

Actions of groups on categories:

①

Motivation: Recall memes in mathematics: non-linear \rightsquigarrow linear \rightsquigarrow numbers.

Categories come in many flavours and occupy a wide range of positions between linear and non-linear.

Example of a very non-linear category: let Γ be a group.

Consider a category $\Sigma\Gamma$ defined as follows: one object $*$,
"suspension of Γ ".
 $\text{End}(*) = \Gamma$.

Example of a very linear category: let A be a finite-dimensional algebra / k .

The category $A\text{-mod}$ of f.g. A -modules is an additive, abelian category.

Exercise: ① $\text{Fun}(\Sigma\Gamma, k\text{-mod}) \cong \cong = k\text{-representations of } \Gamma / \cong$.

② Can you make this an equivalence of categories?

Definition: A weak action of a group G on a category \mathcal{C} is the

data of ① autoequivalences $F_g: \mathcal{C} \rightarrow \mathcal{C}$ for all $g \in G$ such

that $F_{\text{id}} \cong \text{id}$ and $F_g F_h \cong F_{gh}$ for all $g, h \in G$.

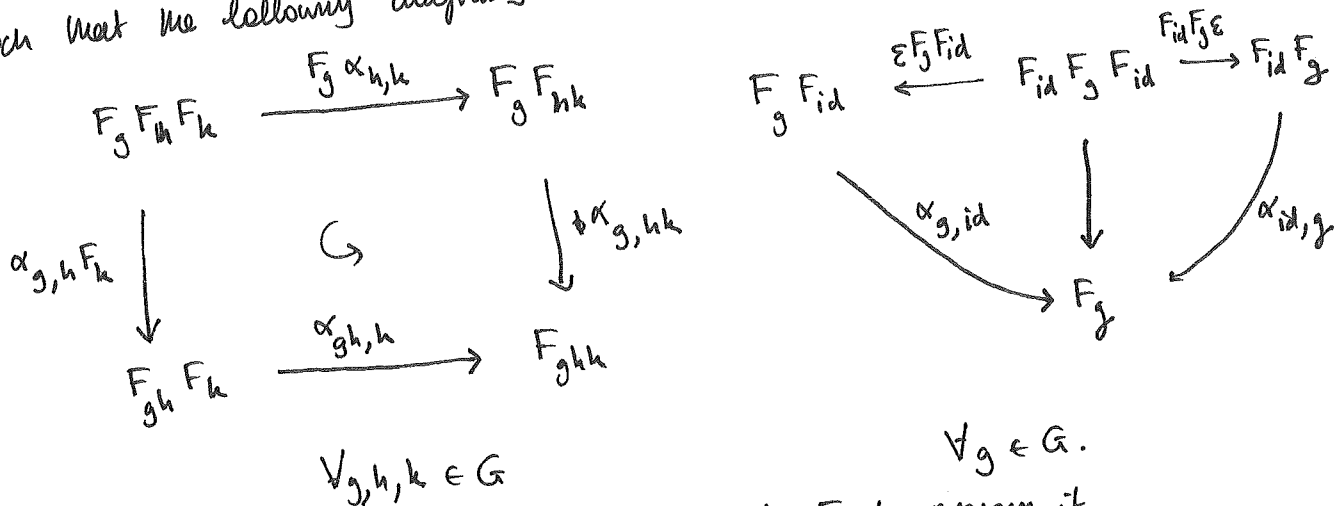
(isomorphisms of functors are not part of data)

Definition: A strict action (or simply action) of a group G on a category \mathcal{C} is the data of

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- ① ~~isomorphisms~~ equivalences $F_g: \mathcal{C} \rightarrow \mathcal{C} \quad \forall g \in G$
- ② isomorphisms $\alpha_{g,h}: F_g F_h \rightarrow F_{gh}$ for all $g, h \in G$
 $\epsilon_g: F_{id} \rightarrow id_{\mathcal{C}}$

such that the following diagrams commute



Remark: If \mathcal{C} has extra structure we usually want F_g to preserve it.
 \hookrightarrow k -linear, additive, abelian, triangulated

Exercises: ① Show that giving an action of \mathbb{Z} on \mathcal{C} is the same thing as an ~~action~~ given a self-equivalence $F: \mathcal{C} \rightarrow \mathcal{C}$.

