Jones polynomial and Temperley-Lieb algebra:

Jones polynomial is a knot invariant $V: \{\text{knots}\} \longrightarrow \mathbb{Z}[q^{\pm 1}]$.

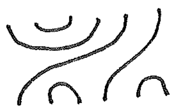
We want to quickly recall its connection to the Temperley-Lieb algebra.

Temperley-Lieb algebra TL_{n+1} is an algebra over $\mathbb{Z}[q^{\pm 1}]$ with generators U_i $i=1,\dots,n$

and relations

$$U_i^2 = (q+q^{-1})U_i \qquad U_iU_{i\pm 1}U_i = U_i \qquad U_iU_j = U_jU_i \quad \text{if } |i-j|>1.$$

It may alternatively be described as having a basis "crossingless $n+1$, $n+1$ matchings":

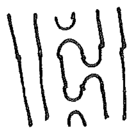
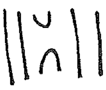


$n=5$.

and multiplication induced by

$a * b = (q+q^{-1})^{\#\text{deleted circles}}$ 

Isomorphism is given by $U_i \longmapsto$ 

$U_iU_{i+1}U_i =$  $=$  $= U_i$.

There is a trace function $Tr: TL_{n+1} \longrightarrow \mathbb{Z}[q+q^{-1}]$

$Tr($  $) = (q+q^{-1})^{\#\text{ of closed circles in}}$  .

E.g. $Tr(||) = (q+q^{-1})^2$, $Tr($  $) = (q+q^{-1})$.

Consider the elements

$$\theta_i := |||||-q\,||\overset{\cup}{\cap}|| = \mathrm{id} - q e_i .$$

~~Exercise: Show that θ_i is invertible~~

$$(\mathrm{id}-q e_i)(\mathrm{id}-q^{-1}e_i) = \mathrm{id}-q^{-1}e_i - q e_i + e_i^2 = \mathrm{id}.$$

Hence $\theta_i^{-1} = \mathrm{id} - q^{-1} e_i$.

Exercise: Show that

$$\theta_1\theta_2\theta_1 = ||| - q\,\overset{\cup}{\cap}| - q\,|\overset{\cup}{\cap} + q^2\,\overset{\cup\ \ }{\diagup\!\!\cap} + q^2\,\overset{\ \ \cup}{\cap\!\!\diagdown} = \theta_2\theta_1\theta_2 .$$

Deduce that there is a homomorphism

"Jones representation"

$$\begin{array}{ccc} Br_{n+1} & \xrightarrow{\phi} & TL_{n+1} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \sigma_i & \longmapsto & \theta_i \end{array} .$$

Up to a power of q one has

$$V_K = \mathrm{Tr}(\phi(\sigma)).$$

("Jones polynomial is evaluated via a trace on the TL algebra")

<u>Rmk:</u> Faithfulness of Jones rep is closely tied to faithfulness question for Burau rep, which in turn is closely tied to question as to whether Jones poly detects unknot.

Quiver algebras and zig-zag algebras

Q a finite quiver (oriented graph).

kQ : quiver algebra, k-basis oriented paths in Q.

$$\pi_1 \cdot \pi_2 = \begin{cases} \pi_1 \pi_2 & \text{if } \mathrm{end}(\pi_1) = \mathrm{start}(\pi_2) \\ 0 & \text{o/w}. \end{cases}$$

Given a vertex i the length zero path at i is denoted e_i.

Example 1: $\overset{1}{\bullet} \xrightarrow{a} \overset{2}{\bullet}$

$f \cdot g$	e_0	e_1	a
e_0	e_0	0	a
e_1	0	e_1	0
a	0	a	0

Example 2: $\begin{tikzcd} \bullet \arrow[loop right, "\alpha"] \end{tikzcd}$

$kQ = k[\alpha]$

kQ is finite-dimensional $\iff$ Q has no oriented loops.

We have $\quad 1 = \sum_{i \in \text{vertices}} e_i$.

Hence, as a left module $\quad kQ = \bigoplus_{i \in I} kQe_i$

$$kQe_i = k\{\text{paths ending in } i\}.$$

Exercise: Show that kQe_i is indecomposable and projective.

