Koszul duality: $\boxed{k=\mathbb{C}}$ $\boxed{\text{H242 F2-1}}$

Last time: Simon explained Koszul duality

$$\begin{array}{ccccc}
\mathcal{K}: D^{mix}_{(B)}(G/B,\mathbb{C}) & \xrightarrow{\sim} & D^{mix}_{(B)}(G/B,\mathbb{C}) & \xrightarrow[\text{Ringel}]{(-)*\Delta_{w_0}} & D^{mix}_{(B)}(G/B,\mathbb{C}) \\
P^{mix}_x & \longmapsto & IC^{mix}_{x^{-1}w_0} & \longmapsto & ? \\
\Delta^{mix}_x & \longmapsto & \nabla^{mix}_{x^{-1}w_0} & \longmapsto & \Delta_{x^{-1}} \\
IC^{mix}_x & \longmapsto & I^{mix}_{x^{-1}w_0} & \longmapsto & T^{mix}_{x^{-1}} \\
\text{Miracle:}\quad \nabla^{mix}_x & & \longmapsto & & \nabla^{mix}_{x^{-1}} \\
T^{mix}_x & & \longmapsto & & IC^{mix}_{x^{-1}} \\
\langle 1\rangle & & \longmapsto & & \langle -1\rangle[1]
\end{array}$$

Remark: One can think about Koszul duality as something like a Fourier transform

On Grothendieck groups: $H_x \longmapsto H_{x^{-1}}$

$v \longmapsto -v^{-1}$.

$\underline{H}_x \longmapsto \underline{H}'_{x^{-1}}$

canonical basis defined via condition $h_{y,x} \in v\mathbb{Z}[v]$

canonical basis defined by $h_{y,x} \in v^{-1}\mathbb{Z}[v^{-1}]$.

Can we upgrade Koszul duality to an (anti-)equivalence of monoidal categories?

$$\begin{tikzcd}
(D^{mix}_B(G/B,\mathbb{C}),*) \arrow[dr, "\text{Forget}"] & (??,*) \arrow[d] \arrow[r, "\sim"] & (??,*) \arrow[d] & ? \arrow[dl] \\
& D^{mix}_{(B)}(G/B,\mathbb{C}) \arrow[r, "\sim"] & D^b_{(B)}(G/B,\mathbb{C}) &
\end{tikzcd}$$

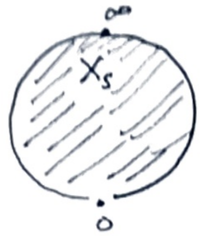
Obvious candidate!

Beilinson-Ginzburg 1999: "the dual of $D^{mix}_B(G/B,\mathbb{C})$ is a completion of the category of unipotent monodromic sheaves on base affine space".

Consider the case of $\mathbb{P}^1$

$\mathrm{Perv}_B(\mathbb{P}^1) \xrightarrow{\text{forget}} \mathrm{Perv}_{(B)}(\mathbb{P}^1)$

$\uparrow$ $\boxed{B \text{ constructible!}}$



$\mathrm{Perv}_{(B)}(\mathbb{P}^1)$ has 5 indecomposable objects.

4 are B-equivariant: $\Delta_0 = \nabla_0 = IC_0 = \underline{\mathbb{Q}}_0$.

$\Delta_s = j_! \underline{\mathbb{Q}}_{X_s}[1]$ $\qquad$ $\nabla_s = j_* \underline{\mathbb{Q}}_{X_s}[1]$

$IC_s = \underline{\mathbb{Q}}_{\mathbb{P}^1}[1]$.

The last one is $T_s = P_0 = \begin{pmatrix} IC_0 \\ \hline IC_s \\ \hline IC_0 \end{pmatrix} = \begin{pmatrix} \Delta_{id} \\ \hline \Delta_s \end{pmatrix}$

$\mathrm{Ext}^m(\Delta_{id}, \Delta_s) = \mathrm{Hom}(i_! \underline{\mathbb{Q}}_0, j_! \underline{\mathbb{Q}}_{X_s}[1+m]) = H^{m+1}(i^! j_! \underline{\mathbb{Q}}_{X_s})$.

To calculate consider: $j_! j^! \underline{\mathbb{Q}}_{\mathbb{C}} \to \underline{\mathbb{Q}}_{\mathbb{C}} \to i_* i^* \underline{\mathbb{Q}}_{\mathbb{C}} \xrightarrow{+1}$

Apply $i^!$ and take cohomology:

2	$\mathbb{Q}$ $\xrightarrow{\sim}$	$\mathbb{Q}$	0
Ext1 $\to$ 1	$\mathbb{Q}$	0	0
Hom $\to$ 0		0	$\mathbb{Q}$

$\rightsquigarrow$ shows existence of T_s.

