arrow[dr, phantom, "\partial^2=0"] & A^2 \arrow[u] \\
0 \arrow[u] \arrow[r] & A^0 \arrow[u] \arrow[r] & A^0 \arrow[u] \arrow[r, "\partial"] \arrow[dr, phantom, "\partial^2=0!"] & A^1 \arrow[u] \\
0 \arrow[u] \arrow[r] & A^{-1} \arrow[u] \arrow[r] & A^{-1} \arrow[u] \arrow[r] & 0 \arrow[u] \\
0 \arrow[u] \arrow[r] & A^{-2} \arrow[u] \arrow[r] & A^{-2} \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}
$$

$\mathcal{D}^{\geq 0}[-1]$ $\quad A^\bullet \quad$ $\in \mathcal{D}^{\leq 0}$

△ triangle

distinguished

"stupid truncation"

t-structure on a derived category

'reverse hom's and triangle'

$\mathcal{D}$ triangulated category. A <u>t-structure</u> on $\mathcal{D}$ is the data of two (full) additive subcategories $\mathcal{D}^{\geq 0}, \mathcal{D}^{\leq 0}$ s.t.

1. $\mathcal{D}^{\geq 0} \subseteq \mathcal{D}^{\geq 0}[1]$, $\mathcal{D}^{\leq 0}[1] \subseteq \mathcal{D}^{\leq 0}$

2. ('Orthogonality') $\mathrm{Hom}(\mathcal{D}^{\leq 0}[1], \mathcal{D}^{\geq 0}) = 0$

3. $\forall X \in \mathcal{D}$, $\exists$ distinguished triangle

$$\begin{tikzcd} A \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & X \arrow[r] & B \arrow[r, "f"] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & A[1] \\ \mathcal{D}^{\leq 0} & & \mathcal{D}^{\geq 0}[-1] & \end{tikzcd}$$

<u>Exercise</u>: (1) Check that $K^{\geq 0}, K^{\leq 0}$ as defined in the previous example does not naturally yield a t-structure.

(2) Define $\mathcal{D}^{\leq 0}$ (resp. ≥ 0) = complexes isom. to complexes whose cohomology is concentrated in degrees ≤ 0 (resp. ≥ 0). Show that this gives a t-structure on $\mathcal{D}$ (... understand how triangles in (3) are built!)

<u>What we want:</u> Using the grading of the objects in Proj-A-mod$^{\mathbb{Z}}$ (or $K^b(\text{Proj } A\text{-mod } \mathbb{Z})$), define a t-structure on $K^b(\text{Proj } A\text{-mod}^{\mathbb{Z}})$.

The following two properties will turn out to be crucial:

1) $\text{Hom}_{\text{Proj } A\text{-mod}\mathbb{Z}}(P_i, P_i) = k$

2) $\text{Hom}_{A\text{-mod}^{\mathbb{Z}}}(P_i, P_j(-k)) = 0 \quad \forall k > 0 \ \forall i, j$

(Exercise!) $\quad (f(e_i) = ?)$

<u>Canonical t-structure induced by the grading on $K^b(\text{Proj } A\text{-mod}^{\mathbb{Z}})$</u>

Let $\Pi = \bigoplus_{\substack{i=1 \\ m \in \mathbb{Z}}}^{n} P_i^{\oplus a_{i,m}}(m) \in$ ~~Proj A-mod$^{\mathbb{Z}}$~~ A-Proj$^{\mathbb{Z}}$, $\quad a_{i,m} \in \mathbb{Z}_{\geq 0}$

Define $\tau_{\leq j} \Pi := \bigoplus_{\substack{i=1 \\ m \geq -j}}^{n} P_i^{\oplus a_{i,m}}(m)$ "take only ind. summands with grading shifts $\geq -j$"

$(\tau_{<j} \Pi := \tau_{\leq j-1} \Pi)$

Perverse filtration of a complex in $K^b(A\text{-}\mathrm{Proj}^{\mathbb{Z}})$

Let F be a minimal complex in $K^b(A\text{-}\mathrm{Proj}^{\mathbb{Z}})$

Define $\ldots \subseteq \tau_{\leq j-1} F \subseteq \tau_{\leq j} F \subseteq \ldots$

$$^n(\tau_{\leq j} F) := \tau_{\leq j-n}(^n F)$$

$=$ " indec. summands in ^nF in graduation shifts $\geq n-j$ "

Gives a w.d. filtration of F by subcomplexes.

