Actions of groups on categories:

Motivation: Recall theme in mathematics: non-linear $\rightsquigarrow$ linear $\rightsquigarrow$ numbers.

Categories come in many flavours and occupy a wide range of positions between linear and non-linear.

Example of a very non-linear category: let Γ be a group.

Consider a category $\Sigma\Gamma$ defined as follows: one object $*$,

$$\mathrm{End}(*) = \Gamma. \qquad \text{"suspension of } \Gamma\text{".}$$

Example of a very linear category: let A be a finite-dimensional algebra / k.

The category A-mod of f.g. A-modules is an additive, abelian category.

Exercise: ① $\mathrm{Fun}(\Sigma\Gamma, k\text{-mod}) /_{\cong} \; = \; k\text{-representations of } \Gamma /_{\cong}$.

② Can you make this an equivalence of categories?

Definition: A *weak action* of a group G on a category $\mathcal{C}$ is the data of ① autoequivalences $F_g : \mathcal{C} \to \mathcal{C}$ for all $g \in G$ such that $F_{id} \cong id$ and $F_g F_h \cong F_{gh}$ for all $g, h \in G$.

(isomorphisms of functors are not part of data)

<u>Definition:</u> A <u>strict action</u> (or simply <u>action</u>) of a group G on a category $\mathcal{C}$ is the data of

1. ~~functors~~ equivalences $F_g : \mathcal{C} \to \mathcal{C}$ $\forall g \in G$
2. isomorphisms $\alpha_{g,h} : F_g F_h \longrightarrow F_{gh}$ for all $g, h \in G$

$$\varepsilon : F_{id} \longrightarrow id_{\mathcal{C}}$$

such that the following diagrams commute

$$\begin{tikzcd}
F_g F_h F_k \arrow[r, "F_g \alpha_{h,k}"] \arrow[d, "\alpha_{g,h} F_k"'] \arrow[dr, phantom, "\circlearrowright"] & F_g F_{hk} \arrow[d, "\alpha_{g,hk}"] \\
F_{gh} F_k \arrow[r, "\alpha_{gh,k}"] & F_{ghk}
\end{tikzcd}
\qquad
\begin{tikzcd}
F_g F_{id} \arrow[dr, "\alpha_{g,id}"'] & F_{id} F_g F_{id} \arrow[l, "\varepsilon F_g F_{id}"'] \arrow[r, "F_{id} F_g \varepsilon"] \arrow[d] & F_{id} F_g \arrow[dl, "\alpha_{id,g}"] \\
& F_g &
\end{tikzcd}$$

$\forall g, h, k \in G$ $\qquad\qquad$ $\forall g \in G$.

<u>Remark:</u> If $\mathcal{C}$ has extra structure we usually want F_g to preserve it.
↳ k-linear, additive, abelian, triangulated

<u>Exercises:</u>

1. Show that giving an action of $\mathbb{Z}$ on $\mathcal{C}$ is the same thing as ~~an~~ ~~as~~ given a self-equivalence $F : \mathcal{C} \to \mathcal{C}$.

2. Show that giving an action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathcal{C}$ is <u>not</u> the same as giving ~~a~~ self-equivalence F together w/ an isomorphism ~~$F: \mathcal{C} \to \mathcal{C}$~~

$\alpha : FF \xrightarrow{\sim} id$.

<u>A little more detail:</u> We want all endomorphisms of functors to be canonical, ($F \to FFF \to F$ may not be identity.)

<u>More detail:</u> Let SVect denote $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces : $V = V_0 \oplus V_1$.

Define $F = F_s (V = V_0 \oplus V_1) = V_1 \oplus V_0$. $\qquad \alpha_{s,s} : F_s F_s \xrightarrow{\begin{pmatrix} id & 0 \\ 0 & -id \end{pmatrix}} id$.

Show that
$$F \xrightarrow{F \alpha_{s,s}^{-1}} FFF \xrightarrow{\alpha_{s,s} F} F \quad \text{is not the identity.}$$

$$F\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \begin{matrix} V_1 \xrightarrow{-1} V_1 \xrightarrow{1} V_1 \\ V_0 \xrightarrow{1} V_0 \xrightarrow{-1} V_0 \end{matrix}$$

Remark: What is really going on here is that $\pi_2(S^1) = \{1\}$

whereas $\pi_2(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$.

