

Highest Weight Categories

Convenient language for discussing the categories we consider.
Based on Beilinson-Ginzburg-Soytel and Chu-Parkash-Schiff.

$\mathcal{A}$ finite length / k a field + $\dim_k \text{Hom}_k(M, N) < \infty \quad \forall M, N \in \mathcal{A}$.

$\Lambda =$ isoclasses of simple objs. in $\mathcal{A}$.

$$\lambda \leftrightarrow L_\lambda.$$

For $I \subset \Lambda$, $\mathcal{A}_I = \langle L_\lambda \mid \lambda \in I \rangle$ (Serre subcat.) $\rightarrow \mathcal{A}_{\leq \lambda}, \mathcal{A}_\lambda$ etc.

Assume given $\Delta_\lambda, \nabla_\lambda$ and maps $\Delta_\lambda \rightarrow L_\lambda \rightarrow \nabla_\lambda$.

Def: $\mathcal{A}$ is highest weight if

$\uparrow$ "standard objects"

① $\{\leq \lambda\}$ is finite $\forall \lambda \in \Lambda$.

② $\text{End}(L_\lambda) = k \quad \forall \lambda$

③ for $I \subset \Lambda$ an ideal, and $\lambda \in I$ maximal

$\Delta_\lambda \rightarrow L_\lambda$ (resp. $L_\lambda \rightarrow \nabla_\lambda$) is a projective cover (resp. injective hull) of L_λ .

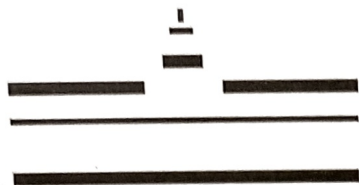
④ $\ker(\Delta_\lambda \rightarrow L_\lambda) \in \mathcal{A}_{< \lambda}$, $\text{coker}(L_\lambda \rightarrow \nabla_\lambda) \in \mathcal{A}_{< \lambda}$.

⑤ $\forall \lambda, \mu \in \Lambda, \text{Ext}^2(\Delta_\lambda, \nabla_\mu) = 0$.

Remark:

$$\Downarrow \quad \text{Ext}^i(\Delta_\lambda, \nabla_\mu) \stackrel{(*)}{=} \begin{cases} k & \text{if } i=0, \lambda=\mu \\ 0 & \text{o/w} \end{cases}$$

VERY IMPORTANT!



key example: ① $\text{Rep } G$, $\Lambda = \mathbb{C}^+$

② $\mathcal{O}$ for a complex semi-simple Lie algebra.

(in both cases the highest point is Ext^2 condition)

③ Certain categories of perverse sheaves ... tomorrow!

standard filtration: $0 = M_0 \subset \dots \subset M_{n-1} \subset M_n = M$ s.t. $M_i/M_{i-1} \cong \Delta_{\lambda_i}$.

costandard filtration:

$\cong \nabla_{\lambda_i}$

Many "naturally occurring" modules admit standard or costandard filtrations.

$T \in \mathcal{A}$ is tilting if it admits standard + costandard filtration.

Thm: ① for every $\lambda \in \Lambda$ there exists (up to iso, unique)

tilting s.t. $(T_\lambda : \Delta_\lambda) = 1$ and $(T_\lambda : \Delta_\mu) = 0$ if $\mu \neq \lambda$.

Any indecomposable tilting module is isomorphic to some T_λ .

② if Λ is bounded above, then there exist $P_\lambda \rightarrow L_\lambda$, $L_\lambda \hookrightarrow I_\lambda$
projective (resp. injective hull) with standard (resp. costandard) filtration.

Main point: $\text{Tilt} =$ additive category of tilting modules.

Use (*) to show: $K^b(\text{Tilt}) \xrightarrow{\sim} D^b(\mathcal{A})$ equivalence.

"homological skeleton" of $\mathcal{A}$

Back to G -modules:

Recall: $\mathcal{X}_1 = \{ \lambda \mid 0 \leq \langle \alpha^\vee, \lambda \rangle < p \ \forall \alpha \}$.

"dominant wts
w/ at most 1
 p -adic digit"

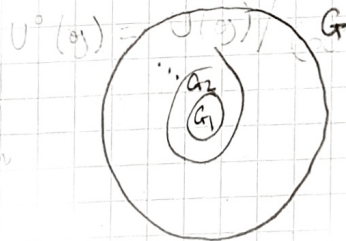
Steinberg $\otimes$ Thm: $\Rightarrow$ knowledge of L_λ w/ $\lambda \in \mathcal{X}_1 \Rightarrow$ complete knowledge of all simples.