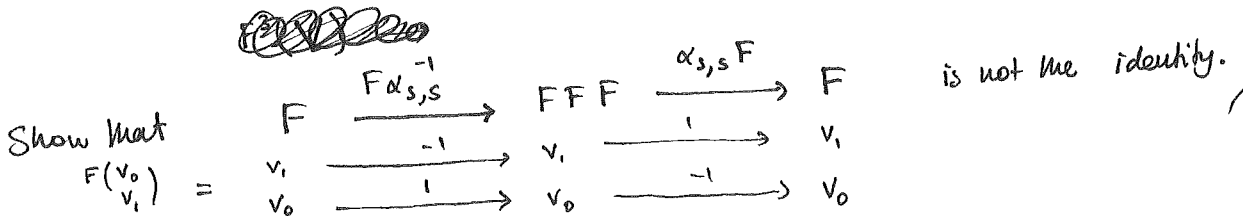
② Show that giving an action of $\mathbb{Z}/2\mathbb{Z}$ on \mathcal{C} is not the same as giving a self-equivalence F together w/ an isomorphism $F^2 \cong id$.

$\alpha: FF \xrightarrow{\sim} id$.

A little more detail: We want all endomorphisms of functors to be canonical, $F \rightarrow FFF \rightarrow F$ may not be identity.)

More detail: Let $SVect$ denote $\mathbb{Z}/2\mathbb{Z}$ -graded vector spaces: $V = V_0 \oplus V_1$.

Define $F = F_s (V = V_0 \oplus V_1) = V_1 \oplus V_0$. $\alpha_{s,s}: F_s F_s \xrightarrow{\begin{pmatrix} id & 0 \\ 0 & -id \end{pmatrix}} id$.



Remark: What is really going on here is that $\pi_2(S^1) = \{1\}$

whereas $\pi_2(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$.

Recall that a representation $(\rho: G \rightarrow GL(V))$ is the same thing as a G-module (linear action of G on V.)

Exercise: Show that an action of G on \mathcal{B} is the same thing as a tensor functor

$$G^{\otimes} \longrightarrow \text{Fun}(\mathcal{B})$$

(G^{\otimes} has objects r_g for each $g \in G$, $\text{Hom}(r_g, r_h) = \begin{cases} \{\text{id}\} & \text{if } g=h \\ \emptyset & \text{o/w.} \end{cases}$
 $r_g \otimes r_h = r_{gh}$.)

Homotopy categories:

Motivation: "Algebraic" abelian ~~categories~~ ~~for~~ usually have very few interesting auto-equivalences. (e.g. simple objects must be preserved).

Whereas triangulated categories can have much richer unexpected symmetry. (no such thing as a simple object in a triangulated category).

Example from algebraic geometry:

X, Y smooth projective varieties

$$\text{Coh}(X) \cong \text{Coh}(Y) \Rightarrow X \cong Y,$$

$$\text{Aut}(\text{Coh}(X)) \cong \text{Aut}(X).$$

"all symmetries already arise from X".

$D^b(X) := D^b(\text{Coh}(X)) \cong D^b(Y)$ still has a mysterious meaning.

Also $\text{Aut}(D^b(X))$ is unknown in many cases.

A, A^v dual abelian varieties $D^b(A) \cong D^b(A^v)$ although $A \not\cong A^v$.

Example when X is an elliptic curve:

