$\boxed{\text{Theme in mathematics}}$: non-linear $\cdots\cdots>$ linear $\cdots\cdot>$ numbers

<u>Algebraic topology</u> : spaces $\rightsquigarrow$ homotopy types $\rightsquigarrow$ Spectra $\rightsquigarrow$ k-linear categories linear algebra

triangulated categories $\rightsquigarrow$ V $\rightsquigarrow$ betti numbers

additive categories

<u>Algebraic geometry:</u> varieties $/\mathbb{Q}$ $\rightsquigarrow$ derived categories $\rightsquigarrow$ Motives $\rightsquigarrow$ Galois representations $\rightsquigarrow$ # points

"soften the landing"

We will see this same pattern play out for braid groups.

G is a group.

presentation, — not very useful. Undecidable if I have the braid group.

faithful action on a space or other non-linear object.

(e.g. $B_n \circlearrowright F_n$ (free group)

faithful action up to homotopy

$B_n \circlearrowright D_n$ via mapping classes

action on triangulated categories

faithful action on a vector space. (<u>linear</u>)

<u>Def:</u> A group G is <u>linear</u> if it admits a faithful finite-dimensional representation.

<u>Example:</u> Any finite group or matrix group is linear.

②

<u>Exercise:</u> ⓐ Show that $(\mathbb{Z}/n\mathbb{Z})^{\infty}$ is not linear.

ⓑ Let X be a countably infinite set. Show that $\mathrm{Perm}(X)$ is not linear.

ⓒ* Show that $\mathbb{Z}^{\infty}$ is not linear. Show no faithful rep into $GL_2(\mathbb{C})$ first.

ⓓ* ~~S~~ Learn what Thompson's group F is, and show that it is not linear.

(f.g. and infinite, "almost" a simple group)

Why do we care about linearity?

① $\Rightarrow$ word problem, conjugacy problem solvable.

② $\Rightarrow$ G is <u>residually finite</u>. (Malcev, 1940)

(Rough idea: $G \hookrightarrow GL_n(\mathbb{Z})$, $G \twoheadrightarrow K \subset GL_n(\mathbb{F}_p)$ $\forall p$.)

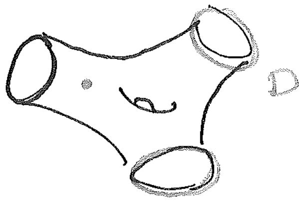
<u>Def:</u> G is <u>residually finite</u> if $\forall g$ in G there exists $\pi: G \to K$ with K finite and $\pi(g) \neq id$.

$\Updownarrow$

$$G \hookrightarrow \hat{G} = \varprojlim_{\substack{H \subset G \\ \text{finite index}}} G/H$$ "profinite completion"

<u>Exercise:</u> Show that $\hat{\mathbb{Z}} = \prod_{p \text{ prime}} \mathbb{Z}_p$.

<u>Mapping class groups</u>



Let X denote a ~~Riemann~~ surface w/ boundary and $Z \subset X$ a closed subset.

$$MCG(X, Z) = \text{orientation preserving self-homeomorphisms of } X \text{ which fix } Z \text{ pointwise.} \Big/ \text{isotopy}$$

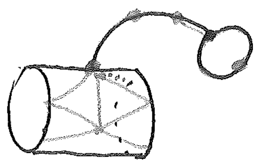
E.g. $X = D_n$ = ◯ (× × × ×) closed disc, $Z = \partial$ = boundary. $MCG(X, B)$ = braid group B_n.

<u>Open question:</u> Are mapping class groups linear?

Many consequences are known and are very difficult.

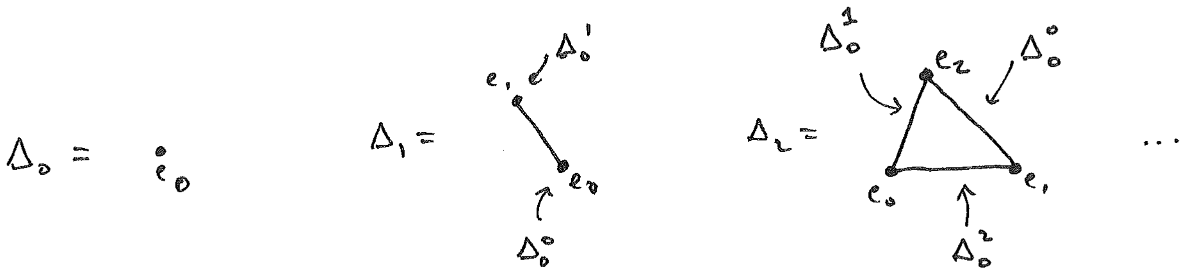
For the next few lectures we will study representations of braid groups.

Faithfulness is often a very difficult question.

