Brav and Thomas' approach to the faithfulness of a categorical action

<u>Recall</u> If S is a finite set, $(m_{st})_{s,t\in S}$ Coxeter matrix ($m_{ss}=1$, $m_{st}\in\{2,3,\dots\}\cup\{\infty\}$ if $t\neq s$, $m_{st}=m_{ts}$)

The corresponding <u>Coxeter group</u> is defined as

$$W = \langle s\in S \mid s^2=1,\ \underbrace{st\cdots}_{m_{st}\text{ factors}} = \underbrace{ts\cdots}_{=m_{ts}} \rangle$$

no relation if $m_{st}=\infty$

braid relations

And the <u>Artin group</u>

$$Br_{\overline{W}} := \langle \underline{s}\in S \mid \text{braid relations of } \overline{W} \rangle$$

<u>Note</u>: $\underline{s}\mapsto s$ induces a canonical surjection

$$p : Br_{\overline{W}} \longrightarrow \overline{W}.$$

It admits a set-theoretic section $\overline{W}\hookrightarrow Br_{\overline{W}}$, $w\mapsto \underline{w}$ where if $s_1s_2\cdots s_k$ is a REX of w, $\underline{w}=\underline{s_1}\,\underline{s_2}\cdots\underline{s_k}$ (w.d. thanks to Matsumoto's lemma). $\underline{\overline{W}}=\{\underline{w}\mid w\in\overline{W}\}$

- $Br_W^+ := \langle \underline{s}\in S \mid \text{bd relations} \rangle_{\text{monoid}}$.

<u>Fact</u> (Brieskorn-Saito, Paris) $Br_W^+ \subseteq Br_{\overline{W}}$ (*)

$$\begin{array}{c} Br_W^+ \\ \cup \\ \underline{W} \end{array}$$

If $w, y \in \overline{W}$, we have

$$\underline{w} \leq \underline{y} \iff \ell(w) + \ell(w^{-1}y) = \ell(y)$$

left divisibility in $Br_{\overline{W}}^+$ (right)

"left weak order". (right)

<u>Right Garside normal form of a positive braid</u>

Let $\beta \in Br_{\overline{W}}$. It can be shown that β has a unique maximal right divisor x_1 in Br_W^+ among the elements in $\underline{W}$.

$\rightsquigarrow$ $\beta = \beta' x_1$ (note: β' is well-defined by (*))

We iterate: $\beta'' = \beta' x_2$, where x_2 is the maximal right divisor of β'' in $\underline{W}$.

$$\beta = x_k x_{k-1} \cdots x_2 x_1 \qquad (x_k \neq 1)$$

Right Garside normal form of β.

<u>Example</u> $W = S_3$, $Br_W \cong B_3$

$\underline{W} = \{ 1, \sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1 \}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\overset{\shortparallel}{\Delta})$

$\beta = \sigma_1 \underline{\sigma_2 \sigma_1 \sigma_2} \sigma_2 \sigma_1$

$= \sigma_1 \sigma_1 \sigma_2 \underbrace{\sigma_1 \sigma_2 \sigma_1}_{\underline{w_0}}$ $\qquad\qquad x_1 = \Delta = \underline{w_0}$

$\beta' = \sigma_1 \sigma_1 \sigma_2$ $\qquad$ <u>no</u> braid relation $\qquad \sigma_1\sigma_2 \in \underline{W}$

$x_2 = \sigma_1\sigma_2$ $\qquad\qquad\qquad\qquad \sigma_1\sigma_1\sigma_2 \notin \underline{W}$

$\beta'' = \sigma_1 \in \underline{W}$ $\quad$ so $\quad x_3 = \sigma_1$

$$\beta = (\sigma_1)(\sigma_1\sigma_2)(\sigma_1\sigma_2\sigma_1)$$

The normal form can be defined ~~and also~~ for positive braids even if W is infinite.

