4. Artin groups and Conjectures ; Garside structures.

<u>Last week</u> B_n is generated by $\sigma_1, \dots, \sigma_{n-1}$ with relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad \forall i = 1, \dots, n-1$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad |i-j| > 1$$

we can show that it gives a presentation of B_n (no other relation is needed).

<u>Note</u> The defining relations of B_n are the braid relations of the Coxeter group G_n !

<u>Def^n</u> Let (W, S) be a Coxeter system. The corresponding <u>Artin-Tits group</u> or <u>braid group</u> Br_W is defined by

$$Br_W = \langle \underline{s},\ s \in S \mid \underbrace{\underline{s}\,\underline{t} \cdots}_{m_{s,t} \text{ factors}} = \underbrace{\underline{t}\,\underline{s} \cdots}_{m_{s,t} \text{ factors}} \rangle$$

$\underline{S}$

Many basic questions about $Br_{\overline{W}}$ can be asked, and remain unsolved. (2)

Conjecture 1 (WORD PROBLEM) The word problem in $Br_{\overline{W}}$ is solvable; i.e., $\exists$ an algorithm allowing one to determine if a word in $\underline{S} \cup \underline{S}^{-1}$ represents the identity in $Br_{\overline{W}}$.

Known in several particular cases (for instance: if $\overline{W}$ is finite, via the theory of Garside monoids).

A CW-Complex X is a <u>classifying space</u> for a group G if $\pi_1(X) \cong G$ and the universal cover of X is contractible.

Recall: W finite Cox. gp $\curvearrowright V$ Tits rep^n

$$\begin{array}{c} V \\ \cup\!\shortmid \\ C \end{array} \quad \text{chamber}$$

Then $V = \bigcup_{w \in W} w\overline{C}$. If W is infinite, one has to take a chamber C rather in the dual rep^n of Tits rep^n and it is not true anymore that $V^* = \bigcup_{w \in W} w\overline{C}$. Set $I := \bigcup_{w \in W} w\overline{C}$ (<u>Tits cone</u>)

Let $M := I \times I \Big/ \bigcup_{r \in \mathrm{Refl}(W)} H_r \times H_r$ "complexify"

$\circlearrowleft W$

<u>Conjecture 2</u> ($K(\pi,1)$-CONJECTURE) $M/_W$ is a classifying space for Br_W.

$\pi_1(M/_W) \cong Br_W$ univ. cover is contractible

Proven by ~~Deligne~~ Brieskorn ~~Saito~~ (1971) and Deligne (1972) for finite W. Van der Lek (1983) showed that $\pi_1(M/_W) \cong Br_W$.

<u>Conjecture 3</u> (TORSION) Artin-Tits groups are torsion-free.

<u>Conjecture 4</u> (CENTER) Let W be infinite and irreducible. Then $Z(Br_W) = 1$.

If W is finite, it can be shown that $Z(Br_W) \cong \mathbb{Z}$.
(Chow, Garside, Brieskorn-Saito)

<u>Exercise 1</u> Let $W = \mathfrak{S}_n$. ($Br_W \cong B_n$).

(1) Show that there is a unique permutation w_0 in W of maximal length, and that $w_0 = s_1(s_2s_1)(s_3s_2s_1)\cdots(s_{n-1}s_{n-2}\cdots s_2s_1)$,

$w_0 s_i = s_{n-i} w_0 \quad \forall\, i = 1, \ldots, n-1.$

(2) Let $\Delta := \underline{s_1}(\underline{s_2}\,\underline{s_1})(\underline{s_3}\,\underline{s_2}\,\underline{s_1})\cdots(\underline{s_{n-1}}\,\underline{s_{n-2}}\cdots\underline{s_2}\,\underline{s_1})$

Show that $\Delta\, \underline{s_i} = \underline{s_{n-i}}\, \Delta \quad \forall\, i = 1, \ldots, n-1$

(Hint: Matsumoto's lemma!)

(3) Show that $\Delta^2 \in Z(B_n)$.

<u>Conjecture 5</u> (LINEARITY) Artin-Tits groups have a faithful representation $Br_W \hookrightarrow GL_A(V)$, where V is a free A-module of finite rank

Bonus we get one free: • Conj 2 $\Rightarrow$ the cohomological dim^n of Br_W is finite $\Rightarrow$ Br_W is torsion free $=$ Conj 3

• Conj 5 $\Rightarrow$ Conj 1

All five Conjectures are known for finite W.