<u>Homology:</u> $X =$  how complicated is X? $\rightsquigarrow$ might try to triangulate it ... but there are many triangulations, "simplest" is difficult to describe.

homology = "easy" to compute, gives powerful invariants of X.

$\Delta_n = n\text{-simplex} = \{ \lambda \in \mathbb{R}^{n+1} = \mathbb{R}e_0 \oplus \mathbb{R}e_1 \oplus \dots \oplus \mathbb{R}e_n \mid \sum \lambda_i = 1,\ 0 \le \lambda_i \le 1 \}.$



A <u>singular n-simplex</u> is a continuous map $f : \Delta_n \to X$.

$C^n(X)$ = free abelian group w/ basis singular n-simplices.

Eg. $C^0(X)$ = " " " " points of X. ("enormous").

For $i = 0, \dots, n$ Δ_n contains a copy Δ_{n-1}^i of Δ_{n-1} $(e_i = 0)$.

Given $f : \Delta_n \to X$, $\partial f = \sum_{i=0}^{n} (-1)^i f|_{\Delta_n^i}$.

Induces $\partial : C^n(X) \to C^{n-1}(X)$.

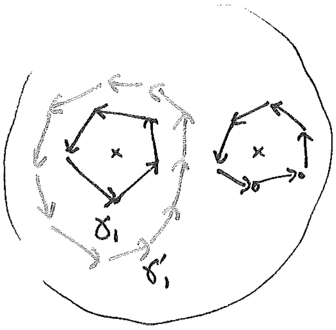
<u>Exercise:</u> $\partial^2 = 0$. ("Most important equation in mathematics." Y-Manin.)
(doesn't count.)

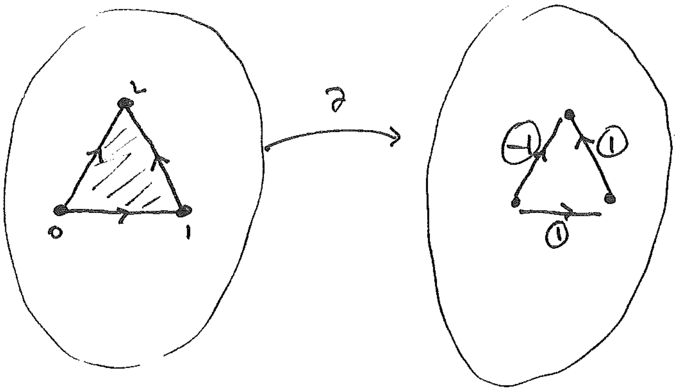
$$H^i(X) := \left(\ker \partial : C^i(X) \to C^{i-1}(X)\right) \Big/ \left(\operatorname{im} \partial : C^{i+1}(X) \to C^i(X)\right)$$

"cycles" "boundaries".

E.g. $H^0(X) = \mathbb{Z} \cdot \{\text{path components of } X\}$.

E.g: $H^1(D_2) = \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2$

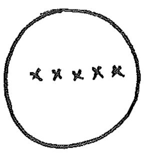




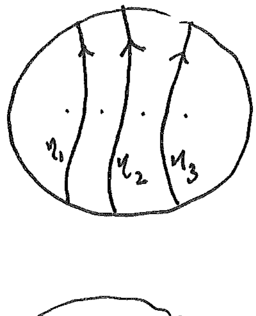
Exercise: $\gamma_1 = \gamma_1'$.

We will also need $H^\bullet_i(X, Z)$

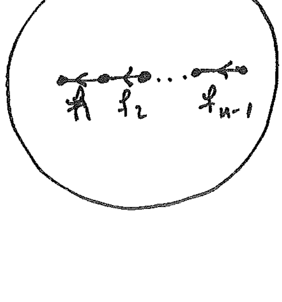
$C^i(X, Z) = C^i(X) \Big/$ subgroup generated by singular n-simplices with ~~verts~~ image in Z.

$X = D_n =$ 

Example: $H_1(X, \partial) = \mathbb{Z}\eta_1 \oplus \dots \oplus \mathbb{Z}\eta_{n-1}$



$H_1(D_n, \{p_1, \dots, p_n\}) = \mathbb{Z}f_1 \oplus \dots \oplus \mathbb{Z}f_{n-1}$

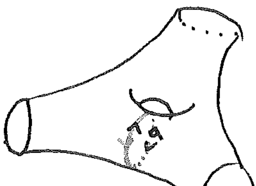


Homological representation:

$$\begin{tikzcd} \mathrm{Homeo}(X \text{ fixing } Z) \arrow[rr] \arrow[dr] & & \mathrm{Aut}(H_1(X,Z)) \\ & MCG(X,Z) \arrow[ur] & \end{tikzcd}$$

rarely faithful.

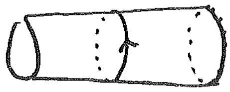
General reason for failure of faithfulness is given by analysing **Dehn twists**.



