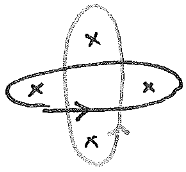
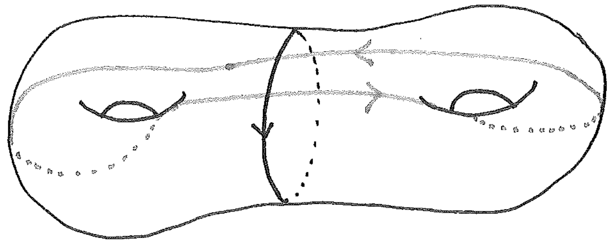
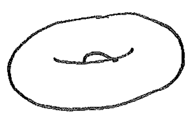
Thus, to find elements in the kernel of homological rep,

just need to find curves which are not disjoint, but whose intersection

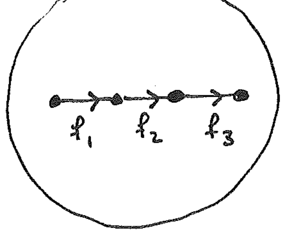
pairing is zero.

E.g. ①  ② 

Exercise: No such curves exist on , in this case

MCG $\longrightarrow SL_2(\mathbb{Z})$ is an isomorphism.

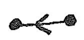
Homological representations of braid groups



$$B_n \circlearrowright \operatorname{Aut}(H_1(D_n, \{p_1, \dots, p_n\}))$$

$$\parallel$$

$$\mathbb{Z}f_1 \oplus \mathbb{Z}f_2 \oplus \cdots$$

$\sigma_1(f_1) =$  $= -f_1$.

$\sigma_1(f_2) =$  $= f_1 + f_2$.

$$\sigma_i(f_j) = \begin{cases} -f_j & \text{if } i = j \\ f_i + f_j & \text{if } |i-j| = 1 \\ f_j & \text{if } |i-j| > 1 \end{cases}$$

Conclusion: Action factors over $B_n \twoheadrightarrow S_n \circlearrowright H_1(D_n, \{p_1, \dots, p_n\})$.

(~~natural~~ reflection representation of S_n)

⑧

Some covering space theory $\underset{x}{\overset{X}{\Downarrow}}$ path connected, locally path connected, "semi-locally simply connected".

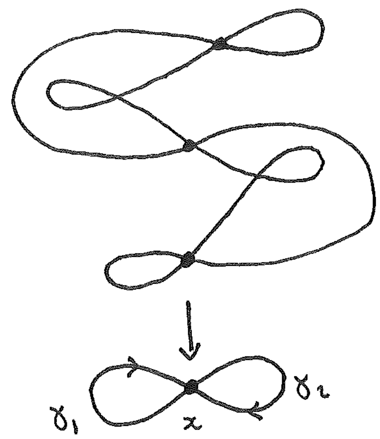
$\pi_1(X, x)$ = ~~isotopy~~ homotopy classes of maps $\begin{matrix} S^1 & \longrightarrow & X \\ \Downarrow & & \Downarrow \\ 1 & \longmapsto & x \end{matrix}$ "fundamental group".

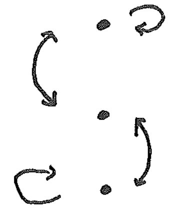
Fundamental theorem: ("Galois correspondence")

$$\{\text{covering spaces of } X\} \xleftrightarrow{\sim} \text{sets w/ } \pi_1(X, x)\text{-action}$$

$\cup$ connected $\qquad\qquad$ $\cup$ transitive

E.g.:





$\gamma_1 \quad \gamma_2$

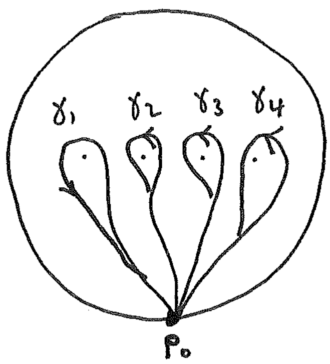
$$\begin{matrix} \tilde{X} \\ \downarrow \\ X \end{matrix} \longrightarrow f^{-1}(x)$$

Moreover: $f : (Y, y) \longrightarrow (X, x)$ lifts to $(\tilde{X}, \tilde{x})$

if and only if the action of $\pi_1(Y, y)$ on $\tilde{x}$ is trivial.

If it lifts the lift is unique.