Definitions ⊛ A monoid M is left-cancellable if for a, b, c in M,

$ab = ac \Rightarrow b = c$. right-cancellable ——

$ba = ca \Rightarrow b = c$ cancellable if both left and right cancellable.

⊛ Let $a, b \in M$ (= monoid). We say that a left-divides b if $\exists\, c \in M$: $ac = b$.

⊛ The divisibility in a monoid M is Noetherian if $\exists\, \lambda : M \longrightarrow \mathbb{Z}_{\geq 0}$ s.t. $\lambda(fg) \geq \lambda(f) + \lambda(g)$ and $g \neq 1 \Rightarrow \lambda(g) \neq 0$. ($\overset{\text{left}}{\Rightarrow}$ "$\leq$" is a partial order)

($\Rightarrow$ every element $\neq 1$ has infinite order; no nontrivial invertible elt.)

A Garside monoid (M, Δ) is a pair where M is a monoid and $\Delta \in M$, s.t.

1. M is left- and right-cancellable
2. Divisibility in M is Noetherian
3. Any two elements of M have a left- and right-gcd's and a left- and right-lcm.
4. Δ is a Garside element in M, i.e., left- and right-divisors of Δ coincide and generate M.
5. The set $\mathrm{Div}(\Delta)$ of divisors of Δ is finite.

If M is a Garside monoid, it can be shown that it admits a group of fractions $G(M)$ in which it embeds

Prove: (5bis)

Exercise 2 (Ore's Theorem) Let Π be a left and right cancellable monoid. Assume that any two elements of Π admit a left common multiple ($\forall x, y \in \Pi$, $\exists a, b \in \Pi : ax = by$). Then Π admits a group of fractions $G(\Pi)$ in which it embeds.

(Define the elements of $G(\Pi)$ as $x^{-1}y$, $x, y \in \Pi$ modulo the equiv. rel^n generated by $x^{-1}y \sim (ax)^{-1}(ay)$ $x, y, a \in \Pi$.)

(Ore's theorem)

Group structure on fractions:

$$x_1 y_1^{-1} x_2 y_2^{-1} \overset{?}{=} ?$$

Consider $a = \mathrm{lcm}(x_2, y_1)$

$\Rightarrow a = x_2 x_2' = y_1 y_1'$

$\Rightarrow$ $\boxed{y_1^{-1} x_2 = y_1' x_2'^{-1}}$ "Reverse fractions"

$\rightsquigarrow$ $x_1 y_1^{-1} x_2 y_2^{-1} = x_1 y_1' x_2'^{-1} y_2^{-1}$

A Garside group $G(M)$ is a group of fractions of a Garside monoids.

Important properties of Garside monoids and groups

(1) Normal forms in M: let $x \in M$ (= Garside monoids)
let $x_1 := \mathrm{gcd}(x, \Delta) \in \mathrm{Div}(\Delta) \Rightarrow x = x_1 \hat{x}_1$; by cancellativity, $\hat{x}_1$ is well-defined.

If $\hat{x}_i \neq 1$, define x_{i+1} as $\mathrm{gcd}(\hat{x}_i, \Delta) \in \mathrm{Div}(\Delta)$ & $\hat{x}_{i+1}$ by $\hat{x}_i = x_{i+1} \hat{x}_{i+1}$

This gives the normal form $x = x_1 x_2 \cdots x_k$ of $x \in M$ (it can be effectively computed starting from any word $x = y_1 y_2 \cdots y_\ell$ with $y_i \in \mathrm{Div}(\Delta)$; we will see how in the case of braids). $\Rightarrow$ THE WORD PROBLEM IN M IS SOLVABLE.

(2) Δ has a power which is central: Lemma Conjugation by Δ induces an autom. of M and $G(M)$ which ~~is central~~ stabilizes $\mathrm{Div}(\Delta)$.

