Character formulas I

Ref: Soergel - KL polynomials and a combinatorics for tilting modules

Hecke algebra and Kazhdan-Lusztig basis

(W,S) Coxeter system. (In practice: finite or affine Weyl gp)

Recall the Bruhat order:

- order generated by $w' < w$ if $w = w't$ with $t \in T = \bigcup_{w\in W} wSw^{-1}$ and $\ell(w) > \ell(w')$
- if $w = s_1 \cdots s_r$ reduced expression, then $\{y \in W \mid y \leq w\} = \{ s_{i_1} \cdots s_{i_k} : 1 \leq i_1 < \cdots < i_k \leq r \}$

Hecke algebra of (W,S):

$$\mathcal{H}_{(W,S)} = \bigoplus_{w\in W} \mathbb{Z}[v,v^{-1}] \cdot H_w$$

with the unique $\mathbb{Z}[v,v^{-1}]$-algebra structure s.th.

- $H_x H_y = H_{xy}$ if $\ell(xy) = \ell(x) + \ell(y)$
- $(H_s + v)(H_s - v^{-1}) = 0$ if $s \in S$.

Note: each H_s is invertible (with $H_s^{-1} = H_s + (v - v^{-1})$), hence each H_w is invertible.

Kazhdan-Lusztig involution: unique ring involution $H \mapsto \overline{H}$ of $\mathcal{H}_{(W,S)}$ with $\overline{v} = v^{-1}$, $\overline{H_w} = (H_{w^{-1}})^{-1}$ for $w \in W$.

Theorem (Kazhdan-Lusztig, 1979)

For all $w \in W$, there exists a unique element $\underline{H}_w$ s.th. $\overline{\underline{H}_w} = \underline{H}_w$ and $\underline{H}_w \in H_w + \sum_{y\in W} v\mathbb{Z}[v] H_y$.

Moreover, $\underline{H}_w \in H_w + \sum_{y<w} v\mathbb{Z}[v] H_y$, and $(\underline{H}_w : w \in W)$ is a $\mathbb{Z}[v,v^{-1}]$ basis of $\mathcal{H}_{(W,S)}$ (called the KL or canonical basis)

In practice: $\underline{H}_s = H_s + v$ for $s \in S$

Then if $w \in W$, $s \in S$ s.th. $ws < w$, then we write $\underline{H}_{ws}\underline{H}_s = H_w + \sum_{y<w} h_y H_y$

with $h_y \in \mathbb{Z}[v]$ and get $\underline{H}_w = \underline{H}_{ws}\underline{H}_s - \sum_{y<w} h_y(0)\underline{H}_y$.

KL polynomials: Determined by

$$\underline{H}_w = \sum_{y\in w} h_{y,w} H_y .$$

KL basis for spherical and antispherical modules

$S_0 \subset S$ subset, $W_0 = \langle S_0 \rangle \subset W$

(In practice W = affine Weyl group, W_0 = finite Weyl gp)

$${}^{0}W = \{ w \in W \mid \forall s \in S_0,\ sw > w \}$$

Then we have $\begin{cases} W_0 \times {}^0W \xrightarrow{\sim} W \\ (x,y) \longmapsto xy \end{cases}$

$\mathcal{H}_0 = \mathcal{H}_{(W_0,S_0)} \subset \mathcal{H} = \mathcal{H}_{(W,S)}$

We have 2 algebra maps $\mathcal{H}_0 \longrightarrow \mathbb{Z}[v,v^{-1}]$:

$\varphi_{v^{-1}}$ determined by $\varphi_{v^{-1}}(H_s) = v^{-1}$ $\forall s \in S_0$

φ_{-v} ——— $\varphi_{-v}(H_s) = -v$ $\forall s \in S_0$.