If kQ is finite-dimensional, show that all ind. projectives are of this form.

A <u>representation</u> of a quiver Q consists of

- vector spaces V_i for all $i \in I$;
- morphisms $V_i \to V_j$ for all edges $i \to j$ in Q.

Representations form an abelian cat. $\mathrm{Rep}_k Q$ in an obvious way.

<u>Important exercise:</u>

Show that $\mathrm{Rep}_k Q \cong$ right kQ-modules.

<u>Rmk:</u> The theory of quiver representations is rich and fascinating. We won't be able to say more here due to lack of time.

<u>Zig-zag rings</u> ZZ_n ~~Q~~:

$$\overset{\bullet}{1} \rightleftarrows \overset{\bullet}{2} \rightleftarrows \overset{\bullet}{3} \rightleftarrows \overset{\bullet}{4}$$

We can interpret oriented paths as sequences $(i_0 | i_1 | i_2 | \dots | i_m)$

with $|i_j - i_{j+1}| = 1 \ \forall j$.

$$A = A_n = kZZ_n \Big/ \left(\begin{array}{l} (i|i+1|i+2) \\ (i|i+1|i) = (i|i-1|i) =: X_i \end{array} \right)$$

↑
"Zig zag algebra"

<u>Remark:</u> Let us determine a basis for A.

Length zero paths	(i)	$i = 1, 2, \dots, n$.
Length one paths right	$(i\|i+1)$	$i = 1, \dots, n-1$.
" " left	$(i+1\|i)$	$i = 2, \dots, n$.
Length 2 paths	X_i	$i = 1, \dots, n$.

Given an algebra $\overset{S}{A}$, $\overset{S}{A}$-bim is a monoidal category under $\otimes_A$.

If S is graded, we consider

Recall the Temperley-Lieb relations: $u_i^2 = (q+q^{-1})u_i$, $u_i u_{i+1} u_i = u_i = u_i u_{i-1} u_i$

$u_i u_j = u_j u_i$ if $|i-j| > 2$.

Consider zig-zag algebra A (graded wrt path length). Consider the A-bimodules

$$U_i := Ae_i \otimes_k e_i A\,(1)$$

degree shift: $M(1)^i := M^{i+1}$.

Hence $e_i \otimes e_i$ sits in degree -1 in U_i.

Important observation: $e_i A \otimes_A e_j Ae_j \cong e_i A e_j =$ vector space of paths $i \to j$ / relations.

Hence:
$$e_i A \otimes_A Ae_j = \begin{cases} ke_i \oplus kX_i & \text{if } i=j \\ k(i|j) & \text{if } |i-j|=1 \\ 0 & \text{o/w.} \end{cases}$$

Hence:
$$\begin{aligned} U_i \otimes_A U_i &= Ae_i \otimes_k e_i A \otimes_A Ae_i \otimes_k e_i A\,(2) \\ &= Ae_i \otimes_k (ke_i \oplus kX_i) \otimes_k e_i A\,(2) \\ &= U_i(1) \oplus U_i(-1) = U_i^{\oplus(q+q^{-1})}. \end{aligned}$$

$$\begin{aligned} U_i \otimes_A U_{i+1} \otimes_A U_i &= \overbrace{Ae_i \otimes_k e_i A} \otimes_A \overbrace{Ae_{i+1} \otimes_k e_{i+1} A} \otimes_A \overbrace{Ae_i \otimes_k e_i A}\,(3) \\ &= Ae_i \otimes_k k(i|i+1) \otimes_k k(i+1|i) \otimes_k e_i A\,(3) \\ &= U_i. \end{aligned}$$

$$U_i \otimes_A U_j = Ae_i \otimes_k e_i A \otimes_A Ae_j \otimes_k e_j A = 0 \quad \text{if } |i-j| > 2.$$

These rel$^{\text{ns}}$s remind us heavily of Temperley-Lieb!

Now we want to categorify $\quad B_{n+1} \xrightarrow{\phi} TL_{n+1} \quad$ Jones rep.

