Affine Weyl groups and their braid groups

Fantastic reference: Iwahori-Matsumoto, Publ. IHES, 1965.

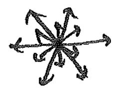
Fix a root system: $\Phi \subset \mathfrak{h}^*$, $\Phi^\vee \subset \mathfrak{h}$, Φ_+ positive roots.

$\cup$

Σ simple roots.

A_1  A_2 

Assume that our root system is irreducible.

G_2 

Weight lattice: $\mathfrak{X} = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \ \forall \alpha\}$.

For $\alpha \in \Phi$ have reflection $s_\alpha(\lambda) := \lambda - \langle \alpha^\vee, \lambda \rangle \alpha$.

<u>Remark</u>: We can and do equip $\mathfrak{h}^*$ with a Euclidean inner product s.t. s_α is orthogonal.

$$W_f = \langle s_\alpha \mid \alpha \in \Phi \rangle = \langle s_\alpha \mid \alpha \in \Sigma \rangle. \qquad \text{"finite Weyl group"}$$

$$\cap$$

$$W = W_f \ltimes \mathbb{Z}\Phi = \mathbb{Z}\Phi \rtimes W_f \qquad \text{"affine Weyl group"}$$

$$\cap$$

$$W_e = W_f \ltimes \mathfrak{X} \qquad \text{"extended affine Weyl group"}$$

Given $\lambda \in \mathfrak{X}$ write t_λ for translation by λ.

For $\alpha \in \Phi_+$, $m \in \mathbb{Z}$ consider $H_{\alpha,m} := \{\lambda \mid \langle \lambda, \alpha^\vee \rangle = m\}$.

$$s_{\alpha,m}(\lambda) := \lambda - \langle \alpha^\vee, \lambda \rangle \alpha + m\alpha.$$

Note that $s_{\alpha,m}$ is a affine reflection and fixes $H_{\alpha,m}$.

This characterises $s_{\alpha,m}$ uniquely.

Also $s_{\alpha,m} = t_{m\alpha} \circ s_\alpha = s_\alpha \circ t_{-m\alpha}$. $(*)$

<u>Lemma</u>: W is generated by $s_{\alpha,m}$.

<u>Proof</u>: Clear from $(*)$. ~~because $\mathbb{Z}\Phi$ is generated by~~

<u>Lemma:</u> W is a normal subgroup of W_e.

<u>Proof:</u> Clearly $t_\lambda(H_{\alpha,m}) = H_{\alpha,m+\langle\lambda,\alpha^\vee\rangle}$. Hence

$$t_\lambda s_{\alpha,m} t_\lambda^{-1} = s_{\alpha,m+\langle\lambda,\alpha^\vee\rangle}.$$

$\square$

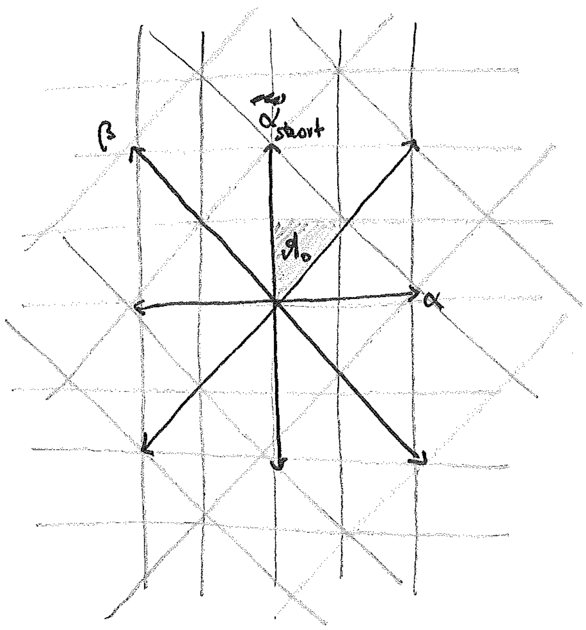
$\mathcal{H}$ = set of reflecting hyperplanes $H_{\alpha,m}$, $\alpha\in\Phi$, $m\in\mathbb{Z}$.

W_e, W act on $\mathcal{H}$.

$\rightsquigarrow$ W_e, W act on $\mathfrak{h}^* \setminus \bigcup_{H\in\mathcal{H}} H$. Connected components are called alcoves.

$$\mathcal{A}_0 = \{\lambda \mid \langle\lambda,\alpha^\vee\rangle > 0\ \forall\alpha\in\Sigma,\ \langle\lambda, \overbrace{(\underbrace{\tilde{\alpha}_{short}}_{\text{highest short root}})^\vee}^{\text{highest coroot}}\rangle < 1\}$$ fundamental alcove.