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There is an exact sequence

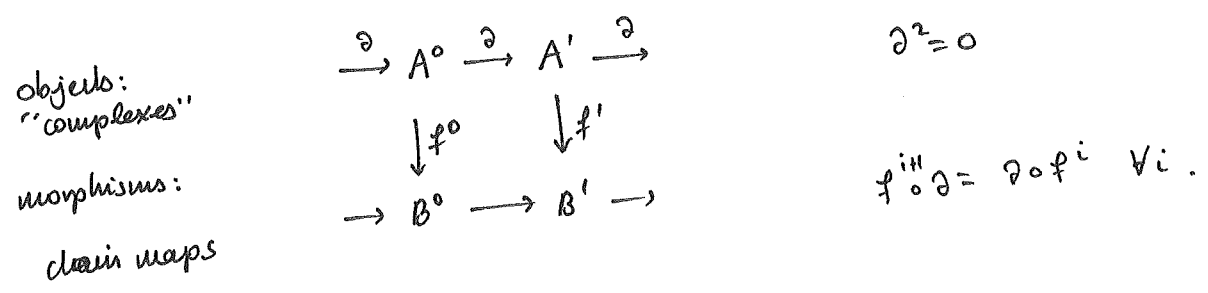
• see Bridgeland "Derived categories of coherent sheaves".

$$1 \longrightarrow \mathbb{Z} \times (\text{Aut} \times \text{Pic}^0(X)) \longrightarrow \text{Aut} D(X) \longrightarrow \text{SL}_2(\mathbb{Z}) \longrightarrow 1$$

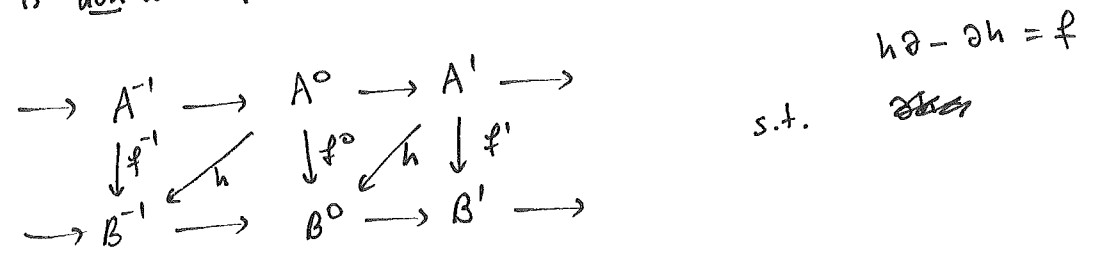
\mathcal{A} an additive, k -linear category. \oplus

E.g.: A -mod for an algebra A , any abelian category, projective objects in an abelian category, even-dimensional vector spaces, ...

$\text{Kom}(\mathcal{A}) =$ category of chain complexes in \mathcal{A}



f is null-homotopic if there exists a maps



Exercise: (1) Formulate a reasonable notion of "ideal" in a category, and show that null-homotopic morphisms form an ideal.

We define $K(\mathcal{A}) = \text{Kom}(\mathcal{A}) / (\text{null-homotopic morphisms})$, the "homotopy category".

(2) Given chain complexes A, B form

$$\text{Hom}^0(A, B) = \bigoplus_{j \in \mathbb{Z}} \prod_i \text{Hom}(A^i, B^{i+j})$$

Make this into a complex s.t.

0-cycles = chain maps

0^{th} cohomology = $\text{Hom}_{K(\mathcal{A})}(A, B)$.

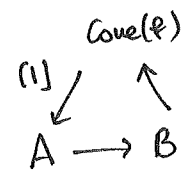
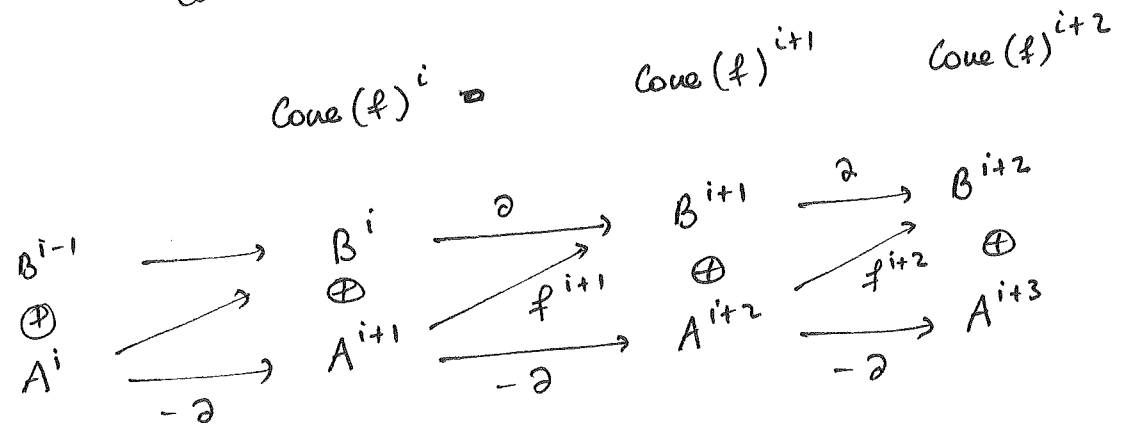
Fundamental structures on $K(\mathcal{A})$:

① additive: $A, B \rightsquigarrow (A \oplus B)^i = A^i \oplus B^i$ w/ induced differential.

② shift: given $A \in K(\mathcal{A})$ form $A[1]$ via $A[1]^i = A^{i+1}$, $\partial_{A[1]} = -\partial_A$.

③ cones: given $f: A \rightarrow B$ in $K^0(\mathcal{A})$ form

$\text{Cone}(f)$ as follows:



We have obvious maps:

$$\cdots \rightarrow A \xrightarrow{f} B \rightarrow \text{Cone}(f) \rightarrow A[1] \rightarrow \cdots$$

Triples $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ isomorphic to such cones are distinguished triangles.

Exercises: ① This exercise shows that ~~com~~^{d.t.s} are something like short exact sequences:

Suppose that $A \xrightarrow{f} B \rightarrow C$ are maps of complexes such that $A^i \rightarrow B^i \rightarrow C^i$ is a split short exact sequence for all i . Show that there exists a map $C \rightarrow A(1)$ and two isomorphisms of distinguished triangles.

$$\begin{array}{ccccccc} A & \rightarrow & B & \rightarrow & C & \rightarrow & A(1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A & \rightarrow & B & \rightarrow & \text{cone}(f) & \rightarrow & A(1) \end{array}$$

② This exercise shows that "size" doesn't make much sense in a Δ ed cat:

Suppose $A \rightarrow B \rightarrow C \rightarrow A(1)$ is a d.t.
Then so is $B \rightarrow C \rightarrow A(1) \rightarrow B(1)$. "forming triangles".

Extended example $K(\mathcal{A})$ for $\mathcal{A} = \text{f.g. free } \mathbb{Z}\text{-modules}$.

Lemma: Any submodule of a f.g. free \mathbb{Z} -module is free. (\mathbb{Z} -mod is "hereditary").

Suppose $A \in \mathcal{A}$:

$$\begin{array}{ccccccc} \dots & \rightarrow & A^{-1} & \rightarrow & A^0 & \rightarrow & A^1 & \rightarrow & \dots \\ & & \text{im}^{-1} \oplus C^{-1} & & \text{im}^0 \oplus C^0 & & \text{im}^1 \oplus C^1 & & \end{array}$$

Hence we may assume A has only two non-zero terms ~~up to shifts~~.
(up to shifts) that

Theory of elementary divisors: $A^0 \xrightarrow{\vartheta} A^1$ (7)

$$\begin{pmatrix} m_1 & & & 0 \\ & m_2 & & \\ & & \ddots & \\ 0 & & & m_{i-1} & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}.$$

$$\Rightarrow A \cong \mathbb{Z}^{\oplus m_1} \oplus \mathbb{Z}[-1]^{\oplus m_2} \oplus \bigoplus_{i=1}^j Y_{m_i}(-i)$$

(*)

where $Y_m: 0 \rightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \rightarrow 0$

Workable exercise: If $(m,n)=1$ show that $Y_{mn} \cong Y_m \oplus Y_n$.
 (Give proof if time permits.)

Lemma: Indecomposable objects in $K(\mathcal{A})$ (up to shifts) are

\mathbb{Z} and Y_{p^r} for p prime, $r \geq 1$.
 (*)

$$\text{Hom}(\mathbb{Z}, Y_{p^r}) = \mathbb{Z}/p^r.$$

$$\begin{array}{c} \mathbb{Z} \\ \downarrow a \\ \mathbb{Z} \xrightarrow{p^r} \mathbb{Z} \end{array}$$

$\text{Hom}(\mathbb{Z}, \mathbb{Z}Y_{p^r}(-i)).$

$\text{Hom}(\mathbb{Z}, Y_{p^r}[m]) = 0$ unless $m=0$. $\dots \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow 0 \rightarrow \dots$
 $\quad \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow$
 $\dots \rightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{p^r} \mathbb{Z} \rightarrow \dots$ no chain maps.