Pf: let $x \in \mathrm{Div}(\Delta)$

$$\Delta x \Delta^{-1} = \Delta \underbrace{(\Delta x^{-1})}_{\in \mathrm{Div}(\Delta)}{}^{-1} \in \mathrm{Div}(\Delta)$$

Hence since $\mathrm{Div}(\Delta)$ is finite, $\exists\, c$: $\Delta^c x \Delta^{-c} = x \quad \forall x \in \mathrm{Div}(\Delta)$

$\Longrightarrow \quad \Delta^c g = g \Delta^c \quad \forall g \in G(M)$.

$\mathrm{Div}(\Delta)$ generates M and $G(M)$

$\boxed{\Longrightarrow \text{INFORMATION ON THE CENTER OF } G}$

③ $\forall g \in G(M), \exists i : \Delta^i g \in M$ Exercise Prove it using ②

④ Normal forms in $G(M)$: Let $g \in G(M)$. Let i be minimal s.t. $x := \Delta^i g \in M$ (note: i can be <0 !)

The normal form of g is $\Delta^{-i} x_1 x_2 \cdots x_k$, where $x_1 x_2 \cdots x_k$ is the n.f. of x in M (see ① !).

$\Longrightarrow$ $\boxed{\text{THE WORD PROBLEM IN } G(M) \text{ IS SOLVABLE}}$

⑤ Garside groups are torsion free : extend $\overset{(\text{left})}{\leq}$ to $G(M)$ by setting: $x \leq y \iff x^{-1} y \in M$; then $(G(M), \leq)$ is still a lattice : if $x, y \in G$ and i is s.t. $\Delta^i x, \Delta^i y \in M$, then $\gcd(x, y) = \Delta^{-i} \gcd(\Delta^i x, \Delta^i y)$; same for lcm's

Assume $x^p = 1$ in $G(M)$. Let $c := \mathrm{lcm}\{1, x, \dots, x^{p-1}\}$

$\Longrightarrow \quad x^{-1} c$ is lcm of $x\{1, x, \dots, x^{p-1}\} = \{1, x, \dots, x^{p-1}\}$

$\Longrightarrow \quad x^{-1} c = c \quad \Longrightarrow \quad x = 1$.

<u>Exercise 4</u> Show that $((\mathbb{Z}^n_{\geq 0}, +), (1,1,\dots,1))$ is a Garside monoid

<u>Note</u> Given a group G, finding $M \leq G$ (Garside monoid) s.t. $G = G(M)$ gives
- a solution to the word pblem in G
- information on the center of G
- a proof that G is torsion free

A Garside monoid (M, Δ) for $G(M) = G$ is not unique in general! (It can be shown that (M, Δ^k) is a G.M. for $G(M)$ $\forall\, k \geq 1$; but one may have a G.M. (M', Δ') with $M' \not\cong M$)

<u>The (classical) Garside structure on Artin-Tits groups of spherical type</u>

Let (W, S) be a Coxeter system. We say that Br_W has <u>spherical type</u> if W is finite.

<u>Proposition</u> Let (W,S) be a Coxeter system and let w_0 be an element of W. TFAE

(i) $\ell(w_0 s) < \ell(w_0)$ $\forall\, s \in S$

(ii) $\ell(w_0 w) = \ell(w_0) - \ell(w)$ $\forall\, w \in W$

(iii) w_0 has maximal length among elt of W

Moreover, if such an elt exists, it is unique and W is finite

<u>Proof</u> <u>Exercise 5</u>!

~~Assume th~~

Let Br_W^+ be the monoid with the same presentation as Br_W.
It is not clear that $Br_W^+ \hookrightarrow Br_{\tilde{W}}$.

The surjective map $Br_{\tilde{W}} \twoheadrightarrow W$ (or $Br_W^+ \twoheadrightarrow W$) has a set-theoretic section; given $w \in \tilde{W}$, let $w = s_1 s_2 \cdots s_k$ be a REX of w. Set $\underline{w} := \underline{s_1}\,\underline{s_2} \cdots \underline{s_k} \in Br_W^+ / Br_W$. By Matsumoto's lemma, $\underline{w}$ is well-defined.

Let $\underline{W} := \{ \underline{w} \mid w \in \tilde{W} \} \subseteq Br_{W^+} / Br_W$ "<u>positive simple braids</u>"

Theorem (W finite: Brieskorn–Saito 1972, Deligne 1972
W arbitrary: Paris 2002, Jensen 2017)
(categorical actions!)