If W is finite: (Br_W^+, Δ) is a Garside monoid, and every $\beta \in Br_W$ can be expressed as xy^{-1}, $x, y \in Br_W^+$.

<u>Exple</u> $\quad \bullet \; W = \langle s, t \mid s^2 = 1 = t^2 \rangle \qquad (D_\infty)$

$\underline{s}\, \underline{t^{-1}}\, \underline{s}$ is never a fraction (Br_W ~~in the monoid~~ id
xy^{-1}, $x, y \in Br_W^+$

- $W = Br_3$ $\qquad \sigma_1 \sigma_2^{-1} \sigma_1$

$$= \underbrace{\sigma_1 \sigma_2^{-1} \sigma_1^{-1}} \sigma_1 \sigma_1 = \sigma_2^{-1} \sigma_1^{-1} \sigma_2 \sigma_1 \sigma_1$$

Hence if $\overline{W}$ is finite, to show that a homomorph. $\varphi : Br_W \to G$ is injective, it suffices to show that $\varphi|_{Br_W^+}$ is injective.

Now let $\overline{W}$ be of type ADE.
Let A be the corresponding zigzag algebra.

$$F_i := 0 \to \underset{\substack{\| \\ Ae_i \otimes_k e_i A}}{U_i(-1)} \to \overset{*}{A} \to 0 \quad \in K^b(A\text{-bim}^{\mathbb{Z}})$$

$$F_i \otimes_A - : \quad K^b(A\text{-Proj}^{\mathbb{Z}}) \longrightarrow K^b(A\text{-Proj}^{\mathbb{Z}})$$

satisfy the braid relations (of $\overline{W}$)

Hence $\quad Br_{\overline{W}} \circlearrowright K^b(A\text{-Proj}^{\mathbb{Z}})$ (categorical action)

Recall from last week

If $F \in K^b(A\text{-Proj}^{\mathbb{Z}})$ is a minimal complex

Perverse filtration of F.

$$\cdots \subseteq \tau_{\leq j-1} F \subseteq \tau_{\leq j} F \subseteq \cdots$$

$\tau_{\leq j-1} F$: "indec. summands in homol. degree n have grad. shifts $\geq n-j+1$"

$\tau_{\leq j} F$: " ——— ——— ——— $\geq n-j$"

$$\boxed{\mathcal{H}^j(F) := \Big(\underbrace{\tau_{\leq j}(F) \big/ \tau_{<j}(F)}_{\text{"indec. summands in degree } n \text{ have grad. shifts } = n-j\text{"}}\Big)[j]}$$

jth perverse cohomology group

Example Recall from Geordie's lecture:

$m \geq 0$

$$X_m := \quad 0 \to P_1(-2m+1) \to P_1(-2m+3) \to \cdots$$

$$\text{deg-}m \qquad \to P_1(-1) \to \underset{*}{P_2} \to 0$$

$\beta = \sigma_1^m$

$F_\beta = F_1^m$ We compute $\mathcal{H}^j(F)$ for some complexes F.

⊛ $F_\beta(\overset{*}{P_2}) \cong X_m$

$$0 \to \overset{-m}{P_1(-2m+1)} \to \overset{-m+1}{P_1(-2m+3)} \to \cdots \to \overset{-1}{P_1(-1)} \to \overset{*}{P_2} \to 0$$

$$\boxed{\begin{array}{lll} \mathcal{H}^0(X_m) = & 0 \to \overset{-1}{P_1(-1)} \to \overset{*}{P_2} \to 0 & \\ \mathcal{H}^j(X_m) = & 0 \to \overset{-1-j}{P_1(-1-2j)} \to 0 & j = 1, \dots, m-1 \\ & & \mathcal{H}^j = 0 \text{ otherw.} \end{array}}$$

⊛ $F_\beta(\overset{*}{P_1}) \cong \quad 0 \to \overset{\deg -m}{P_1(-2m)} \to 0$ (exercise!)

