7. The Lawrence-Krammer-Bigelow representation

<u>Recall</u> A <u>Garside monoid</u> (M, Δ) is a monoid M with an element $\Delta \in M$ s.t.

1. M is left- and right-cancellable
2. The divisibility in M is Noetherian $\exists \lambda : M \to \mathbb{Z}_{\geq 0}$ s.t. $\lambda(fg) \geq \lambda(f) + \lambda(g)$ & $g \neq 1 \Rightarrow \lambda(g) \neq 0$
3. Every two elements in M have left- and right-lcm's and gcd's
4. The set $\mathrm{Div}(\Delta)$ of left divisors of Δ coincides with the set of right divisors of Δ and generates M.
5. $\mathrm{Div}(\Delta)$ is finite

In such a situation, M has a group of fractions $G(M)$ in which it embeds, and the word problem in M and $G(M)$ is solvable.

$W \xleftrightarrow{1:1} \mathrm{Div}(\Delta)$, $w \to \underline{w}$

<u>Example</u> $M = Br_W^+$, where W finite Cox. gp., $\Delta = \underline{w_0}$, $w_0 \in W$ longest element in W, is a G.M. with $G(M) \cong Br_W$.

Open problem (M. Elder) Is every Garside group linear ?

$n=3$ cf last week.

B_n : Krammer, $n=4$ (2000)
Bigelow : all n (top. pf) (2001)
Krammer : all n (uses the Garside struct., 2002)

[can be used to generalize faithfulness to every Artin group attached to a finite crystallographic Coxeter group.
(Cohen–Wales 2002, Digne 2003)]

Today : Explain Krammer's proof & ideas.

Garside normal form Let (M, Δ) be a Garside monoid.

Let $x \in M$. Define $x_1 = \gcd(x, \Delta)$, $\tilde{x}_1$ by $x = x_1 \tilde{x}_1$ (well-defined by cancellability)

$x_{i+1} = \gcd(\tilde{x}_i, \Delta)$ if $\tilde{x}_i \neq 1$, $\tilde{x}_i = x_{i+1} \tilde{x}_{i+1}$

$\rightsquigarrow$ $x = x_1 x_2 \cdots x_k$ is the Garside normal form of $x \in M$. Write $LF(x) := x_1 = \gcd(x, \Delta)$
"Leftmost factor"

Important relation $\forall x, y \in M$

$$\boxed{LF(xy) = LF(x\, LF(y))} \quad (*)$$

Exercise 1 ① Show (*) !

② Let $x, y \in \mathrm{Div}(\Delta)$. Give a formula for the Garside normal form of xy

③ Show that $x = x_1 x_2 \cdots x_k$ is a n.f. iff $x_i x_{i+1}$ is a n.f. for all "i".

Note: (*) is used to compute the G. n. f. of an element $x \in M$.

Example: $x = x_1 x_2 x_3 x_4$ $\qquad x_i \in \mathrm{Div}(\Delta)$

$$LF(x) = LF(x_1 x_2 x_3 x_4)$$

$$= LF(x_1\, LF(x_2 x_3 x_4))$$

$$= \underbrace{LF(x_1 \underbrace{LF(x_2 \underbrace{LF(x_3 x_4)}_{②})}_{②})}_{②}$$

$LF(x_4) = x_4$ since $x_4 \in \mathrm{Div}(\Delta)$

→ just need to know what $LF(ab)$ for $a, b \in \mathrm{Div}(\Delta)$ is (② above)

Then: Kill $LF(x)$! $x' = LF(x)^{-1} x$, go on computing $LF(x')$ it has to terminate (by Noetherianity)

Faithful representations of Garside groups

Lemma Let $\overset{\rho : M \to}{V}$ be a faithful representation of a Garside monoid M, s.t. $\rho(m)$ is invertible $\forall m \in M$.
Then ρ extends to a faithful rep. of $G(M)$.