Br_W^+ embeds into $Br_{\tilde{W}}$. (embed Br_W^+ in $Br_{W'}^+$ for which it holds)

If one believes Conjecture 4, there should be no Garside structure on Br_W if W is infinite. But we have

Theorem (Garside 1969 (Bn), BS '72) Let (W,S) be finite. Let w_0 be the longest element in $\tilde{W}$. Let $\Delta := \underline{w_0}$. Then (Br_W^+, Δ) is a Garside monoid, with corresponding Garside group $Br_{\tilde{W}}$.

($\Rightarrow$ Conj. 1, 3 for finite $\tilde{W}$. Conj. 2 can also be proven using the Garside theoretic properties)

[For infinite $\tilde{W}$, two approaches: ① embed Br_W in a Garside gp for Conj 1
② use a ~~weaker~~ more general theory of 'Garside families'.]

The only point in the def^n of Garside monoid which is clearly true is (2) (because the defining presentation of Br_W^+ is homogeneous)

<u>Exercise 6</u> (1) Let $u, v \in W$ (not nec. finite). Show that

$$\underline{u} \leq \underline{v} \ (\text{in } Br_W^+) \iff \ell(u) + \ell(u^{-1}v) = \ell(v)$$

"left weak order on W"

(2) Conclude that $\mathrm{Div}(\Delta) = \underline{W}$ if W is finite.

Given $\beta \in Br_W^+$, let $L(\beta)$ (resp. $R(\beta)$) be the set $\{\underline{s} \mid s \in S, \ \underline{s}\beta' = \beta$ for some β' in $Br_W^+\}$

———— $\beta'\underline{s} = \beta$ ————

More generally, we can define the normal form of $x \in Br_W^+$ (W not nec. finite) as a dec^n $x = x_1 x_2 \cdots x_k$, $1 \neq x_i \in \underline{W}$ and $\forall i, \ \mathcal{L}(x_{i+1}) \subseteq R(x_i)$. For finite W it can be shown that it is equivalent to the given definition.

⚠ It is not clear that such a decomposition is unique! We will show it later as an application of categorification.

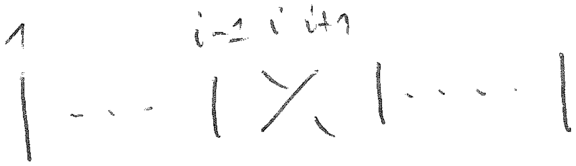
For now we will assume it.

<u>Exercise 7</u> Compute the Garside normal form of

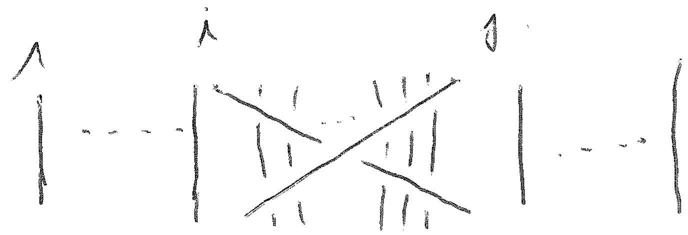
$\sigma_2\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1\sigma_1\sigma_1$ in $Br^+_{\mathfrak{S}_4}$. (using the second def^n of normal form)

Alternative Garside structures on Br_W

Let $W = \mathfrak{S}_n$. Then Br_W^+ is generated by

σ_i  $i = 1, \dots, n-1$

Consider, for $i<j$, the (bigger) set of ~~generators~~ elements of Br_W

$\sigma_{i,j}$ 

<u>Theorem</u> (Birman–Ko–Lee '98) The $\sigma_{i,j}$ generate a monoid M which is a Garside monoid for Br_W (with Garside element $\cancel{\sigma_1\sigma_2\cdots\sigma_n}$ $\sigma_n\sigma_{n-1}\cdots\sigma_1$)

→ improves the sol^n for the word problem

(They generalize to 'dual braid monoids', which give alternative Garside structures on Br_W for finite W (Bessis '04); and for cplex reflection groups)

<u>Open problem</u>: Is the monoid $\langle \sigma_1, \sigma_1\sigma_2, \dots, \sigma_1\sigma_2\cdots\sigma_{n-1}\rangle$ $\subseteq B_n$ a Garside monoid for B_n? (Known only for $n=3$)

→ classification of Garside structures on B_n?