$\text{Hom}(\mathbb{Z}, \mathbb{Z}(m)) = 0$ for $m \neq 0$

$$\text{Hom}(Y_{p^r}, \mathbb{Z}) = 0$$

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{p^r} & \mathbb{Z} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z} \end{array}$$

$$\begin{array}{ccc} \mathbb{Z}(-1) & \longrightarrow & Y_{p^r}(-1) \\ \swarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & Y_{p^r} \\ \swarrow & & \downarrow \\ \mathbb{Z}(1) & \longrightarrow & Y_{p^r}(1) \\ \swarrow & & \downarrow \\ & & \vdots \end{array}$$

$$\text{Hom}(Y_{p^r}, \mathbb{Z}(1)) = \mathbb{Z}/p^r.$$

Exercise: (1) $\text{Hom}(Y_{p^r}, Y_{q^{r'}}) = 0$ if $(p,q)=0$.

(2) $\text{Hom}(Y_{p^r}, Y_{p^{r'}}) = \begin{cases} \mathbb{Z}/p^{\min(r,r')} & \text{if } m=0, 1 \\ 0 & \text{otherwise} \end{cases}$

Minimal complexes:

M is indecomposable if $M \cong M_1 \oplus M_2$

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$$\Downarrow \\ M_1 = 0 \text{ or } M_2 = 0$$

Suppose A is a f.g. k -algebra.

Then objects in $\text{mod } A$ finitely generated A -modules have the following properties:

① ~~Any~~ Any $M \in A\text{-mod}$ can be written $M = M_1 \oplus \dots \oplus M_k$ (*)
w/ M_i indecomposable.

② $M \in A\text{-mod}$ is indecomposable $\Leftrightarrow \text{End}(M)$ is local.

③ Summands and multiplicities in (*) are well-defined
iso. ("Krull-Schmidt theorem").

In fact ① + ② \Rightarrow ③. An additive category \mathcal{A} is Krull-Schmidt
if it satisfies ① and ②.

E.g. $\mathbb{Z}\text{-mod}$, even-dimensional vector spaces fail ②
~~fail ②~~

Suppose \mathcal{A} is an additive category. The complex

$$0 \rightarrow M \rightarrow M \rightarrow 0 \quad (*)$$

is isomorphic to 0 in $K(\mathcal{A})$ (its identity is null-homotopic).

A contractible summand ~~is~~ of a complex N^\bullet is a summand
isomorphic to (*).

Exercise: (Gaussian elimination)

Suppose $M \in K(\mathcal{A})$ has the form

$$\begin{matrix} C \\ \oplus \\ \tilde{M}^i \end{matrix} \xrightarrow{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}} \begin{matrix} C' \\ \oplus \\ \tilde{M}^{i+1} \end{matrix}$$

with α an iso.

$$\dots \rightarrow M^i \rightarrow M^{i+1} \rightarrow \dots$$

Then M is isomorphic to a complex of the form

$$\dots \rightarrow M^{i-1} \rightarrow \begin{matrix} C \\ \oplus \\ \tilde{M}^i \end{matrix} \xrightarrow{\begin{pmatrix} \text{id} & 0 \\ 0 & \delta \end{pmatrix}} \begin{matrix} C \\ \oplus \\ \tilde{M}^{i+1} \end{matrix} \rightarrow M^{i+2} \rightarrow \dots$$

in particular, M contains a contractible summand.

~~sep~~ A minimal complex is a complex without contractible summands.

Lemma: If \mathcal{A} is Krull-Schmidt then any complex $M \in K(\mathcal{A})$ contains a summand $M_{\min} \subset M$ s.t.

- ① $M_{\min} \hookrightarrow M$ is an isomorphism
- ② M_{\min} is minimal.

Moreover, any two minimal complexes are isomorphic as complexes.