If we instead do this equivariantly:

		$H^{\bullet+2}_{\mathbb{C}^\times}(pt)$	$H^{\bullet}_{\mathbb{C}^\times}(pt)$
			$\vdots$
			0
4	0	$\mathbb{Q}$ $\xrightarrow{\sim}$	$\mathbb{Q}$
3	0	0	0
Ext^1=0 $\to$ 2	0	$\mathbb{Q}$ $\xrightarrow{\sim}$	$\mathbb{Q}$
1	$\mathbb{Q}$	0	0
Hom $\to$ 0	0	0	$\mathbb{Q}$

$\rightsquigarrow$ T_s does not lift!

Remarks: ① Under localisation, Soergel equivalence corresponds to fact that Z^+ does not act by zero on $P_{-2} \twoheadrightarrow \Delta_{-2}$.
$Z^+ = \ker(Z \to \mathbb{C})$

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② Another "explanation": microlocal stalk ~~at~~ of T_s at id is $\mathcal{L}$ on $\mathbb{C}^\times$ w/ monodromy $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which is not $\mathbb{C}^\times$-equivariant.

③ For $G = SL_2$, $C = k[\varepsilon]/(\varepsilon^2)$. $D_{id} = k$, $D_s = C[1]$.

In $D^b_{mix}(\mathbb{P}^1)$, $T_s \longleftrightarrow$

$$\begin{array}{ccccccccc} & & -1 & & 0 & & 1 & & \\ \hline 0 & \to & D_{id}(-1) & \to & D_s & \to & D_{id}(1) & \to & 0 \end{array}$$

Each map admits a unique lift, hence lift would have to be:

$$0 \to B_{id}(-1) \xrightarrow{\text{ɗ}} B_s \xrightarrow{\text{ƥ}} B_{id}(1) \to 0$$

but this is not a complex!

However, as U is a unipotent group: $D^b_{(B)}(\mathbb{P}^1) \simeq D^b_U(\mathbb{P}^1)$.

$\rightsquigarrow$ T lifts to $U \backslash G / B$.

$\rightsquigarrow$ Seem to be forced to work w/ $U \backslash G / U$. $\rightsquigarrow$ explains "basic affine space"

Next problem already apparent w/ $G = \mathbb{C}^\times$:

Expect: weight zero $\longleftrightarrow$ perverse.
Hence $\circledast$ on RHS should be t-exact!

Exercise: ⓐ Let $\mathcal{L}$ be a f.d. local system on $\mathbb{C}^\times$. Show that $H^0(\mathcal{L}) \neq 0 \Leftrightarrow H^1(\mathcal{L}) \neq 0$.

ⓑ Deduce that if $\mathcal{L}, \mathcal{L}'$ are f.d. local systems then $m_*(\mathcal{L} \boxtimes \mathcal{L}')$ is never concentrated in one degree.

$m : \mathbb{C}^\times \times \mathbb{C}^\times \to \mathbb{C}^\times$ mult.

ⓒ Show that if $\mathcal{L}$ det. by $\begin{pmatrix} 1 & 1 & & \\ 0 & 1 & 1 & \\ 0 & 0 & 1 & \ddots \\ & & & \ddots \end{pmatrix}$, then $H^{i}(\mathbb{C}^\times, \mathcal{L}) = \begin{cases} \mathbb{Q} & i = 1 \\ 0 & o/w. \end{cases}$

Upshot: No exact convolution on f.d. local systems on $\mathbb{C}^\times$.

Need to pass to completions.

Дача F2-4

Bezrukavnikov–Yun: $\pi : G/U \to G/B$.

$\hat{D}(B \backslash G/B) =$ certain pro objects in $\langle \pi^* D^b_{(B)}(G/B) \rangle \subset D^b_U(G/U)$

equipped w/ $\tilde{*}$ $\left(m_!(\mathcal{F} \boxtimes \mathcal{G})[r] \right)$ rank of T.

<u>Thm:</u> (BY) Equivariant monodromic duality:

$$(D^b_{mix}(B\backslash G/B), *) \xrightarrow{\sim} (D^b_m(\check{B} \backslash \check{G} / \check{B}), \tilde{*})$$

$$IC_x \longmapsto \check{T}_x$$

$$\vdots$$