D_n :

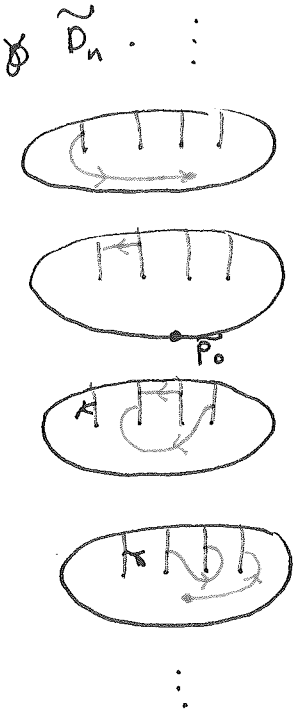


$\pi_1(D_n, p_0)$ is a free group with generators $\gamma_1, \gamma_2, \gamma_3, \gamma_4$.

Consider $\mathbb{Z}$ as a $\pi_1(D_n, p_0)$ set in which γ_i, ~~$\gamma_2, \gamma_3, \gamma_4$~~ all act by translation by 1.

i.e. $F_n \longrightarrow \mathbb{Z}$ mapping each $\gamma_i \mapsto 1$.

This determines a covering space of $\widetilde{D}_n$.



"infinite carpark".

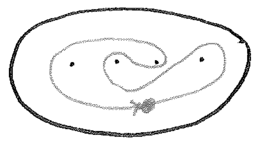
Explicit description:

① take $\mathbb{Z}$ many copies of D_n.

② cut each D_n vertically from p_i to the boundary

③ Glue RHS of each cut to left hand side above.

$\mathbb{Z} \curvearrowright \widetilde{D}_n$ via deck transformations



Let K denote $\ker(F_n \to \mathbb{Z})$, $\quad K = \pi_1(\widetilde{D}_n, \tilde{p}_0)$

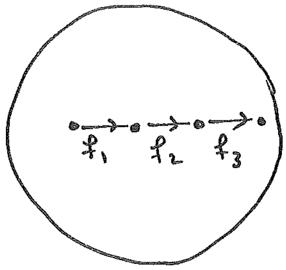
$B_n = MCG(D_n, \partial) \curvearrowright \pi_1(D_n) = F_n$, preserves K.

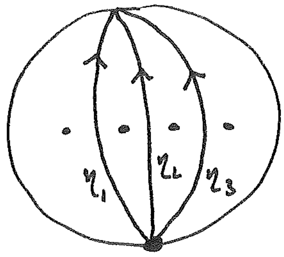
$\Rightarrow$ Any $\sigma \in B_n$ has a unique lift $\tilde{\sigma} : \widetilde{D}_n \to \widetilde{D}_n$ s.t. $\tilde{\sigma}(\tilde{p}_0) = \tilde{p}_0$.

$\rightsquigarrow$ Burau representation $\quad B_n \curvearrowright H_1(\widetilde{D}_n, \tilde{\partial})$

$\curvearrowright H_1(\widetilde{D}_n, \widetilde{\{p_1, \dots, p_n\}})$.

Recall: $H_1(D_\bullet, \{p_1,\dots,p_n\}) = \mathbb{Z}f_1 \oplus \mathbb{Z}f_2 \oplus \dots \oplus \mathbb{Z}f_{n-1}$





$H_1(D_n, \partial) = \mathbb{Z}\eta_1 \oplus \dots \oplus \mathbb{Z}\eta_{n-1}$

Moreover, $\langle f_i, \eta_j \rangle = \delta_{ij}$, hence the two spaces are in duality.

Let q denote the deck translation "up 1".

q acts on $H_1(\tilde{D}_n, \tilde{\partial})$, $H_1(\tilde{D}_n, \widetilde{\{p_1,\dots,p_n\}})$, $\mathbb{Z}[q,q^{-1}]$-modules.

$\eta_1,\dots,\eta_n$ lift to a basis of $H_1(\tilde{D}_n, \dots, \tilde{\partial})$.

$f_1,\dots,f_n$ lift to a basis of $H_1(\tilde{D}_n, \widetilde{\{p_1,\dots,p_n\}})$.

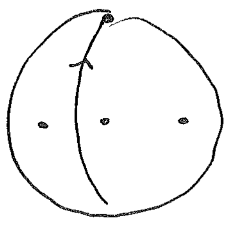
We add a handle to f_i to remember in which layer we are.