$\rightsquigarrow$ 2 module structures on $\mathbb{Z}[v,v^{-1}]$: $\mathcal{L}(v^{-1})$, $\mathcal{L}(-v)$

We set $\begin{cases} {}^0\mathcal{M} = \mathcal{L}(v^{-1}) \otimes_{\mathcal{H}_0} \mathcal{H} \\ {}^0\mathcal{N} = \mathcal{L}(-v) \otimes_{\mathcal{H}_0} \mathcal{H} \end{cases}$ (right $\mathcal{H}$-modules)

${}^0\mathcal{M}$ has a "standard basis" $(M_w = 1\otimes H_w : w \in {}^0W)$

${}^0\mathcal{N}$ ——— $(N_w = 1\otimes H_w : w \in {}^0W)$

The maps $a\otimes H \mapsto \bar{a}\otimes \bar{H}$ determine involutions $M \mapsto \bar{M}$ of ${}^0\mathcal{M}$ and ${}^0\mathcal{N}$

Theorem (Deodhar, 1987)

1 - For all $w \in {}^0W$, there exists a unique $\underline{M}_w \in {}^0\mathcal{M}$ s.t. $\overline{\underline{M}_w} = \underline{M}_w$ and $\underline{M}_w \in M_w + \sum_y v\mathbb{Z}[v] M_y$.

Moreover, $\underline{M}_w \in M_w + \sum_{y<w} v\mathbb{Z}[v] M_y$, and $(\underline{M}_w : w \in {}^0W)$ is a $\mathbb{Z}[v,v^{-1}]$-basis of ${}^0\mathcal{M}$

2 - Same for ${}^0\mathcal{N}$ $\rightsquigarrow$ basis $(\underline{N}_w : w \in {}^0W)$

Corresponding KL polynomials: determined by

$$\underline{M}_w = \sum_y m_{y,w} M_y$$

$$\underline{N}_w = \sum_y n_{y,w} N_y$$

Relation with ordinary KL basis:

- We have $\xi : \begin{cases} \mathcal{H} \twoheadrightarrow {}^0\mathcal{N} \\ H \longmapsto 1 \otimes H \end{cases}$

Fact: $\xi(\underline{H}_w) = \begin{cases} \underline{N}_w & \text{if } w \in {}^0W \\ 0 & \text{otherwise} \end{cases}$

- Assume that W_0 is finite, with longest element w_0. Then we have $\zeta : \begin{cases} \mathcal{M} \hookrightarrow \mathcal{H} \\ 1 \otimes H \longmapsto \underline{H}_{w_0} \cdot H \end{cases}$

Fact: $\zeta(\underline{M}_w) = \underline{H}_{w_0 w}$ for all $w \in {}^0W$.

Lusztig's conjecture

Setting: G connected reductive gp (with simply-connected derived subgp)
W = affine Weyl group
S = simple reflections (= reflections along the walls of the fundamental alcove)
$p \geq h$ = Coxeter number
$\mathrm{Rep}_0(G)$ = principal block
$= \langle L(w \cdot_p 0) : w \in W \text{ s.t. } w \cdot_p 0 \in \mathbb{X}^{\vee +} \rangle_{\text{Serre}}$

Here, $w \cdot_p 0 \in \mathbb{X}^\vee \iff w \in {}^f W$

minimal representatives for parabolic subgp W_f = finite Weyl gp $\subset W$.

$$[\mathrm{Rep}_0(G)] = \bigoplus_{w \in {}^f W} [H^0(w \cdot_p 0)]$$

Goal: express $[L(w \cdot_p 0)]$ in this basis for "sufficiently many" ~~st~~ w's (i.e. enough so that one can recover all simple characters using Steinberg's tensor product theorem)

<u>Conjecture</u> (Lusztig, 1980)

Assume that $w \in {}^f W$ is such that
$$\langle w \cdot_p 0 + \rho, \alpha^\vee \rangle \leq p(p-h+2) \text{ for all } \alpha \in R^+$$
Then $$[L(w \cdot_p 0)] = \sum_{y \in {}^f W} (-1)^{\ell(w)+\ell(y)} h_{w_0 y, w_0 w}(1) \cdot [H^0(y \cdot_p 0)]$$

<u>Rmks:</u>

(1) For $y, w \in {}^f W$ we have $h_{w_0 y, w_0 w} = m_{y,w}$
for "m" defined wrt $W_f \subset W$

(2) <u>Very</u> important aspect: the formula is <u>independent of p</u> in its region of validity

