Affine Weyl group

Fix a root system + positive roots $R^+ \subset R \subset \mathcal{X}$.

$\alpha \in R \rightsquigarrow s_\alpha(\lambda) := \lambda - \langle \alpha^\vee, \lambda \rangle \alpha$ reflection.

$W_f = \langle s_\alpha \mid \alpha \in R \rangle = \langle s_\alpha \mid \alpha \in R^s \rangle$ "simple reflections", S_f

W_f is a Coxeter system w/ Coxeter generators S_f.

$W = W_f \ltimes \mathbb{Z}R \curvearrowright \mathcal{X}$, "affine Weyl group".

Rmk: of dual root system!

$s_{\alpha,m}(\lambda) = \lambda - (\langle \alpha^\vee, \lambda \rangle - m)\alpha = \lambda - \langle \alpha^\vee, \lambda \rangle \alpha + m\alpha$

affine reflection through the hyperplane $H_{\alpha,m} = \{\lambda \mid \langle \alpha^\vee, \lambda \rangle = m\}$.

Exercise: $t_{m\alpha} = s_{\alpha,m} \circ s_{\alpha,0}$, hence

$W = \langle s_{\alpha,m} \mid \alpha \in R, m \in \mathbb{Z} \rangle$ is generated by reflections.

General theory of affine reflection groups:

$W \curvearrowright \mathcal{X}_{\mathbb{R}} = \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{R}$

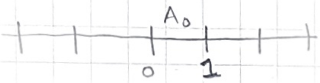
$\mathcal{A}$ = "alcoves" = connected components of $\mathcal{X}_{\mathbb{R}} \setminus \bigcup H_{\alpha,m}$

$W \curvearrowright \mathcal{A}$ simply transitively.

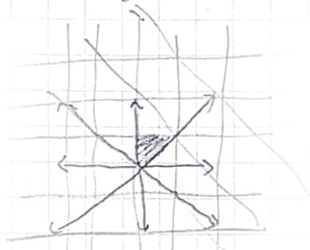
Fix an alcove $A_0 \subset \mathfrak{A}$ (usually $A_0 = \{\lambda \mid 0 \leq \langle \alpha^\vee, \lambda\rangle < 1 \quad \forall \alpha \in R^+\}$)

- $\overline{A_0}$ is a fundamental domain for $W_{\dot{p}}$
- Reflections in the walls of $\overline{A_0}$ yield Coxeter/generators $S \subset W$.

E.g.: sl_2:



Sp_4:



Linkage principle:

Recall for $\mathfrak{g}$ complex semi-simple Lie algebra:

$$Z = Z(U(\mathfrak{g})) = \mathcal{O}(\mathfrak{h}^*/W_f \cdot) \quad \text{Harish-Chandra isomorphism.}$$

$$\Rightarrow \quad \mathfrak{g}\text{-mod}^{Z\text{-fin}} = \bigoplus_{\lambda \in \mathfrak{h}^*/W_f \cdot} \mathfrak{g}\text{-mod}_\lambda^{Z\text{-fin}}$$

$\boxed{\text{We want something analogous for algebraic representations.}}$

$W \overset{\cdot}{\underset{p}{\circlearrowright}} \mathcal{X}$ "p-dilated affine Weyl group":

$$W_f \ni x \underset{p}{\cdot} \lambda = x(\lambda + \rho) - \rho,$$

$$W \ni t_\gamma \cdot \lambda = \lambda + p\gamma.$$

$\boxed{\text{Alternatively, take above arrangement, dilate by } p\text{, and shift by } -\rho.}$

$$Rep_\lambda = \langle L_\gamma \mid \gamma \in (W \underset{p}{\cdot} \lambda) \cap \mathcal{X}_+ \rangle$$

Linkage principle: $\mathrm{Rep}\, G = \bigoplus_{\gamma \in \mathcal{X}/W_p^{\cdot}} \mathrm{Rep}_\gamma$

Rmks: ① γ is p-regular $\iff$ stab in $W_p^{\cdot}$ action is trivial.

Coxeter number.

p-Regular weights exist $\iff$ 0 is p-regular $\iff$ $\langle \overbrace{\underbrace{\tilde{\alpha}^\vee}_{\text{highest coroot}}, \rho}^{h-1 \,=} \rangle < p$

② Rep_γ is not indecomposable in general, but Rep_0 for $p \geq h$ is.

$\boxed{\text{Why do we believe this?}}$

③ If p is not too small, then HC mod p gives

$$\mathrm{Ext}^1(L_\lambda, L_\mu) = 0 \quad \text{if} \quad \lambda \notin W \cdot \bar{\lambda} \qquad \bar{\lambda} \in \mathcal{X}/p\mathcal{X}.$$

+ action of centre of G gives the result.

$\boxed{\text{For small primes proof (Andersen, Jantzen) is rather tricky.}}$

④ If $p \geq h$, highest weights in Rep_0 are

$$\{ w \cdot_p 0 \mid w \cdot_p 0 \in \mathcal{X}^+ \} = \{ w \mid w \text{ minimal in } W/W_f \}$$

"independent of p".

Translation functors:

$$C_+^p := \{ \lambda \in X \mid 0 \le \langle \alpha^\vee, \lambda+\rho \rangle \le p \quad \forall \alpha \in R^+ \}.$$

$$C_+^p \xrightarrow{\sim} X/(W_p^\circ)$$ system of representatives.

$-\rho$

$\lambda, \mu \in C_+^p$: $\quad T_\lambda^\mu := pr_\mu(L(\nu) \otimes M) : Rep_\lambda \longrightarrow Rep_\mu$.

$\uparrow$ unique dominant weight in $W(\mu-\lambda)$.

$\otimes$ on general reps is very hard, but $L(\nu)$ are relatively easy, hope to use T_λ^μ to study Rep G.

Thm: If λ, μ have same stabilizer in W_p° (e.g. both p-regular) then $T_\lambda^\mu : Rep_\lambda \to Rep_\mu$ is an equivalence.

Also, one understands  passage to "more singular weights".

Upshot: For character Q's, + many structural Q's it is enough to understand Rep_0 (if $p \ge h$).