<u>Example:</u>



<u>Claim</u>: $\mathcal{A}_0$ is a connected component of $\mathfrak{h}^* \setminus \bigcup_{H \in \mathcal{H}} H$.

Suppose some $H_{\alpha,m}$ intersects $\mathcal{A}_0$. We can find $\alpha^\vee + \beta^\vee = \tilde{\alpha}^\vee$.

Choose v in this intersection.

Now:

$$\underbrace{m}_{m \geq 1} \leq \underbrace{\langle \alpha^\vee, v \rangle}_{>0} + \underbrace{\langle \beta^\vee, v \rangle}_{>0} = \langle \tilde{\alpha}^\vee, v \rangle < 1 \implies \text{contradiction.}$$

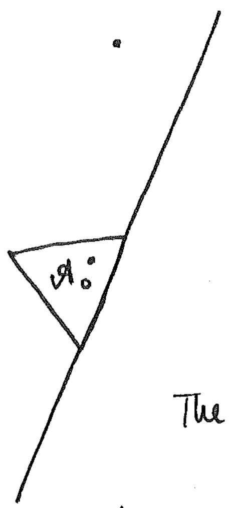
because $\alpha^\vee, \beta^\vee$ are sums of simple roots and $v \in \mathcal{A}_0$. $\square$

Let $S := \{ s_\alpha \mid \alpha \in \Delta \} \cup \{ s_{\alpha_{\text{short}}, 1} \}$ be the reflections in the walls of $\bar{\mathcal{A}}_0$.

<u>Claim</u>: ~~$\mathcal{A}_0$ is a fundamental domain for W acting on $\mathfrak{h}^*$~~

$$\bar{\mathcal{A}}_0 \twoheadrightarrow \mathfrak{h}^* / \langle S \rangle .$$

<u>Proof</u>: Choose $v \in \mathfrak{h}^*$. If $v \in \bar{\mathcal{A}}_0$ then we're done, otherwise there exists a ~~hyperplane through corresponding~~ $s \in S$ whose reflecting hyperplane separates $\mathcal{A}_0$ and v.



If ς denotes a point in the interior of $\mathcal{A}_0$ then

$$\| s(v) - \varsigma \| < \| v - \varsigma \| .$$

The set of W orbits of v is discrete, hence $\| w.(v) - \varsigma \|$ obtains a minimum.

This point must lie in $\mathcal{A}_0$.

Now we're done.

• $W = \langle S \rangle$

NOW PRINTED NOTES.

~~Corollary: W acts simply transitively on $\mathcal{A}$.~~
~~Pf: W consists of isometries, if W fixes $\mathcal{A}$~~

• Two different interpretations of length function, simply transitive.

$$||v-\rho|| > ||s(v)-\rho||.$$

Because W_S is discrete, there are finitely many points in the W_S orbit of v which are of distance at most $||v-\rho||$ from ρ. Hence, using reflections from W_S we can keep reducing the distance from ρ to v until this is no longer possible, i.e. until $v \in \Delta$.

Lemma 2.2. *$W = W_S$, i.e. W is generated by S.*

Proof. Because W is generated by the reflections it contains, it is enough to show that any reflection in W belongs to W_S. To this end, fix $\alpha \in \Phi$ and let r denote the corresponding affine reflection. Choose an alcove $A \in \mathscr{A}$ such that $F := A \cap \alpha$ is of dimension one less than V. By the previous lemma, there is an element $w \in W_S$ such that $wA = \Delta$. Let $s \in S$ be the reflection in the wall $wF \subset wA = \Delta$. Then

$$w^{-1}sw = r.$$

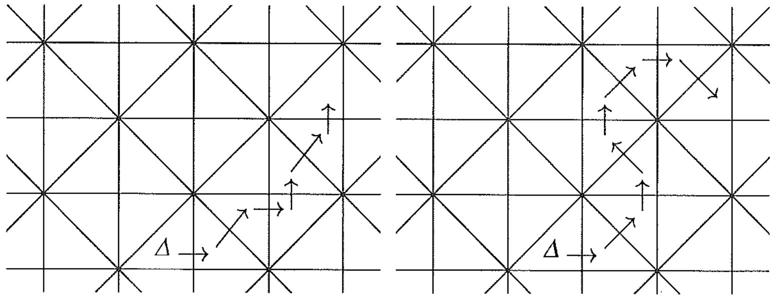
(Indeed, the left hand side is a reflection which fixes F and hence α, and hence must be r.)

