# The Finkelberg - Mirković conjecture

## 1 - Reminder on the Satake equivalence

$G$ connected reductive gp over $\mathbb{F}$ alg. closed field
$B \subset G$ Borel subgp
$T \subset B$ max$^{l}$ torus
$\mathcal{K} = \mathbb{F}((x))$ , $\mathcal{O} = \mathbb{F}[[x]]$.

Loop group: $G_{\mathcal{K}}$ = ind-gp scheme over $\mathbb{F}$ which represents the functor $R \mapsto G(R((t)))$

Arc gp: $G_{\mathcal{O}}$ = gp scheme ~~(not of~~ over $\mathbb{F}$ which represents the functor $R \mapsto G(R[[t]])$.

NB: $G_{\mathcal{O}}$ is a scheme, but NOT of finite type.

Affine Grassmannian: $Gr_G := G_{\mathcal{K}}/G_{\mathcal{O}}$ (fpqc quotient)
This is an ~~ind~~-projective ind-scheme of ind-finite type.
We have a ~~closed~~ embeddings

$$\begin{array}{ccc} Gr_T = T_{\mathcal{K}}/T_{\mathcal{O}} & \hookrightarrow & Gr_G = G_{\mathcal{K}}/G_{\mathcal{O}} \\ \uparrow & [x^\lambda] & \\ & \uparrow & \\ X_*(T) & \ni \lambda & \end{array}$$

with $x^\lambda$ = image of $x$ under $\mathcal{K}^\times \longrightarrow T_{\mathcal{K}}$ associated with $\lambda$.

Cartan decomposition: $Gr_G = \bigsqcup_{\lambda \in X_*(T)^+} Gr_G^\lambda$ with $Gr_G^\lambda = G_{\mathcal{O}} x^\lambda G_{\mathcal{O}}$

$X_*(T)^+$ — dominant coweights

Each $\overline{Gr_G^\lambda}$ is a scheme (projective, of finite type)

Satake category: $k$ coefficient ring
2 possible settings: (1) $\mathbb{F}$ arbitrary, $k$ s.t. étale $k$-sheaves ~~make~~ ~~sense~~ behave reasonably (e.g. finite field of char $\neq$ char$(\mathbb{F})$, or finite extension of $\mathbb{Q}_\ell$ with $\ell \neq$ char$(\mathbb{F})$, or ring of integers in such an extension)

(2) $\mathbb{F} = \mathbb{C}$, $k$ commutative Noetherian ring of finite global dimension.

↳ $\mathrm{Perv}_{G_O}(Gr_G, k)$ = category of $G_O$-equivariant $k$-perverse sheaves on $Gr_G$ (either in the étale sense in setting (1), or in the classical sense in setting (2))

More precisely: For any $X \subset Gr_G$ finite closed union of $G_O$ orbits, there exists $K \subset \ker(G_O \to G)$ s.th. $G_O/K$ is of finite type and the $G_O$-action on $X$ factors through $G/K$ ⇝ set $\mathrm{Perv}_{G_O}(X, k) := \mathrm{Perv}_{G_O/K}(X, k)$ (does not depend on choice of $K$)

And the $\mathrm{Perv}_{G_O}(Gr_G, k) = \varinjlim_{\substack{X \subset Gr_G \\ \text{closed finite} \\ \text{union of } G_O\text{-orbits}}} \mathrm{Perv}_{G_O}(X, k)$.

Convolution product:

$$Gr_G \times Gr_G \xleftarrow{p} G_K \times Gr_G \xrightarrow{q} G_K \times^{G_O} Gr_G \xrightarrow{m} Gr_G$$

Given $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}_{G_O}(Gr_G, k)$, there exists a unique perverse sheaf $\mathcal{F} \tilde{\boxtimes} \mathcal{G}$ on $G_K \times^{G_O} Gr_G$ s.th.

$$\cancel{p^*(\mathcal{F} \overset{L}{\boxtimes}_k \mathcal{G})} \; p^* \mathcal{H}^0(\mathcal{F} \overset{L}{\boxtimes}_k \mathcal{G}) \simeq q^*(\mathcal{F} \tilde{\boxtimes} \mathcal{G})$$

↳ $\boxed{\mathcal{F} * \mathcal{G} := m_*(\mathcal{F} \tilde{\boxtimes} \mathcal{G})}$

Facts: 1) $\mathcal{F} * \mathcal{G}$ is a perverse sheaf (follows from the fact that $m$ is a stratified semismall map)

2) the monoidal product $*$ admits a canonical associativity constraint. (Again, uses the fact that $m_*$ is t-exact.)