$G, \sigma = \text{Fr} : G \rightarrow G$ Frobenius map.

$G_r \subseteq \ker \text{Fr}^r$ "Frobenius kernel".
"infinitesimal group scheme."

Picture:

conjugation



"has many small
subgroups" " $G = \bigcup G_r$ "

$\mathfrak{g} = \text{Lie } G =$ left-invariant vector fields on G

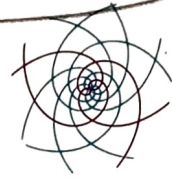
$U^0(\mathfrak{g}) = U(\mathfrak{g}) / (x^{(p)} - x)$ restricted enveloping algebra.
finite-dim of dim = $\langle \text{dim } \mathfrak{g} \rangle!$.

$\text{Rep } G_r \cong \text{Rep } U^0(\mathfrak{g})$

($U^0(\mathfrak{g})^* = \mathcal{O}(G_r)$ canonically).

Curtis' Theorem: $\lambda \in \mathcal{X}_1 \Rightarrow L_\lambda|_{G_r}$ is irreducible.

Moreover, all irreducible G_r -modules occur
in this way.



simple

Reduces character χ to question for G_1 .

Problem: weights of G_1 -modules belong to $\mathcal{X}/p\mathcal{X}$.

Instead work with G_1T -modules, where $G_1T = \text{Fr}^{-1}(T)$.

$\text{Rep } G_1T =$ G_1 -modules w/ compatible T -grading.

"Harish-Chandra modules for G_1, T "

E.g: $\forall \lambda \in \mathcal{X}$, $p\lambda$ is a G_1T -module.

Given $\lambda \in \mathcal{X} \rightsquigarrow \widehat{\Delta}_\lambda = U^0(\mathfrak{g}) \otimes_{U^0(\mathfrak{b})} \lambda$ "baby Verma module".

$$\widehat{\nabla}_\lambda = \text{ind}_{B_1T}^{G_1T} \lambda$$

$$\begin{matrix} \widehat{T}_\lambda \\ \widehat{P}_\lambda \\ \widehat{I}_\lambda \end{matrix}$$

all exist, all $\mathcal{X}$ is unbounded.

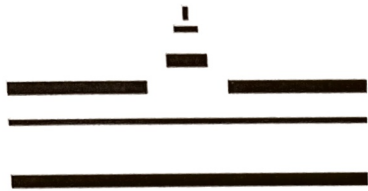
$$\widehat{\Delta}_\lambda \rightarrow \widehat{L}_\lambda \rightarrow \widehat{\nabla}_\lambda \quad \text{simple h.w. module.}$$

All simple G_1T -modules are of this form.

$$\widehat{\text{Rep}}_\lambda = \langle \widehat{L}_\gamma \mid \gamma \in W_p \circ \lambda \rangle.$$

Linkage principle: $\text{Rep } G_1T = \bigoplus \widehat{\text{Rep}}_\lambda$

Again, translation functors exist and reduce to $\mathcal{Q}$'s on principal block $\widehat{\text{Rep}}_0$.



Lusztig's periodic Hecke module:

identity $W \xrightarrow{\sim} \mathcal{A}$
 $w \mapsto wA_0$

$$A \cdot H_S = \begin{cases} A_S + vA & \text{if } A_S \geq A, \\ A_S + v^{-1}A & \text{if } A_S \leq A. \end{cases}$$

Periodic order: $A \leq B$ - if $A + \lambda \leq B + \lambda$ for λ sufficiently deep in $\mathcal{A}_+$.

Lusztig periodic module: $P = (\bigoplus \mathbb{Z}[v^{\pm 1}])A$

$$A \cdot H_S = \begin{cases} A_S + vA & \text{if } A_S \geq A, \\ A_S + v^{-1}A & \text{if } A_S \leq A. \end{cases}$$

$\leadsto$ P has $\mathbb{Z}R_+$ symmetry.

Lusztig defined $-$ and canonical basis $\{q_A\}$ in P .

E.g. For SL_3 :



$$q_A = \sum q_{B,A} B$$

Lusztig's periodic conjecture:

$$[P_A] = [T_A] = \sum q_{B,A}(1) [\Delta_B]$$

Remark: ① $[P_A] \cong [P_A]$

② Brauer-Thrall reciprocity:

$$[P_A : \Delta_B] = [\Delta_B : L_A]$$

+ Curtis' theorem implies that

LPC also gives answer to char. Q.