2.3 Expressions and strolls

Fix an affine reflection group W acting on V, together with a choice of fundamental alcove Δ as above. Let S denote the set of reflections in the walls of Δ. An *expression* for x is a word $\underline{x} = (s_1, s_2, \dots, s_m)$ in S such that $x = s_1 s_2 \dots s_m$. The *length* $\ell(\underline{x})$ of an expression is its length as a word. An expression for x is reduced if it is of minimal length amongst all possible expressions for x. The length $\ell(x)$ is the length of a reduced expression.

A *stroll* is a sequence $\underline{A} := (A_0, A_1, \dots, A_k)$ of elements of $\mathscr{A}$ such that $A_0 = \Delta$ and A_{i-1} and A_i share a codimension 1 face F_i for all $1 \le i \le k$. We think of a stroll as a path in V beginning in Δ and only passing through codimension 1 parts of the hyperplane arrangement Φ (see the examples below). The *length* $\ell(\underline{A})$ is the number of hyperplanes crossed by the path (i.e. if $\underline{A}$ is as above then $\ell(\underline{A}) = k$). A stroll is *reduced* if F_i and F_j are never contained in the same hyperplane for $i \neq j$, i.e. if our stroll "never crosses the same reflecting hyperplane twice".

Example 2.2. Two strolls ending in the same element; one is reduced, one is not:



Remark 2.2. Starting in §3.3.5, we will redefine a stroll so that it also allows $A_i = A_{i-1}$. That is, a stroll is like a walk from alcove to alcove, where one might pause to admire the scenery. For the rest of this chapter, however, $A_i \neq A_{i-1}$.

An expression $\underline{x} = (s_1, s_2, \ldots, s_m)$ determines a stroll $\underline{A}(\underline{x})$ via

$$\underline{A}(\underline{x}) := (A_0 = \Delta, A_1 = s_1\Delta, A_2 = s_1 s_2 \Delta, \ldots, A_k = s_1 s_2 \ldots s_k \Delta).$$

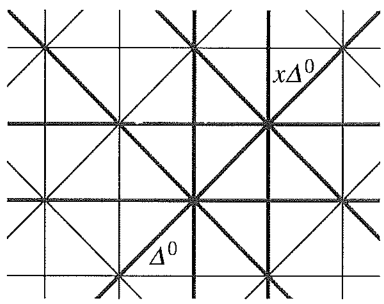
(Obviously Δ and $s\Delta$ meet in a codimension 1 face, and hence so do $x\Delta$ and $xs\Delta$ for any $x \in W$.) The following proposition tells us that (reduced) expressions and (reduced) strolls are essentially the same thing:

{2_prop:length}

Proposition 2.1. *An expression $\underline{x}$ for $x \in W$ is reduced if and only if the corresponding stroll $\underline{A}(\underline{x})$ is reduced. Moreover, we have*

$$\ell(x) = \#\{\alpha \in \Phi \mid \alpha \text{ separates } \Delta^0 \text{ and } x\Delta^0\}.$$

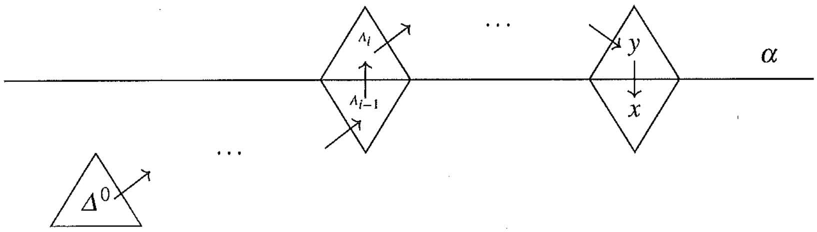
Example 2.3. The geometric meaning of $\ell(x)$:



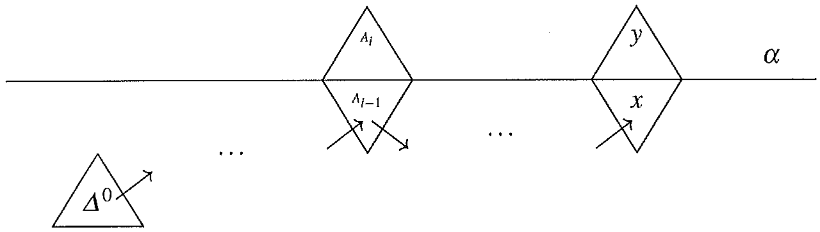
Proof. Let us temporarily define

$$\ell'(x) := \#\{\alpha \in \Phi \mid \alpha \text{ separates } \Delta^0 \text{ and } x\Delta^0\}.$$

We will argue by induction on $\ell(x)$ that $\ell(x) = \ell'(x)$ and that any reduced expression for x yields a reduced stroll. Let $\underline{x} = (s_1, \ldots, s_k)$ denote a reduced expression for x and let $\underline{y} = (s_1, \ldots, s_{k-1})$. Then $\underline{y}$ is a reduced expression for $y = s_1 \ldots s_{k-1}$ (an expression of length $< k-1$ for y would yield an expression of length $< k$ for x, contradicting $\ell(x) = k$). Thus we can apply induction to conclude that $\ell(y) = \ell'(y)$ and that $\underline{A}(y)$ crosses $k-1$ distinct hyperplanes. Now consider $\underline{A}(x)$. Either $\ell(x) = \ell'(x)$ or the hyperplane α crossed from $y\Delta^0$ to $x\Delta^0$ has already been crossed in $\underline{A}(y)$:



Let A_{i-1} and A_i with $i < k$ be two alcoves where this hyperplane is crossed earlier. Then $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{k-1})$ is an expression for x which is shorter than k. The corresponding stroll is obtained by reflecting the stroll between i and $k-1$ in the hyperplane α:



This contradicts the fact that $\ell(x) = k$. Hence $\ell(x) = \ell'(x)$ and we are done.

Corollary 2.1. $x\Delta = \Delta$ *if and only if* $x = \mathrm{id}$.

Proof. If $x\Delta = \Delta$ then x is of length zero in the generators, and hence $x = \mathrm{id}$.

Combining this result with Lemma 2.1 yields:

Corollary 2.2. Δ *is a fundamental domain for the* W*-action on* V.

In particular the map $x \mapsto x\Delta$ is a bijection. We can use this bijection to identify W and $\overline{\mathscr{A}}$. This is particularly useful as it allows us to deduce properties of W via the geometry of V and its decomposition into the sets $\overline{\mathscr{A}}$.

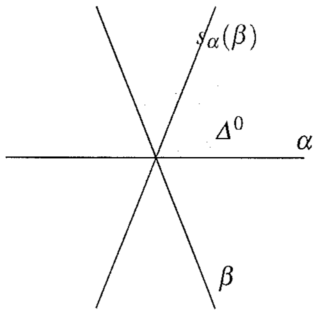
Exercise 2.3. Modify the proof of Proposition 2.1 to prove the Exchange Condition and the Deletion Condition for W (see §1.2.3).

2.4 The Coxeter presentation

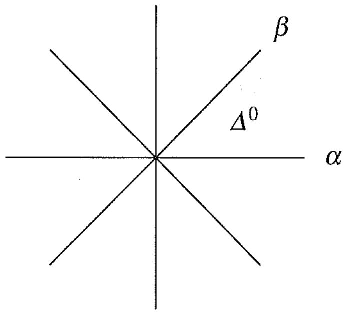
Suppose that α and β belong to Φ_Δ (i.e. α and β constitute walls of Δ).

Lemma 2.3. *If α and β intersect, then they do so at an angle $\leq \pi/2$. Moreover, this angle is of the form π/m for some $m \in \mathbb{Z}_{\geq 0}$.*

Proof. Suppose for contradiction that α and β intersect at an angle $> \pi/2$. Then reflecting β in the hyperplane α would yield a hyperplane in the interior of Δ, which is a contradiction:



To see the second claim is a piece of cake (by properness, the cake is cut into finitely many pieces):



If s and t denote the reflections in the hyperplanes $\alpha, \beta \in \Phi_\Delta$ then we define

$$m_{st} := \begin{cases} m \text{ (of the previous lemma)} & \text{if } \alpha \text{ and } \beta \text{ meet,} \\ \infty & \text{if } \alpha \text{ and } \beta \text{ do not meet.} \end{cases}$$

The composition of two reflections in distinct, parallel hyperplanes is a non-trivial translation. Meanwhile, the composition of two reflections in hyperplanes meeting at an angle of π/m is a rotation through $2\pi/m$. Hence:

Lemma 2.4. *For s, t as above, the order of $st \in W$ is m_{st}.*

We have established the easy part (i.e. that the relations are satisfied) of the following fundamental theorem:

{2_thm:Coxeter}

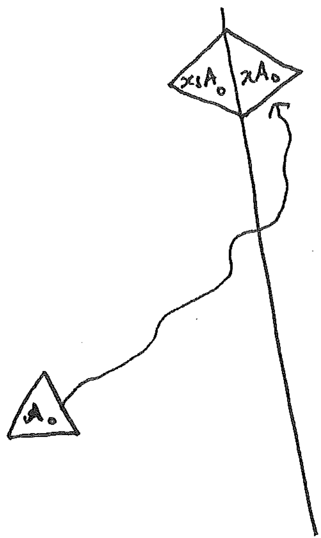
Theorem 2.1. *W admits the following "Coxeter" presentation:*

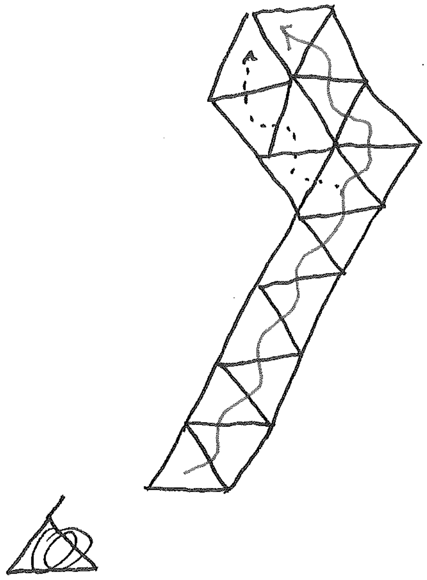
$$W = \langle s \in S \mid s^2 = \mathrm{id} \textit{ for all } s \in S, (st)^{m_{st}} = \mathrm{id} \textit{ for all distinct } s,t \in S \rangle.$$

<u>Exchange condition:</u> Suppose $x = s_1 s_2 \dots s_m$ is reduced and $s \in S$ w/ $\ell(xs) < \ell(x)$. (4)

Then there exists $1 \le i \le m$ s.t. $xs = s_1 s_2 \dots \hat{s}_i \dots s_m$.

<u>Proof:</u>





□

Definition of ℓ extends to W_{ext}:

$$\ell : W_{ext} \longrightarrow \mathbb{Z}_{\geq 0}$$

$$x \longmapsto \# \{ H \in \mathcal{H} \mid H \text{ separates } \mathcal{A}_0 \text{ and } x\mathcal{A}_0 \}.$$

$$\Omega := \text{"length zero elements"} = \ell^{-1}(0) = \{ x \mid x\mathcal{A}_0 = \mathcal{A}_0 \}.$$

Vertices of $\overline{\mathcal{A}_0}$ are $\{0, \varpi_1, \varpi_2, \dots, \varpi_{rank}\}$

and $\Omega \hookrightarrow \mathrm{Sym}(\{0, \varpi_1, \dots, \varpi_{rank}\})$ hence it is

a finite group.

~~Also,~~ $\cancel{\Omega \circlearrowright \{ H_{\alpha,0} \mid \alpha \in \Delta \} \cup \{ H_{\alpha_{short}}, 1 \}}$

$\Omega \circlearrowright$ walls of fundamental alcove $\rightarrow \Omega \circlearrowright$ affine Dynkin diagram.

Lemma: $W_{ext} = \Omega \ltimes W$

Proof: ① $\Omega \cap W = \{1\}$ follows from simple transitivity

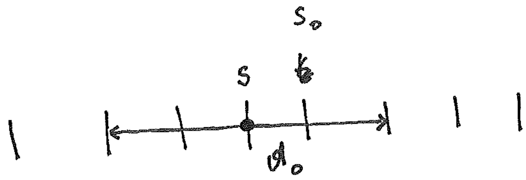
② W normal in W_{ext} explained above.

③ $W_{ext} = \langle \Omega, W \rangle$: take $x \in W_{ext}$. Because W is transitive on alcoves, $\exists\, y \in W$ s.t. xy^{-1} preserves A_0.

