Achar–Makisumi–Riche–W.

gara S2-1

Towards tilting characters:

$\mathcal{G}$ Kac-Moody group / $\mathbb{C}$, $\mathcal{B} \subset \mathcal{G}$ Borel. $\Bbbk$ a field.

Thm: "modular BY equivalence"

There exists an equivalence of monoidal categories

$$\tilde{\mathcal{K}} : (\mathrm{Parity}(\mathcal{B}\backslash\mathcal{G}/\mathcal{B}, \Bbbk), *) \xrightarrow{\sim} (\mathrm{Tilt}(\mathcal{B}^\vee \backslash\!\backslash \mathcal{G}^\vee /\!/ \mathcal{B}^\vee_{\Bbbk}), \tilde{*})$$

By killing deformation parameters and passing to K^b we obtain

$$\mathcal{K} : D^{mix}_{(\mathcal{B})}(\mathcal{G}/\mathcal{B}) \xrightarrow{\sim} D^{mix}_{(\mathcal{B})}(\cancel{\mathcal{G}^\vee/\mathcal{B}^\vee}\ \mathcal{B}^\vee\backslash\mathcal{G}^\vee)$$

$$\Delta_w \longmapsto \Delta^\vee_w$$
$$\nabla_w \longmapsto \nabla^\vee_w$$
$$\mathcal{E}_w \longmapsto T^\vee_w$$
$$T_w \longmapsto \mathcal{E}^\vee_w$$

$$\boxed{\mathcal{K}\circ\langle 1\rangle = \langle -1\rangle[-1]\circ\mathcal{K}}$$

(I haven't yet said what both sides are!)

Remark on the proof:

Bezrukavnikov-Yun's approach:

$$\mathrm{Parity} \xrightarrow{\mathbb{IH}} \mathrm{SBim} \xleftarrow{\text{log of monodromy}} \mathrm{Tilt}$$

Our proof uses generators and relations, and reverses the arrows:

$$\mathrm{Parity} \longleftarrow \text{diagrammatic SBim} \longrightarrow \mathrm{Tilt}$$

"easy", already done in RW

~~[illegible], uses [illegible]~~

Both arrows use that rel^{n}s are limited to finite parabolic subgroups.

Why does Koszul duality solve our problems?

"Classical" picture of Lusztig, Bezrukavnikov, Soergel, ...

$$\begin{array}{ccc} & & D^{mix}_{(I)}(Gr^\vee) \;(\text{spherical}) \\ & \overset{ABG}{\underset{\text{Arider, AMR}}{\sim}} \nearrow & \\ \mathrm{Rep}_0\, U_\varepsilon G & \xleftarrow[AR]{ABG} \; D^b \mathrm{Coh}^{G\times \mathbb{G}_m}(\tilde{\mathcal{N}}) & \\ & \underset{AB}{\sim} \searrow & \\ & & D^{mix}_{IW}(Fl^\vee) \;(\text{anti-spherical}) \end{array}$$

Finkelberg-Mirković conjecture: simples $\leftrightarrow$ IC stalks (not computable)

Lusztig's char. formula (simples)

"parabolic-Whittaker duality" BY

Still not available in char. p

Soergel's char. formula (tiltings)

RW-conjecture

Tilting $\xleftrightarrow{[AMRW]}$ Parity $\leftarrow$ computed via p-canonical basis.

$\sim$: the modular world.

Rmk: $v \leftrightarrow -v^{-1}$ interchanges spherical and anti-spherical modules. Invisible at $v=1$.

<u>Monodromy action</u>: Slogan: "Lefschetz action" $\leftrightarrow$ "monodromy action" (B: reminiscent of mirror symmetry).

$$\begin{array}{c} G/U \\ \pi \downarrow \\ G/B \end{array} \qquad T = B/U\text{-torsor}. \qquad D^b_c(G/\!/B) = \langle \pi^* D^b_c(G/B) \rangle \subset D^b_c(G/U).$$

"complexes w/ unipotent monodromy along the fibres"

Verdier: functorial action of $\pi_1(T)$ on objects of $D^b_c(G/\!/B)$.

BY: take logs $\rightsquigarrow$ $S(\mathrm{Lie}\,T)$-action.

big problem in char. p

ojara S2-3

Instead we seek an algebraic model:

"geometry is exchanged for homological algebra!"

$D^{mix}(B\backslash G/B)$ left-monodromic category. (complicated)

$D^{mix}(B\backslash G/B)$ free-monodromic category. (very complicated!)

$D_{(B)}(G/B) \xleftarrow{\sim} D_U(G/B) = D^b(B\backslash G/B)$

$D^{mix}_{(B)}(G/B) = K^b(\text{Soergel modules})$

Where has the monodromy gone?

<u>Beautiful idea</u> (Achar, Makisumi independently)

$M \in K^b(\text{Soergel modules})$ $\to M^i \to M^{i+1} \to$

$\rightsquigarrow$

$\widetilde{M}$ $\cdots \to \widetilde{M}^i \to \widetilde{M}^{i+1} \to \cdots$ sequence of Soergel ~~modules~~ bimodules.

failure of $\widetilde{M}$ to be a complex gives monodromy action!

<u>Eg</u> $0 \to B_{id}(-1) \to B_s \to B_{id}(1) \to 0$

$\rightsquigarrow$

$B_{id}(-1) \xrightarrow{\bullet} B_s \xrightarrow{\bullet} B_{id}(1)$

~~Reflecting~~

<u>Theorem</u>: [AMRW] There exist categories

$D^{\mathrm{comb}}(B\backslash G/B)$ — objects "complexes" of parity sheaves, w/ differential $\partial \in \Lambda \otimes \bigoplus \mathrm{Hom}^{\bullet}(\mathcal{F}^i, \mathcal{F}^j)$ satisfying a "Koszul condition".

$D^{\mathrm{comb}}(B\backslash G/B)$: objects " " with differential in $\Lambda \otimes \mathrm{Hom}^{\bullet}(\mathcal{F}^i, \mathcal{F}^j) \otimes S$

$V = X_*(T) \otimes_{\mathbb{Z}} k$

$\Lambda = S(V^*)$

$S = S(V)$

Such that

1. $D^{\mathrm{comb}}(B\backslash G/B)$ has a left-monodromy action,
2. $D^{\mathrm{comb}}(B\backslash G/B)$ has a LR monodromy action,
3. $D^{\mathrm{comb}}(B\backslash G/B)$ contains a subcategory of convolutive objects + nice morphisms.

$\rightsquigarrow$ Koszul duality.