3) the convolution product admits a commutativity constraint (construction due to Drinfeld - uses interpretation of $Gr_G$ in terms of $G$-bundles on a curve.)

Notation: For $H$ a $k$-gp scheme of finite type, $\mathrm{Rep}(H)$ = abelian cat. of $H$-modules which are of finite type over $k$.

CENTRE HENRI LEBESGUE

CENTRE DE MATHÉMATIQUES



**Theorem** (Lusztig, Ginzburg, Mirković-Vilonen)

There exists an equivalence of symmetric monoidal categories $(\mathrm{Perv}_{G_O}(Gr_G, k), *) \xrightarrow{\sim} (\mathrm{Rep}(G_k^\vee), \otimes_k)$ where $G_k^\vee$ is the ~~split~~ ~~cover~~ ~~reductiv~~ base-change to $k$ of the unique split ~~Grothendie~~ reductive gp $G_{\mathbb{Z}}^\vee$ over $\mathbb{Z}$ s.th. $G_{\mathbb{Z}}^\vee \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{C})$ has root datum $(X_*(T), X^*(T), \text{coroots}, \text{roots})$.

Moreover, under this equivalence, the forgetful functor $\mathrm{Rep}(G_k^\vee) \to \mathrm{Mod}_k^{fg}$ corresponds to $\mathbb{H}^*(Gr_G, -)$

For $\lambda \in X_*(T)$ we have $j^\lambda : Gr_G^\lambda \hookrightarrow Gr_G$

$$\rightsquigarrow \quad J_*(\lambda, k) := {}^p\mathcal{H}^0((j^\lambda)_* \underline{k}_{Gr_G^\lambda}[\dim Gr_G^\lambda])$$
$$J_!(\lambda, k) := {}^p\mathcal{H}^0((j^\lambda)_! \underline{k}_{Gr_G^\lambda}[\dim Gr_G^\lambda])$$
$$J_{!*}(\lambda, k) := (j^\lambda)_{!*} \underline{k}_{Gr_G^\lambda}[\dim Gr_G^\lambda].$$

Under the Satake equivalence we have

$$J_*(\lambda,k) \longleftrightarrow H^0(\lambda)$$
$$J_!(\lambda,k) \longleftrightarrow V(\lambda) = (H^0(-w_0\lambda))^*$$
$$J_{!*}(\lambda,k) \longleftrightarrow L(\lambda) \quad \text{(if } k \text{ is a field)}.$$

~~The Finkelberg Mirković Conjecture~~

**Tilting objects**

Assume that $k$ alg. closed field.

↳ Q: What are the objects in $\mathrm{Perv}_{G_O}(Gr_G, k)$ corresponding to the indecomposable tilting objects in $\mathrm{Rep}(G_k^\vee)$?

**Conjecture** (Juteau-Mautner-Williamson, 2008)

1) For all $k$, the tilting objects correspond to the direct sums of direct summands of objects ${}^p\mathcal{H}^0(\mathcal{E}_\lambda)$ for $\lambda \in X_*(T)^+$ with $\mathcal{E}_\lambda$ = indecomposable parity object labelled by $\lambda$ for the stratification $Gr_G = \bigsqcup_{\lambda \in X_*(T)^+} Gr_G^\lambda$

2) If char(k) is good for $G$ then $\mathcal{E}^\lambda$ is perverse and corresponds to $T(\lambda)$ for all $\lambda \in X_*(T)^+$!

<u>Recall</u>: bad primes: $\emptyset$ in type A, 2 for B/C/D, 2&3 for $E_6, E_7, F, G$ and 2,3&5 for $E_8$

Status:

⊗ JMW (paper published in 2016):

- If char(k) is bad then not all $\mathcal{E}^\lambda$'s are perverse
- If $\mathcal{E}^\lambda$ is perverse then it corresponds to $T(\lambda)$
- All $\mathcal{E}^\lambda$'s are perverse if $p$ bigger than

| type of $G_k^\vee$ | A | B/D | $C_n$ | $G_2, F_4, E_6$ | $E_7$ | $E_8$ |
|---|---|---|---|---|---|---|
| bound | 1 | 2 | n | 3 | 19 | 31 |

⊗ Mautner–R (paper published in 2018):
if char(k) is good, then all $\mathcal{E}^\lambda$'s are perverse

⊗ Bezrukavnikov–Gaitsgory–Mirković–R.–Rider:
In general, the tilting objects correspond to the direct sums of direct summands of ${}^p\mathcal{H}^0(\mathcal{E}^\lambda)$.