__Example__

$$F = 0 \longrightarrow \underset{\deg 2}{P_1(2)} \xrightarrow{\begin{pmatrix}\mathrm{id}\\ \cdot(2|3)\end{pmatrix}} \underset{\deg 3}{\cancel{P_1(2)} \oplus P_2(3)} \longrightarrow 0$$

Let $j=0$. Then $^n(\tau_{\leq 0} F) =$ " indec. summands with ^nF with grad. shifts $\geq n =$ hom. degree "

$$\rightsquigarrow \quad \tau_{\leq 0} F = 0 \longrightarrow \underset{\deg 2}{P_1(2)} \longrightarrow \underset{\deg 3}{P_2(3)} \longrightarrow 0$$

$$\begin{tikzcd}
\tau_{\leq 0} F) & 0 \arrow[r] \arrow[d] & \underset{e_2}{P_1(2)} \arrow[r, "\cdot(2|3)"] \arrow[d] & P_2(3) \arrow[r] \arrow[d, "\begin{pmatrix} 0 \\ \mathrm{id} \end{pmatrix}"] & 0 \arrow[d] \\
F) & 0 \arrow[r] & P_1(2) \arrow[r, "\begin{pmatrix} \mathrm{id} \\ \cdot(2|3) \end{pmatrix}"'] & P_1(2) \oplus P_2(3) \arrow[r] & 0
\end{tikzcd}$$

m

<u>not</u> a morphism of complexes!

$\boxed{\text{What is wrong?}}$

F is <u>not</u> minimal: $\quad P_1(2) \xrightarrow{\mathrm{id}} P_1(2)$

contractible summand (see Geordie's lecture!)

we should have taken

$$F: \quad 0 \longrightarrow 0 \longrightarrow \underset{\text{deg. 2}}{P_2(3)} \longrightarrow \underset{\text{deg. 3}}{0}$$

$$\tau_{\leq 0} F = F$$

in general we <u>do</u> have $\tau_{\leq j-1} F \subseteq \tau_{\leq j} F$

because: ① $\mathrm{Hom}(P_i, P_k(-m)) = 0$ if $m > 0$

② $\mathrm{Hom}(P_i, P_i) = k$, hence if a

map $P_i \longrightarrow P_i$ in the cplex is nonzero,
it is an iso, and the cplex is not minimal.

This defines a t-structure: set $K^{\geq 0}$ = cplexes having a minimal complex F s.t. $\tau_{<0} F = 0$

$K^{\leq 0}$ .... $\tau_{>0}(F) := F/\tau_{\leq 0}F = 0$

$K^{\geq 0}$: "shifts are bounded above by cohom. degree"

$K^{\leq 0}$: ———— "below" ————

Exercise $\exists$ distinguished triangle

(take diff'tial)

$$\begin{array}{ccccccc} \tau_{\leq 0}F & \longrightarrow & F & \longrightarrow & \tau_{>0}F & \longrightarrow & \tau_{\leq 0}F[1] \\ \cap & & & & \cap & & \\ K^{\leq 0} & & & & K^{\geq 0}[-1] & & \end{array}$$

"bounded below by ~~0~~ hom. degree"

"bounded above by hom. degree −1"

"bounded below by hom. degree +1"

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Note: $K^{\leq 0}[1] =$ " $\geq n+1$ in $\overset{\text{above}}{\vee}$ degree n "

$K^{\geq 0} =$ " $\leq n$ in degree n "

$$\boxed{\begin{array}{l} \mathrm{Hom}(K^{\leq 0}[1], K^{\geq 0}) = 0 \text{ because} \\ \qquad \mathrm{Hom}(P_i, P_j(-m)) = 0 \text{ for } m > 0! \end{array}}$$