(3) <u>Meaning of the bound</u> on w:
Write $w \cdot_p 0 = \lambda_0 + p\lambda_1$ with $\lambda_0 \in X_1$, $\lambda_1 \in X^+$
Then $L(w \cdot_p 0) \simeq L(\lambda_0) \otimes Fr^*(L(\lambda_1))$
The bound says that $\langle \lambda_0 + p\lambda_1 + \rho, \alpha^\vee \rangle \leq p(p-h+2)$ for all $\alpha \in R^+$

$$\Rightarrow \quad p\langle \lambda_1, \alpha^\vee \rangle < p(p-h+2)$$
$$\Rightarrow \quad \langle \lambda_1 + \rho, \alpha^\vee \rangle < (p-h+2) + (h-1) = p+1$$

Hence λ_1 is in the closure of the fundamental alcove, so that <u>$L(\lambda_1) = H^0(\lambda_1)$.</u>

This bound should be seen as a simple condition that guarantees that this property holds, which seems necessary if we expect a formula which is independent of p.
(This condition is usually called "Jantzen's condition" since it appeared earlier in work of Jantzen.)

(4) If $\lambda \in X_1$ and $\alpha \in R^+$ we have
$$\langle \lambda + \rho, \alpha^\vee \rangle \leq \langle (p-1)\rho + \rho, \alpha^\vee \rangle = p(h-1)$$
$\Rightarrow$ if $p-h+2 \geq h-1$ (i.e. $p \geq 2h-3$) then all w's s.th.

$w \cdot_p 0 \in X_1$ satisfy Jantzen's condition.
In this case, from Lusztig's conjecture one can compute the characters of <u>all</u> simple G-modules.

<u>History</u>:

× <u>1980</u>: Statement of the conjecture by Lusztig:
"The evidence for this character formula is very strong. I have verified it in the cases where G is of type A_2, B_2 or G_2. (In these cases, $ch(L_w)$ has been computed by Jantzen.)"
[In fact, A_3 was also treated by Jantzen at that time.]

× <u>1982</u>: Kato proves that Lusztig's conjecture holds iff the formula holds for all $w \in {}^fW$ s.th. $w \cdot_p 0 \in X_1$ ~~[illegible]~~ (and $w \cdot_p 0$ satisfies Jantzen's condition)
(→ some authors state the conjecture for all w s.th. $w \cdot_p 0 \in X_1$)
↳ gives a finite number of cases to check, indep. of p.

× <u>~1990</u>: Lusztig proposes a program to prove the conjecture, involving quantum gps at a root of unity

× <u>1995/96</u>: Realization of the program:

$\mathrm{Rep}_0(G) \longleftrightarrow \left\{\begin{array}{l}\text{principal block} \\ \text{of Lusztig's quantum} \\ \text{gp at a root of } 1\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Rep}^{\text{ns}} \text{ of} \\ \text{affine} \\ \text{Lie alg.}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Perverse sheaves} \\ \text{on affine} \\ \text{flag variety}\end{array}\right\}$

Andersen-Jantzen-Soergel

Kazhdan-Lusztig (+Lusztig)

Kashiwara-Tanisaki

KL (1980)

$\left\{\text{Affine Hecke alg.}\right\}$

<u>But</u>: the 1st step works only for $p \gg 0$ (with no explicit bound)
× <u>2012</u>: Fiebig produces an "explicit" but enormous bound on p over which the conjecture holds ($\sim 10^{40}$ for GL_9)
× <u>2016</u>: Williamson gives families of examples for which Lusztig's formula does <u>not</u> hold (for $G = GL_n(k)$).
Consequence: there <u>cannot</u> exist a polynomial P s.th. the conjecture holds for $p \geq P(h)$.
(First counterexample with $p > h$: for $GL_{22}(k)$.)

Examples

① $G = SL_2$.

~~Restricted~~ $X_1 = \{0, \dots, p-1\}$

$\rightsquigarrow$ Only 1 restricted dominant weight in $W \cdot_p 0$,

and $H^0(0) = L(0) = k$.

Jantzen's condition: $n+1 \leq p^2 \iff n < p^2$

In this case we have $h_{x,y}(1) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$

(W = infinite dihedral group).