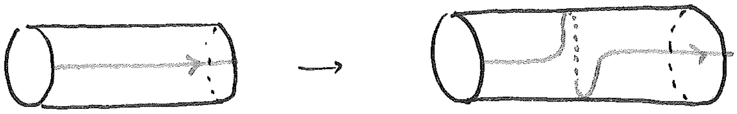
γ

γ a simple closed curve on X.

Neighbourhood looks like $(-1,2)\times S^1$

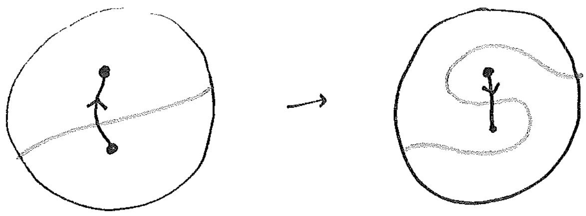


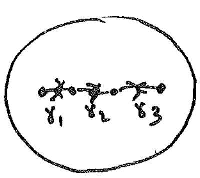
$$(\lambda,\gamma) \longmapsto \begin{cases} (\lambda,\gamma) & \text{if } \lambda \notin [0,1] \\ (\lambda, e^{2\pi i\lambda}\cdot\gamma) & \lambda\in[0,1]. \end{cases}$$



$\rightsquigarrow \tau_\gamma \in MCG.$

Similarly: γ simple closed curve between two punctures $\rightsquigarrow$ ½ Dehn twist σ_γ



E.g. $D_n =$ 

$\sigma_{\gamma_i} = \sigma_i$ generator of braid group.

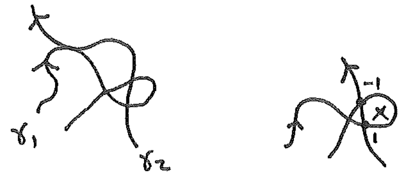
Fact: MCG is generated by Dehn twists.

Relations are very difficult to describe in general.

$H_1(X)$ carries "intersection form".

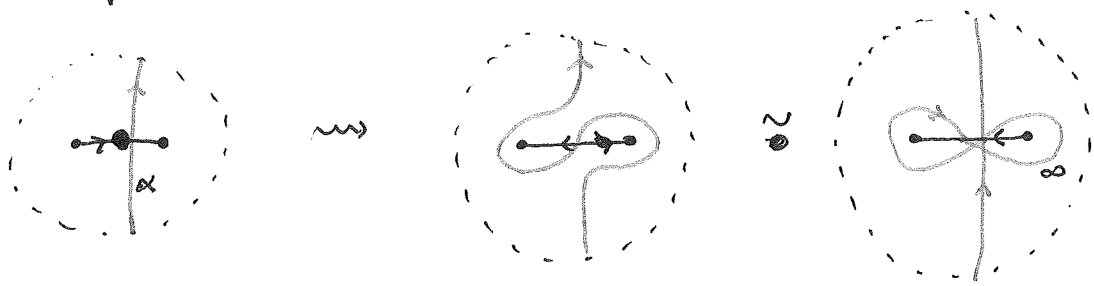
$$\langle \gamma_1, \gamma_2 \rangle = \langle \gamma_1', \gamma_2' \rangle = \sum_{p \in \gamma_1 \cap \gamma_2} \varepsilon_p$$

move so that they intersect transversally

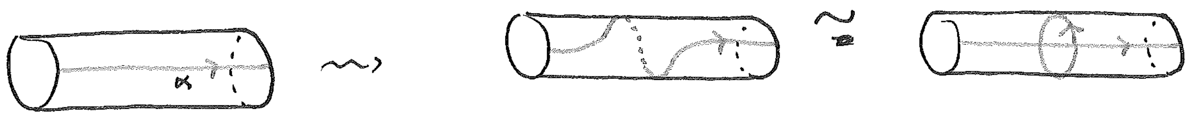


well-defined.

Let us compute how ½-Dehn twists act on homology.



$$[\alpha] \longmapsto [\alpha] + \langle \gamma, \alpha \rangle [\infty].$$

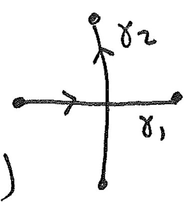


$$[\alpha] \longmapsto [\alpha] + \langle \alpha, \gamma \rangle [\gamma].$$

<u>Moral</u>: If $\langle \gamma, \gamma' \rangle = 0$, then the actions of τ_γ and $\tau_{\gamma'}$ commute in homology.

<u>Fact</u>: τ_γ and $\tau_{\gamma'}$ commute in MCG $\Longleftrightarrow$ γ is isotopic to a curve <u>disjoint</u> from γ'.

<u>Exercise</u>: (Very worthwhile!)



Show that σ_{γ_1} and σ_{γ_2} do not commute in MCG.