Proof

$$\begin{tikzcd} M \arrow[r, hook, "\iota"] \arrow[dr, "\rho"'] & G(M) \arrow[d, dotted, "\varphi"] \\ & GL(V) \end{tikzcd} \quad \exists \text{ gp homomorphism } \varphi$$

To show: φ injective

Let $g \in G(M)$. $\exists\ x, y \in M$: $g = x^{-1}y$

$\varphi(x^{-1}y) = 1 \Rightarrow \varphi(x) = \varphi(y) \Rightarrow \rho(x) = \rho(y) \Rightarrow x = y$

$\Rightarrow g = 1.$ $\square$

$\boxed{\text{"To show that a rep}^n\text{ of a Garside gp is faithful, it suffices to show that its restriction to the Garside monoid is faithful!"}}$

Proposition (Krammer's criterion) Let $G(M)$ be a Garside group acting on a set U. Suppose that $\{C_x\}_{x \in Div(\Delta)}$ is a collection of nonempty disjoint subsets of U s.t.

$$(**) \quad x C_y \subseteq C_{LF(xy)} \quad \forall\, x, y \in Div(\Delta).$$

Then the action of $G(M)$ on U is faithful.

Proof Claim 1: $(**)$ extend to $x C_y \subseteq C_{LF(xy)}$

$\forall\, x \in M,\ y \in Div(\Delta)$.

Proof Let $x = x_1 x_2 \cdots x_k$, $\quad x_i \in Div(\Delta)$.

We argue by induction on k. If $k=1$ there's nothing to prove. Assume $k>1$. Then

$$x C_y = x_1 x_2 \cdots x_k C_y \underset{ind}{\subseteq} x_1 C_{LF(x_2 \cdots x_k y)}$$

$$\underset{(**)}{\subseteq} C_{LF(x_1^{LF}(x_2 \cdots x_k y))} \underset{(*)}{=} C_{LF(\underbrace{x_1 x_2 \cdots x_k}_{x} y)}$$

$\square$ Cl. 1

Claim 2: Let $\rho: G(M) \to U$ be the representation. Let $x, y \in M$. Then $\rho(x) = \rho(y)$ $\Rightarrow$ $x = y$.

Note: by the previous lemma it implies the proposition.

<u>Proof</u> By induction on $\ell(x)+\ell(y)$, where $\ell(x)$ is the length of a maximal chain of left divisors of x.

Let $x, y \in$ ~~B~~ $G(M)$ with $\pi(x)=\pi(y)$

$\ell(x)+\ell(y)=0 \Rightarrow x=y=1$. Hence assume $\ell(x)+\ell(y)>0$.

$C_1 \neq \emptyset$ by assumption; hence let $u \in C_1$

<u>Claim 1</u> $\Rightarrow \quad xu \in C_{LF(x)}, \; yu \in C_{LF(y)}$

$\rho(x)=\rho(y) \quad \Rightarrow \quad C_{LF(x)} \cap C_{LF(y)} \neq \emptyset$

$\Rightarrow LF(x)=LF(y) =: z$

Define $x', y' \in$ ~~B~~ $G(M)$ by $x=zx'$, $y=zy'$ ($z \neq 1$, otherwise $x=y=1$). Hence $\ell(x')+\ell(y')<\ell(x)+\ell(y)$.

We have $\rho(x')=\rho(y')$ since $\rho(x)=\rho(y)$ $\underset{\text{ind.}}{\Rightarrow} x'=y'$

$\Rightarrow x=y$.

$\blacksquare$

The representation

$R = \mathbb{R}[t^{\pm 1}]$, $0<q<1$

Let $V = \bigoplus_{1\le i<j\le n} R\, x_{i,j}$. Define

$(\sigma_k = s_k)$

$$
\begin{cases}
\sigma_k x_{k,k+1} = t q^2 x_{k,k+1} & \\
\sigma_k x_{i,k} = \underbrace{(1-q)}_{>0} x_{i,k} + \underbrace{q}_{>0} x_{i,k+1} & \text{if } i<k \\
\sigma_k x_{i,k+1} = x_{i,k} + t q^{k-i+1}(q-1)\, x_{k,k+1} & \text{if } i<k \\
\sigma_k x_{k,j} = t q (q-1) x_{k,k+1} + \underbrace{q}_{>0} x_{k+1,j} & \text{if } k+1<j \\
\sigma_k x_{k+1,j} = x_{k,j} + \underbrace{(1-q)}_{>0} x_{k+1,j} & \text{if } k+1<j \\
\sigma_k x_{i,j} = x_{i,j} & i<j<k \text{ or } k+1<i<j \\
\sigma_k x_{i,j} = x_{i,j} + t q^{k-i}(q-1)^2 x_{k,k+1} & i<k<k+1<j
\end{cases}
$$

NB

1. Defines an action of B_n on V (straightforward but annoying computation).
2. All entries of the matrix of σ_k are in $\mathbb{R}_{>0} + t\mathbb{R}[t]$
3. The basis is indexed by reflections (= transpositions) in S_n. $((i,j) \mapsto x_{i,j}$, $i<j)$ Let $A \subseteq \mathrm{Ref}(S_n)$.