Recall that a representation ($\rho: G \to GL(V)$) is the same thing as a G-module (linear action of G on V.)

Exercise: Show that an action of G on $\mathcal{C}$ is the same thing as a tensor functor

$$G^{\otimes} \longrightarrow \mathrm{Fun}(\mathcal{C})$$

($G^{\otimes}$ has objects r_g for each $g \in G$, $\mathrm{Hom}(r_g, r_h) = \begin{cases} \{id\} & \text{if } g=h \\ \phi & \text{o/w.} \end{cases}$

$r_g \otimes r_h = r_{gh}$.)

Homotopy categories:

Motivation: "Algebraic" abelian categories usually have very few interesting auto-equivalences.

(e.g. simple objects must be preserved).

Whereas triangulated categories can have much richer unexpected symmetry.

(no such thing as a simple object in a triangulated category).

Example from algebraic geometry:

X, Y smooth projective varieties

$\mathrm{Coh}(X) \cong \mathrm{Coh}(Y) \Rightarrow X \cong Y$, $\quad \mathrm{Aut}(\mathrm{Coh}(X)) \subseteq \mathrm{Aut}(X)$.

"all symmetries already arise from X".

$D^b(X) := D^b(\mathrm{Coh}(X)) \cong D^b(Y)$ still has a mysterious meaning.

Also $\mathrm{Aut}(D^b(X))$ is unknown in many cases.

$A, A^\vee$ dual abelian varieties $D^b(A) \cong D^b(A^\vee)$ although $A \ncong A^\vee$.

Example when X is an elliptic curve:

There is an exact sequence • see Bridgeland "Derived categories of coherent sheaves".

$$1 \longrightarrow \mathbb{Z} \times (\mathrm{Aut} \ltimes \mathrm{Pic}^\circ(X)) \longrightarrow \mathrm{Aut}\, D(X) \longrightarrow SL_2(\mathbb{Z}) \longrightarrow 1$$

$\mathcal{A}$ an additive, k-linear category. $\oplus$

E.g: A-mod for an algebra A, any abelian category, projective objects in an abelian category, even-dimensional vector spaces, ...

$\mathrm{Kom}(\mathcal{A})$ = category of chain complexes in $\mathcal{A}$

objects: "complexes"

morphisms: chain maps

$$\begin{tikzcd} \arrow[r, "\partial"] & A^0 \arrow[r, "\partial"] \arrow[d, "f^0"] & A^1 \arrow[r, "\partial"] \arrow[d, "f^1"] & {} \\ \arrow[r] & B^0 \arrow[r] & B^1 \arrow[r] & {} \end{tikzcd}$$

$\partial^2 = 0$

$f^{i+1} \circ \partial = \partial \circ f^i \quad \forall i.$

f is null-homotopic if there exist a maps

$$\begin{tikzcd} \arrow[r] & A^{-1} \arrow[r] \arrow[d, "f^{-1}"] & A^0 \arrow[r] \arrow[d, "f^0"] \arrow[dl, "h"] & A^1 \arrow[r] \arrow[d, "f^1"] \arrow[dl, "h"] & {} \\ \arrow[r] & B^{-1} \arrow[r] & B^0 \arrow[r] & B^1 \arrow[r] & {} \end{tikzcd}$$

s.t. $h\partial - \partial h = f$

Exercise: ① Formulate a reasonable notion of "ideal" in a category, and show that null-homotopic morphisms form an ideal.

We define $K(\mathcal{A}) = \mathrm{Kom}(\mathcal{A}) / (\text{null-homotopic morphisms})$, the "homotopy category".

② Given chain complexes A, B form

$$\mathrm{Hom}^\bullet(A, B) = \bigoplus_{j \in \mathbb{Z}} \prod_{i} \mathrm{Hom}(A^i, B^{i+j})$$

Make this into a complex s.t.

$$\text{0-cycles} = \text{chain maps}$$

$$0^{\text{th}} \text{ cohomology} = \operatorname{Hom}_{K(\mathcal{A})}(A, B).$$

Fundamental structures on $K(\mathcal{A})$:

① <u>additive</u>: $A, B \rightsquigarrow (A \oplus B)^i = A^i \oplus B^i$ w/ induced differential.