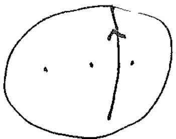
We define a new intersection form $\langle -,- \rangle$

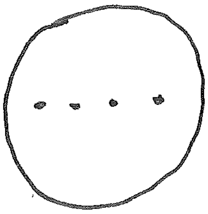
$$H_1(\tilde{D}_\bullet, \widetilde{\{p_1,\dots,p_n\}}) \times H_1(\tilde{D}_n, \tilde{\partial}) \longrightarrow \mathbb{Z}[q,q^{-1}].$$

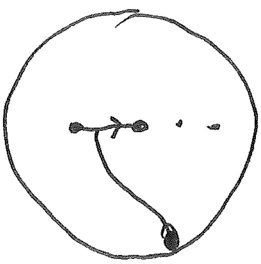
$$\langle \gamma, \delta \rangle := \sum \langle \gamma, q^i \delta \rangle q^i.$$

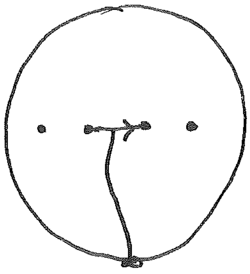
Example:

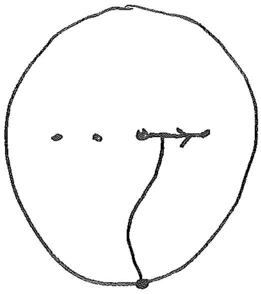
$\langle$  ,  $\rangle = -q^2$.

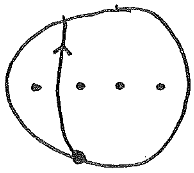
 $= -q$.

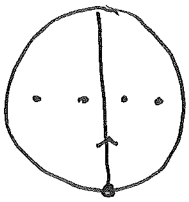
$D_4 =$ 

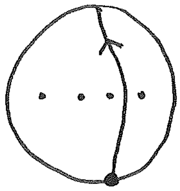

f_1


f_2

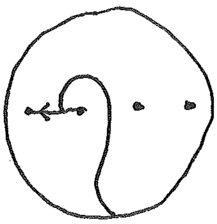

f_3

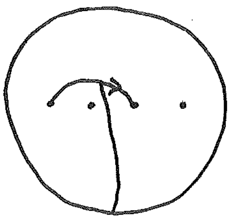

n_1


n_2


n_3 .

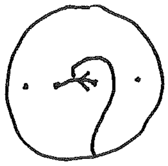
$$\langle f, g \rangle := \sum \langle f, q^i g \rangle q^i \in \mathbb{Z}[q^{\pm 1}]. \qquad \langle f_i, n_j \rangle = \delta_{ij}.$$

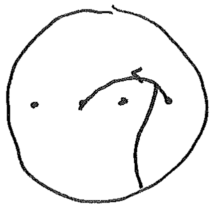
$\sigma_1(f_1) =$  $\begin{matrix}-q\\0\\0\end{matrix}$

$\sigma_1(f_2) =$  $\begin{matrix}q\\1\\0\end{matrix}$

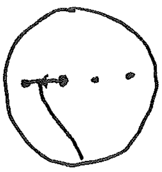
$$\sigma_1 = \begin{pmatrix} -q & q & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

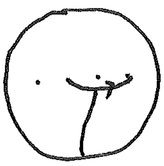
$\sigma_2(f_1) =$ 

$\sigma_2(f_2) =$ 

$\sigma_2(f_3) =$ 

$$\sigma_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -q & q \\ 0 & 0 & 1 \end{pmatrix}$$

$\sigma_3(f_1) =$ 

$\sigma_3(f_2) =$ 

$\sigma_3(f_3) =$ 

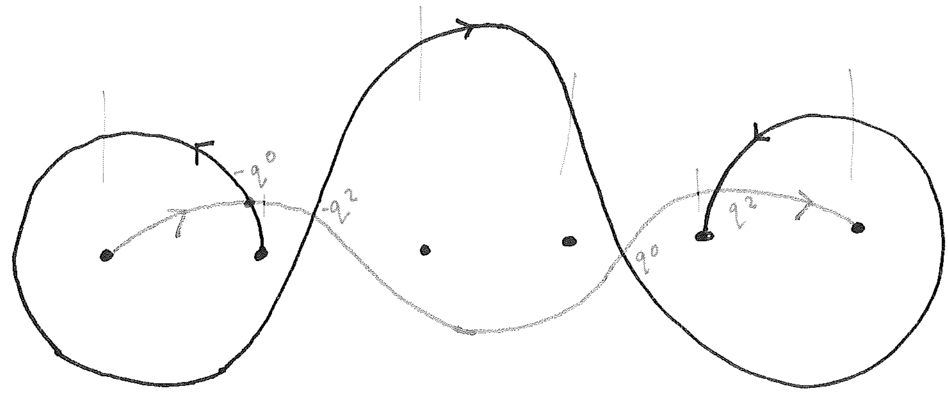
$$\sigma_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -q \end{pmatrix}$$

The question of faithfulness of the Burau representation was open for 50 years.