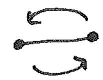
Hence $xy^{-1} \in \Omega$. □

Some examples of Ω:

① $\tilde{A}_1$

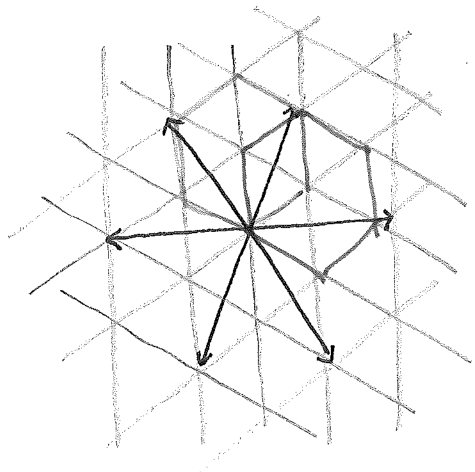


$s_0 t_{\varpi_1}$ preserves A_0

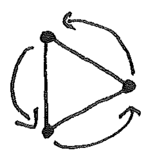


$\Omega = \mathbb{Z}/2\mathbb{Z}$

② $\tilde{A}_2$



$s_\alpha s_0 t_{\varpi_1}$ preserves A_0.



$\Omega = \mathbb{Z}/3\mathbb{Z}$

③ $\tilde{B}_2$: $\Omega = \mathbb{Z}/2\mathbb{Z}$, $\tilde{G}_2$: $\Omega = \{1\}$

Affine braid groups:

$$\mathcal{B} = \langle T_x, x \in W \mid T_x T_y = T_{xy} \text{ if } \ell(xy) = \ell(x) + \ell(y) \rangle$$

$$\mathcal{B}_e = \langle T_x, x \in W_{ext} \mid \ \dots \ \rangle.$$

$\Omega \hookrightarrow \mathcal{B}_e$ and $\mathcal{B}_e = \Omega \ltimes \mathcal{B}$.

Seek a presentation for $\mathcal{B}$, $\mathcal{B}_e$ akin to the lattice presentation of W, W_e.

For $\lambda \in \mathcal{X}$ write $\lambda = \gamma - \mu$, $\gamma, \mu \in \mathcal{X}_+$ (dominant).

Lemma: $\theta_\lambda := T_{t_\gamma} T_{t_\mu}^{-1}$ depends only on λ.

Proof: # of hyperplanes $H_{\alpha,m}$ for $m \in \mathbb{Z}$ separating $\varepsilon\varsigma$ and $\lambda + \varepsilon\varsigma$.

For $\lambda \in \mathcal{X}_+$, $\alpha \in \Phi_+$

$$= \langle \lambda, \alpha^\vee \rangle.$$

Hence: $$\ell(t_\lambda) = \sum_{\alpha \in \Phi_+} \langle \lambda, \alpha^\vee \rangle = 2 \langle \lambda, \varsigma^\vee \rangle.$$

In particular, if $\lambda, \gamma \in \mathcal{X}_+$, $T_{t_\lambda} T_{t_\mu} = T_{t_{\lambda+\mu}}$ because

$$\ell(t_\lambda t_\mu) = \ell(t_{\lambda+\mu}) = 2\langle \lambda+\mu, \varsigma^\vee \rangle = \ell(t_\lambda) + \ell(t_\mu).$$

Hence: $\Lambda_+ \hookrightarrow \mathcal{B}_e$ via $\lambda \mapsto T_{t_\lambda}$.

For $\lambda \in \mathcal{X}$ write $\lambda = \lambda' - \lambda''$, $\lambda', \lambda'' \in \mathcal{X}_+$.

$$\theta_\lambda := T_{t_{\lambda'}} T_{t_{\lambda''}}^{-1}.$$

$\square$

Loop presentation of $\mathcal{B}_{ext}$:

$$T_s \qquad s \in S_f \qquad\qquad \theta_\lambda \qquad \lambda \in \mathcal{X}$$

$$\underbrace{T_s T_t \cdots}_{m_{st}} = \underbrace{T_t T_s \cdots}_{m_{st}} \qquad \text{(finite braid relations)}$$

$$\theta_\lambda \theta_\mu = \theta_{\lambda+\mu} \qquad \text{(lattice part)}$$

$$\theta_\lambda T_s = T_s \theta_\lambda \quad \text{if} \quad \langle \lambda, \alpha_s^\vee \rangle = 0 .$$

$$\theta_\lambda T_s^{-1} = T_s \theta_{\lambda-\alpha} \quad \text{if} \quad \langle \lambda, \alpha_s^\vee \rangle = 1 .$$