② <u>shift</u>: given $A \in K(\mathcal{A})$ form $A[1]$ via

$$A[1]^i = A^{i+1}, \quad \partial_{A[1]} = -\partial_A.$$

③ <u>cones</u>: given $f : A \to B$ in $K^{\bullet}(\mathcal{A})$ form

$\operatorname{Cone}(f)$ as follows:

$$\begin{tikzcd}
 & \operatorname{Cone}(f)^i & \operatorname{Cone}(f)^{i+1} & \operatorname{Cone}(f)^{i+2} \\
B^{i-1} \arrow[r] & B^i \arrow[r, "\partial"] & B^{i+1} \arrow[r, "\partial"] & B^{i+2} \\
\oplus & \oplus & \oplus & \oplus \\
A^i \arrow[r, "-\partial"'] \arrow[uur] & A^{i+1} \arrow[r, "-\partial"'] \arrow[uur, "f^{i+1}"] & A^{i+2} \arrow[r, "-\partial"'] \arrow[uur, "f^{i+2}"] & A^{i+3}
\end{tikzcd}$$

$$\begin{tikzcd}
 & \operatorname{Cone}(f) \arrow[dl, "{[1]}"'] \\
A \arrow[r] & B \arrow[u]
\end{tikzcd}$$

We have obvious maps:

$$\xrightarrow{\cdots} A \xrightarrow{f} B \longrightarrow \operatorname{Cone}(f) \longrightarrow A[1] \xrightarrow{\cdots}$$

Triples $X \to Y \to Z \to X[1]$ isomorphic to such cones are <u>distinguished</u> <u>triangles</u>.

<u>Exercises:</u> (1) This exercise shows that ~~cones~~ d.t.'s are something like short exact sequences:

Suppose that $A \xrightarrow{f} B \to C$ are maps of complexes such that $A^i \to B^i \to C^i$ is a split short exact sequence for all i. Show that there exists a map $C \to A(1)$ and an isomorphism of distinguished triangles.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & A(1) \arrow[d] \\ A \arrow[r] & B \arrow[r] & \mathrm{Cone}(f) \arrow[r] & A(1) \end{tikzcd}$$

This exercise shows that "size" doesn't make much sense in a Δed cat:

(2) Suppose $A \to B \to C \to A(1)$ is a d.t.

Then so is $B \to C \to A(1) \to B(1)$. "turning triangles".

<u>Extended example</u> $K(\mathcal{A})$ for $\mathcal{A} =$ f.g. free $\mathbb{Z}$-modules.

<u>Lemma:</u> Any submodule of a f.g. free $\mathbb{Z}$-module is free. ("$\mathbb{Z}$-mod is hereditary").

Suppose $A \in \mathcal{A}$:

$$\cdots \to A^{-1} \to A^0 \to A^1 \to \cdots$$

$$\mathrm{im}^{-1} \oplus C^{-1} \qquad \mathrm{im}^0 \oplus C^0 \qquad \mathrm{im}^1 \oplus C^1$$

Hence we may assume (up to shifts) that A has only two non-zero terms ~~up to shifts~~.

Theory of elementary divisors: $A^0 \xrightarrow{\partial} A^1$

$$\begin{pmatrix} m_1 & & & \\ & m_2 & & 0 \\ & & \ddots & \\ 0 & & & m_n \ 0 \ \ddots \ 0 \end{pmatrix}.$$

$$\Rightarrow \quad A \cong \mathbb{Z}^{m_1} \oplus \mathbb{Z}[-1]^{m_2} \oplus \bigoplus_{i=1}^{j} Y_{m_i}[-1]$$

(*)

where $\quad Y_m : \quad 0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to 0$

<u>Worthwhile exercise:</u> If $(m,n) = 1$ show that $Y_{mn} \cong Y_m \oplus Y_n$.

(Give proof if time permits.)

<u>Lemma:</u> Indecomposable objects in $K(\mathcal{A})$ (up to shifts) are

$\mathbb{Z}$ and Y_{p^r} for p prime, $r \geq 1$.

(*)

$\mathrm{Hom}(\mathbb{Z}, Y_{p^r}) = \mathbb{Z}/p^r$.

$$\begin{tikzcd} & \mathbb{Z} \arrow[d, "a"] \\ \mathbb{Z} \arrow[r, "p^r"] & \mathbb{Z} \end{tikzcd}$$

$\mathrm{Hom}(\mathbb{Z}, Y_{p^r}[-1])$.