Define
$$\boxed{D_A := \sum_{r\notin A} \mathbb{R}_{>0}\, x_r + \sum_{r\in \mathrm{Ref}(S_n)}{}' \; t\,\mathbb{R}[t]\, x_r}$$

"vectors which are positive linear combinations of the x_r, $r \notin A$ modulo t"

The formulas above show if $v \in D_A$, then $\sigma_K v \in D_B$ for a unique $B \subseteq Ref(S_n)$ and B depends only on A: $\sigma_K D_A \subseteq D_B$.

(Example: coeff of $x_{i,k}$ in $\sigma_k v$ lies in $t\mathbb{R}[t]$

$\iff$ coeff's of $x_{i,k}$ and $x_{i,k+1}$ in v both lie in $t\mathbb{R}[t]$ etc.)

$\rightsquigarrow$ $\boxed{\text{This defines an action of } B_n^+ \text{ on } \mathcal{P}(Ref(S_n))}$

(This action is not easy to describe, a formula can be derived from the definition of the σ_k-action)

(Follows from (1): the formulas define a rep. of B_n)

Note $D_A \cap D_B = \emptyset$ if $A \neq B$.

Want: Find sets to apply Krammer's criterion.

C_y, $y \in W^{(Div(\Delta))}$ nonempty, disjoint

s.t. $\underline{x} C_{\underline{y}} \subseteq C_{LF(\underline{x}\underline{y})}$ $\forall x \in \overline{W}$ $(\underline{x} \in Div(\Delta))$

How can we associate a set of reflections to an element $y \in \widetilde{W}$?

$$y \rightsquigarrow N(y) = \{ (i,j) \mid i<j,\ y^{-1}(i) > y^{-1}(j) \}$$

Naive candidate: $C_y = D_{N(y)}$

Exercise 2 Show that the B_n^+-action on $\mathcal{P}(\mathrm{Ref}(S_n))$ does not preserve $\{\{N(y)\}_{y\in\widetilde{W}}\}$

But it preserves a property satisfied by the $N(y)$

Definition A set $A \subseteq \mathrm{Ref}(S_n)$ is closed if whenever $1 \le i < j < k \le n$, if $(i,j), (j,k) \in A$, then $(i,k) \in A$

Exercise 3 ① Show that $\forall y \in S_n$, $N(y)$ is biclosed, i.e., both $N(y)$ and $\mathrm{Ref}(S_n) \setminus N(y)$ are closed

② Show that every biclosed $A \subseteq \mathrm{Ref}(S_n)$ is equal to $N(y)$ for some $y \in S_n$.

$\underset{A}{\underset{\parallel}{\{(1,3)\}}}$ is closed, but not biclosed: $(1,2), (2,3) \in \mathrm{Ref} \setminus A$ but $(1,3) \notin \mathrm{Ref} \setminus A$
in S_n

Problem Too many D_A's ! But:

Lemma (Kramer) ① if A is closed, then $\sigma_k A$ is closed.

2. If A is closed, $\exists!$ maximal $N(\underline{y}) \subseteq A$ (for inclusion). Set $\mathrm{Pro}(A) := N(\underline{y})$

$(\mathrm{Pro}^2 = \mathrm{Pro})$

$y \in W$ Set $C_{\underline{y}} := \coprod_{\substack{A \text{ closed} \\ \mathrm{Pro}(A) = N(\underline{y})}} D_A$ (nonempty + disjt)

<u>Proposition</u> Let $x, y \in W$. Then

$$\underline{x}\, C_{\underline{y}} \subseteq C_{LF(\underline{x}\,\underline{y})}$$

<u>Corollary</u> The B_n action on V is faithful.

$(\mathrm{Pro}(\underline{x} A) = N(LF(\underline{x}\,\underline{y}))$, where $N(\underline{y}) = \mathrm{Pro}(A)$.$)$

$\Rightarrow \underline{x}\, C_{\underline{y}} \subseteq C_{LF(\underline{x}\,\underline{y})}$