$\rightsquigarrow$ Lusztig's conjecture can be checked by hand.

② $w = s \in S \setminus S_f$.

In this case we have $H^0(s \cdot_p 0) = \boxed{\begin{array}{c} L(0) \\ \hline L(s \cdot_p 0) \end{array}}$

(we know that only $L(0)$ can appear, and one can compute $\mathrm{Hom}(H^0(s \cdot_p 0), L(0))$ using translation functors)

$\Rightarrow [L(s \cdot_p 0)] = [H^0(s \cdot_p 0)] - [H^0(0)]$.

$\rightarrow$ For $G = SL_3$ this is the only element in $(W \cdot_p 0) \cap X_1$.

③ For types B_2, G_2, A_3: characters can be computed using Jantzen's sum formula.

II-4

Derivation of weight multiplicities in char. 0 out of Lusztig's conjecture

We fix $\lambda \in X^+$, and assume that $p \gg 0$ so that
$$L(\lambda) = H^0(\lambda)$$
Then we have
$$\begin{aligned} ch(L(p\lambda)) &= ch(Fr^*(L(\lambda))) \\ &= ch(H^0(\lambda))^{[p]} \leftarrow \text{operation s.th. } (e^{\mu})^{[p]} = e^{p\mu} \\ &= \sum_{\mu \in X^+} (\dim H^0(\lambda)_\mu) \sum_{\nu \in W_f(\mu)} e^{p\nu} \end{aligned}$$
For $\eta \in X$ we set
$$\chi_\eta = \frac{\sum_{w \in W_f} (-1)^{\ell(w)} e^{w(\eta+\rho)}}{\sum_{w \in W_f} (-1)^{\ell(w)} e^{w\rho}}$$
Then it is easily checked that for $w \in W_f$ we have $\chi_{w \cdot \eta} = (-1)^{\ell(w)} \chi_\eta$

We also have
$$\begin{aligned} \sum_{w \in W_f} \chi_{w\eta} &= \frac{\sum_{y, w \in W_f} (-1)^{\ell(y)} e^{y(w\eta+\rho)}}{\sum_{z \in W_f} (-1)^{\ell(z)} e^{z\rho}} \\ &= \sum_{x \in W_f} e^{x\eta} \left(\frac{\sum_{y \in W_f} (-1)^{\ell(y)} e^{y\rho}}{\sum_{z \in W_f} (-1)^{\ell(z)} e^{z\rho}} \right) \qquad (x = yw) \\ &= \sum_{x \in W_f} e^{x\eta} \end{aligned}$$
$$\Rightarrow \sum_{\nu \in W_f(\mu)} e^{p\nu} = \sum_{\nu \in W_f(\mu)} \chi_{p\nu}$$
$$= \sum_{w \in W_f^\mu} (-1)^{\ell(w)} \chi_{p\mu - \rho + w^{-1}(\rho)}$$
W_f^μ: rep. for $W_f / \mathrm{stab}_{W_f}(\mu)$

Hence finally
$$\begin{aligned} ch(L(t_\lambda \cdot_p 0)) &= \sum_{\mu \in X^+} (\dim H^0(\lambda)_\mu) \sum_{w \in W_f^\mu} (-1)^{\ell(w)} ch(H^0(p\mu - \rho + w^{-1}\rho)) \\ &= \sum_{\mu \in X^+} (\dim H^0(\lambda)_\mu) \sum_{w \in W_f^\mu} (-1)^{\ell(w)} ch(H^0(t_\mu w^{-1} \cdot 0)) \end{aligned}$$

If $p \gg 0$ then Lusztig's conjecture should apply to $p\lambda$
(if $\lambda \in X^+ \cap \mathbb{Z}R$)

$\leadsto$ We should have $\boxed{h_{w_0 t_\mu, w_0 t_\lambda}(1) = \underbrace{\dim H^0(\lambda)_\mu}_{\text{same as weight multiplicity in char. 0}}}$

for $\lambda, \mu \in X^+ \cap \mathbb{Z}R$

This formula was checked (as an evidence for his conjecture) in 1981.