$\mathrm{Hom}(\mathbb{Z}, Y_{p^r}[m]) = 0$ unless $m = 0$.

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r, "p^r"] & \mathbb{Z} \arrow[r] & \cdots \end{tikzcd}$$

no chain maps.

$\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}[m]) = 0$ for $m \neq 0$

$\mathrm{Hom}(Y_{p^r}, \mathbb{Z}) = 0$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "p^r"] \arrow[d] & \mathbb{Z} \arrow[d] \\ 0 \arrow[r] & \mathbb{Z} \end{tikzcd}$$

$\mathrm{Hom}(Y_{p^r}, \mathbb{Z}[1]) = \mathbb{Z}/p^r$.

$$\begin{tikzcd} & \vdots \arrow[dl] & \\ \mathbb{Z}[-1] \arrow[r] & Y_{p^r}[-1] \arrow[d] \arrow[dl] \\ \mathbb{Z} \arrow[r] & Y_{p^r} \arrow[d] \arrow[dl] \\ \mathbb{Z}[1] \arrow[r] & Y_{p^r}[1] \arrow[dl] \\ \vdots & \end{tikzcd}$$

<u>Exercise:</u> ① $\mathrm{Hom}(Y_{p^r}, Y_{q^{r'}}^{(m)}) = 0$ if $(p,q) = 0$.

② $\mathrm{Hom}(Y_{p^r}, Y_{p^{r'}}^{(m)}) = \begin{cases} \mathbb{Z}/p^{\min(r,r')} & \text{if } m = 0, 1 \\ 0 & \text{otherwise} \end{cases}$

Minimal complexes:

M is indecomposable if $M \cong M_1 \oplus M_2$
$$\Downarrow$$
$M_1 = 0$ or $M_2 = 0$

Suppose A is a f.g. k-algebra.

Then objects in finitely generated A-modules have the following properties:

① Any $M \in A\text{-mod}$ can be written $M = M_1 \oplus \dots \oplus M_k$ $(*)$
w/ M_i indecomposable.

② $M \in A\text{-mod}$ is indecomposable $\iff$ $\mathrm{End}(M)$ is local.

③ Summands/iso. and multiplicities in $(*)$ are well-defined
("Krull-Schmidt theorem").

In fact ① + ② ⇒ ③. An additive category $\mathcal{A}$ is Krull-Schmidt if it satisfies ① and ②.

E.g. $\mathbb{Z}$-mod, even-dimensional vector spaces fail ②

Suppose $\mathcal{A}$ is an additive category. The complex

$$0 \longrightarrow M \longrightarrow M \longrightarrow 0 \qquad (*)$$

is isomorphic to 0 in $K(\mathcal{A})$ (its identity is null-homotopic).

A contractible summand of a complex N° is a summand isomorphic to $(*)$.

Exercise: (Gaussian elimination)

Suppose $M \in K(\mathcal{A})$ has the form

$$\cdots \longrightarrow M^i = \begin{matrix} C \\ \oplus \\ \tilde{M}^i \end{matrix} \xrightarrow{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}} \begin{matrix} C' \\ \oplus \\ \tilde{M}^{i+1} \end{matrix} = M^{i+1} \longrightarrow \cdots$$

with ~~M~~ α an iso.

Then M is isomorphic to a complex of the form

$$\cdots \longrightarrow M^{i-1} \longrightarrow \begin{matrix} C \\ \oplus \\ \tilde{M}^i \end{matrix} \xrightarrow{\begin{pmatrix} \mathrm{id} & 0 \\ 0 & \gamma \end{pmatrix}} \begin{matrix} C \\ \oplus \\ \tilde{M}^{i+1} \end{matrix} \longrightarrow M^{i+2} \longrightarrow \cdots$$

in particular, M contains a contractible summand.

Def A <u>minimal complex</u> is a complex without contractible summands.

<u>Lemma</u>: If $\mathcal{A}$ is Krull-Schmidt then any complex $M \in K(\mathcal{A})$ contains a summand $M_{min} \subset M$ s.t.

1. $M_{min} \hookrightarrow M$ is an isomorphism
2. M_{min} is minimal.

Moreover, any two minimal complexes are isomorphic as complexes.