$$\leadsto \boxed{\begin{array}{l} \mathcal{H}^m(F_\beta(\overset{*}{P_1})) = F_\beta(\overset{*}{P_1}) \\ \mathcal{H}^j(-) = 0 \quad \text{if } j \neq m \end{array}}$$

⊛ $F_\beta(\overset{*}{P_\ell}) \cong \overset{*}{P_\ell}$ if $\ell > 2$

$$\boxed{\begin{array}{l} \mathcal{H}^0(F_\beta(\overset{*}{P_\ell})) = \overset{*}{P_\ell} \\ \mathcal{H}^j(-) = 0 \quad \text{if } j \neq 0 \end{array}}$$

What is the (right) Garside normal form of

$\beta = \sigma_1^m$? $\quad \beta = \sigma_1 \sigma_1 \cdots \sigma_1$

$x_1 = x_2 = \dots = x_m = \sigma_1$

$\beta = x_m x_{m-1} \cdots x_2 x_1$

From the previous computation: if $S = \bigoplus_{i=1}^{n} P_i$ *

Then $\mathcal{H}^j(F_\beta S) \neq 0$ only for $j = 1, 2, \dots, m$

$\mathcal{H}^m(F_\beta P_\ell) \neq 0$ only for $\ell = 1$.

m is the number of factors in the Garside normal form of β. This is a general fact:

Theorem (Brav–Thomas '11) $\beta = x_m x_{m-1} \cdots x_2 x_1$ right Garside normal form of $\beta \in Br_W^+$.

① $\max_{k \in \mathbb{Z}} \{ \mathcal{H}^k(F_\beta S) \neq 0 \} = m$

② $\mathcal{H}^m(F_\beta P_i) \neq 0 \iff \underline{s_i} \in \mathcal{L}(\cancel{\beta}\, x_m)$

($\underline{s_i}$ is a left divisor of $\cancel{\beta}\, x_m$)

Corollary $Br_W \circlearrowright K^b(A\text{-}Proj^{\mathbb{Z}})$ is faithful !

Proof Let $\beta \in Br_W^+$

~~Then β has m Garside factors $x_m x_{m-1} \cdots x_2 x_1$~~

let $m = \max_{k\in\mathbb{Z}} \{ \mathcal{H}^k(F_\beta S) \neq 0 \}$. Then by ①,

β has m Garside factors $x_m x_{m-1} \cdots x_2 x_1$.

choose i_1 s.t. $\mathcal{H}^m(F_\beta P_{i_1}) \neq 0$. Then by ②

$\underline{s_{i_1}}$ is a left descent of x_m. Hence $x_m = \underline{s_{i_1}} x_m'$

But $F_\beta \cong F_{\underline{s_{i_1}}} F_{x_m'} F_{x_{m-1}} \cdots F_{x_2} F_{x_1}$

Consider $F'_{\underline{s_{i_1}}} F_\beta \quad (\simeq F_{x_m'} F_{x_{m-1}} \cdots F_{x_2} F_{x_1})$

Note that since $x_m x_{m-1} \cdots x_2 x_1$ is the right Garside normal form of β, $x_m' x_{m-1} \cdots x_2 x_1$ is also the r. G. n.f of $\underline{s_{i_1}}^{-1} \beta$

1) If $\mathcal{H}^m(F'_{\underline{s_{i_1}}} F_\beta) = 0$, then $\underline{s_{i_1}}^{-1}\beta$ has $m-1$ Garside factors $\Rightarrow x_m' = 1$

2) otherwise, use ② to find a left descent of x_m'

$\Rightarrow$ after finitely many steps, we get a word for $x_m = \underline{s}_{i_1} \underline{s}_{i_2} \cdots \underline{s}_{i_k}$.

We then repeat to find all the Garside factors

$\Rightarrow$ the Garside normal form of β can be recovered from F_β

$\Rightarrow$ the action of Br_W^+ is faithful ~~on Br_W^+~~

$\Rightarrow$ the action of Br_W is faithful

Br_W^+ is a Garside monoid

$\square$