<u>Thm:</u> (Moody '91, Long-Paton '93, Bigelow '99).

Suppose that we can find two simple curves γ_1, γ_2 which such that γ_1 cannot be isotoped off γ_2 but with $\langle \tilde{\gamma}_1, \tilde{\gamma}_2 \rangle = 0$, where $\tilde{\gamma}_i$ denote lifts to $\tilde{D}_n$. Then the Burau representation is not faithful.

<u>Example</u> (Bigelow)



There is a complicated example for $n=5$ (see Bigelow's paper).

<u>Exercise:</u> Show that no such curves exist for $n=3$.

$n=4$ is open! I suspect no such curves exist.

Hecke algebras:

Recall that there is a braid group associated to any Coxeter system:

$$W = \langle\, s \in S \mid \cancel{st}\ s^2 = id,\ (st)^{m_{st}} = 1 \,\rangle$$

$$\underbrace{sts\ldots}_{m_{st}} = \underbrace{tst\ldots}_{m_{st}}$$

$$B_W = \langle\, \sigma_s,\ s \in S \mid \underbrace{\sigma_s \sigma_t \sigma_s \ldots}_{m_{st}} = \underbrace{\sigma_t \sigma_s \sigma_t \ldots}_{m_{st}} \,\rangle$$

Eg. $W = S_n$ $\quad 1 - 2 - 3 \cdots - n-1 \qquad B_W = B_n$, classical braid group.

Suppose we wish to study representations of B_W algebraically.

Idea: W is a well-studied and understood group,

can we "deform away" from W?

Step 1: reps of W satisfy $s^2 = id$ $\rightsquigarrow$ forces eigenvalues of s to belong to $\{\pm 1\}$.

What happens if we require σ_s to be semi-simple w/ eigenvalues $\{a, b\} \subset k^\times$.

Replacing σ_s by $\lambda \sigma_s$ we can assume eigenvalues are in $\{-1, q\}$.

Relation: $(\sigma_s + 1)(\sigma_s - q) = 0$

$$\sigma_s^2 - (q-1)\sigma_s - q \implies \sigma_s^2 = (q-1)\sigma_s + q.$$

H $\qquad\qquad$ $\mathbb{Z}[q^{\pm 1}]$

Def$^{\underline{n}}$ The Hecke algebra is the algebra w/ generators

T_s, $s \in S$ and relations

$$\underbrace{T_s T_t \cdots}_{m_{st}} = \underbrace{T_t T_s \cdots}_{m_{st}} \qquad T_s^2 = (q-1)T_s + q.$$

"braid relation" $\qquad\qquad$ "quadratic relation".

For any $w \in W$, choose a ~~red~~ rex $\underline{w} = (s_1, \dots, s_\ell)$ and define

$T_w := T_{s_1} \cdots T_{s_\ell}$. ~~Then T_w is a~~

Thm: T_w is a basis of H, the "standard basis".

Rmk: Thus H is a flat deformation of the group algebra of W.

Many interesting representations of braid groups factor over H.

Open problem: Is the map $B_W \to H_W$ injective?

True for rank 2 (e.g. S_3), unknown for S_4 and equivalent to faithfulness of Burau rep$^{\underline{n}}$.

Question is very closely tied to the question:

does the Jones polynomial detect the unknot.

Theorem (Benson-Curtis, Lusztig)

$$H \otimes \mathbb{Q}(q^{1/2}) \xrightarrow{\sim} \mathbb{Q}(q^{1/2})W.$$

In particular, any representation of W may be "deformed" to yield a representation of H.

Example: Consider the reflection rep of S_n: 

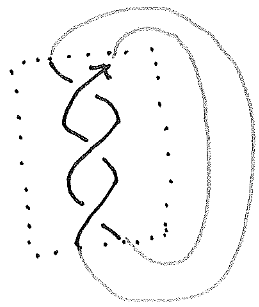
It deforms to give (almost) the Burau representation.

$$T_{s_i} \longmapsto -[\sigma_{s_i}] \circlearrowright H_1(\tilde{D}_n, \{p_1, \dots, p_n\}).$$

Why? K knot

Alexander

We can write K as the closure of a braid:



$J(K) \in \mathbb{Z}[q^{\pm 1}]$ Jones polynomial.

We can evaluate $J(K)$ as follows:

$$B_n \longrightarrow H \xrightarrow{\text{Jones trace}} \mathbb{Z}[q^{\pm 1}]$$

If this map isn't faithful, it is hard to believe that the Jones polynomial can detect the unknot.