## The Finkelberg–Mirković ~~equivalence~~ conjecture

From now on: $k$ alg. closed field of char. $p > h$.
$G$ connected reductive over $k$.

→ Want to realize geometrically the principal block $\mathrm{Rep}_0(G_k^\vee)$, or more generally the "extended principal block":

$$\mathrm{Rep}_0^{ext}(G_k^\vee) = \langle L(w \cdot_p 0) : w \in {}^f W_{ext} \rangle_{Serre}$$

with $W_{ext} = W \ltimes X^*(T_k^\vee)$.

This is a subcategory of $\mathrm{Rep}(G)$
→ But difficult to "see" it in ~~$\mathrm{Perv}_{G_O}(Gr_G, k)$~~ the Satake category.

Other point of view:

$$\mathrm{Rep}_0^{ext}(G) \supset \mathrm{Rep}(\dot{G}) \quad \text{via } V \mapsto Fr^*(V)$$

($\dot{G}$ = Frobenius twist of $G$
$Fr : G \to \dot{G}$ Frobenius map)

→ We need to consider a category that <u>contains</u> $\mathrm{Perv}_{G_O}(Gr_G, k)$ with $G_k^\vee = \dot{G}$.

$Iw \subset$ ~~$G_{\mathcal{O}}$~~ $G_{\mathcal{O}}$ Iwahori subgp associated with the positive Borel in $G$

Conjecture (Finkelberg, Mirković, 1999)

There exists an equivalence of abelian categories

$$\begin{array}{ccc} \mathrm{Rep}_0^{ext}(\mathbb{G}) & \simeq & \mathrm{Perv}_{(Iw)}(Gr_G, k) \\ \uparrow Fr^* & & \uparrow \text{embedding.} \\ \mathrm{Rep}(\dot{\mathbb{G}}) & \underset{\text{Satake}}{\simeq} & \mathrm{Perv}_{G_{\mathcal{O}}}(Gr_G, k) \end{array}$$

Underlying combinatorics:

$$\{ Iw\text{-orbits on } Gr_G \} \overset{1:1}{\longleftrightarrow} X_*(T)$$

$$Iw \cdot x^\lambda G_{\mathcal{O}} \longleftarrow\!\shortmid \lambda$$

$\rightarrow$ get objects
$$\begin{cases} \Delta_\lambda = (j_\lambda)_! \underline{k}_{Gr_{G,\lambda}} [\dim Gr_{G,\lambda}] \\ \nabla_\lambda = (j_\lambda)_* \underline{k}_{Gr_{G,\lambda}} [\dim Gr_{G,\lambda}] \\ IC_\lambda = (j_\lambda)_{!*} \underline{k}_{Gr_{G,\lambda}} [\dim Gr_{G,\lambda}] \end{cases}$$

On the other hand we have

$$\{ \text{simple objects in } \mathrm{Rep}_0^{ext}(G) \}_{/\text{isom.}} \longleftrightarrow {}^f W_{ext}$$

$$[L(w \cdot_p 0)] \longleftarrow\!\shortmid w$$

We have $\quad {}^f W_{ext} \xrightarrow{\sim} W_f \backslash W_{ext} \xleftarrow{\sim} X_*(T)$

Under this bijection, $\Delta_\lambda$ should corresp. to $V(w \cdot_p 0)$

$\nabla_\lambda$ ——— $H^0(w \cdot_p 0)$

$IC_\lambda$ ——— $L(w \cdot_p 0)$

Relation with Lusztig's conjecture

We want to write $[L(w \cdot_p 0)] = \sum_{y \in {}^f W_{ext}} (?)_{y,w} [H^0(y \cdot_p 0)]$

in a certain region. If we believe in the FM conjecture, this is equivalent to writing

$$[\mathcal{IC}_\lambda] = \sum_{\mu} (?)_{\mu,\lambda} [\nabla_\mu]$$

Here we know what to put:

$$\sum_n (-1)^n \dim\left(H^{n+\dim(\mathrm{Gr}_{G,\mu})}\left(i_\mu^! \mathcal{IC}_\lambda\right)\right)$$

KL polynomials compute this alternating sum in case $\mathcal{IC}_\lambda$ is replaced by its analogue for char-0 coeff$^{ts}$

→ the Lusztig conjecture can be rephrased as saying that the Euler characteristic of IC sheaves of Iw-orbits on $\mathrm{Gr}_G$ (in the appropriate region) are the same in char. 0 and in characteristic p...

<u>Status</u>: Conjecture not proved at the moment.

Only a "mixed version" known (where ordinary perverse sheaves are replaced by their "mixed" analogues from the previous lecture).