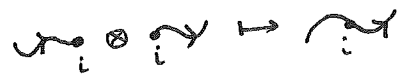
$$\begin{array}{ccc} A & & U_i(-1) \\ \wr & & \wr \\ \sigma_i & = 1 - & qe_i . \end{array}$$

~~Pro~~

Q: How does one make sense of $-qe_i$?

<u>Answer:</u> The homotopy category $K^b(A\text{-bim})$.

Define $\quad F_i := \quad 0 \to \underset{\substack{\| \\ U_i(-1)}}{Ae_i \otimes_k e_i A} \xrightarrow{m} \underset{\text{degree zero}}{A} \to 0$

where m is induced by multiplication in A: 

$*$

Consider: $\quad F_i' := \quad 0 \to A \xrightarrow{\gamma_i} U_i(1) \to 0$

where $\gamma_i : 1 \mapsto (i-1|i) \otimes (i|i-1) + (i+1|i) \otimes (i|i+1) + X_i \otimes (i) + (i) \otimes X_i$

(omit one term in sum if $i = 1$ or $i = n$.)

<u>Lemma:</u> $\quad F_i \otimes_A F_i' \cong F_i' \otimes F_i \cong A \quad$ in $K^b(A\text{-bim})$

Hint: $F_i \otimes_A F_i' = \left(U_i(-1) \longrightarrow \overset{(*)}{A} \right) \otimes_A \left(\overset{(*)}{A} \longrightarrow U_i(1) \right)$

$$\begin{tikzcd}[row sep=tiny]
 & A \arrow[dr] & \\
U_i(-1) \arrow[ur] \arrow[dr] & \oplus & U_i(1) \\
 & U_iU_i \arrow[ur] &
\end{tikzcd}$$

$$\text{\rotatebox{90}{$\cong$}}$$

$$\begin{tikzcd}[row sep=tiny]
 & A & & & \\
 & \oplus & & & \overset{(*)}{A} \\
 & U_i(1) \arrow[r, "\sim"] & U_i(1) & \cong & \\
 & \oplus & & & \\
U_i(-1) \arrow[r, "\sim"] & U_i(-1) & & &
\end{tikzcd}$$

Consider the additive category $\mathcal{A}$ of projective graded A-modules.

Up to shifts the projective A-modules are precisely $P_i := Ae_i$.

Hence $[K^b(\mathcal{A})] = \bigoplus \mathbb{Z}[v^{\pm 1}][P_i]$. (Assume this is true, more detail next time!)

The ~~bimodules~~ complexes of bimodules F_i $\quad Ae_i \otimes e_iA \longrightarrow A$

act on $\mathcal{A}$. Let us compute this action.

Good normalisation

$$F_i \qquad Ae_i \otimes e_i A \longrightarrow \overset{(*)}{A}$$

$$F_1 : \qquad \begin{array}{lcl} P_1 \longmapsto & P_1(-2) \longrightarrow & \overset{(*)}{0} \\ P_2 \longmapsto & P_1(-1) \longrightarrow & P_2 \\ P_3 \longmapsto & 0 \longrightarrow & P_3 \end{array} \qquad \begin{pmatrix} -q^2 & -q & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$F_2 : \qquad \begin{array}{lcl} P_1 \longmapsto & P_2(-1) \longrightarrow & \overset{(*)}{P_1} \\ P_2 \longmapsto & P_2(-2) \longrightarrow & 0 \\ P_3 \longmapsto & P_2(-1) \longrightarrow & P_3 \end{array} \qquad \begin{pmatrix} 1 & 0 & 0 \\ -q & -q^2 & -q \\ 0 & 0 & 1 \end{pmatrix}$$

$$F_3 : \qquad \begin{array}{lcl} P_1 \longmapsto & \cancel{P_3} & \overset{(*)}{P_1} \\ P_2 \longmapsto & P_3(-1) \longrightarrow & P_2 \\ P_3 \longmapsto & P_3(-2) \longrightarrow & 0 \end{array} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -q & -q^2 \end{pmatrix}$$

<u>Exercise:</u> Show that ~~conj~~ this representation is equivalent to the Burau representation after $q \mapsto q^2$.

(<u>Hint:</u> Conjugate by $\begin{pmatrix} q & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{-1} \end{pmatrix}$.)