

Towards tilting characters:

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$\mathcal{G}$ Kac-Moody group / $\mathbb{C}$, $B \subset \mathcal{G}$ Borel. M a field.

Thm: "modular BY equivalence"

There exists an equivalence of monoidal categories

$$\tilde{\mathcal{K}}: (\text{Parity}(\mathcal{B} \backslash \mathcal{G} / \mathcal{B}, k), *) \xrightarrow{\sim} (\text{Tilt}(\mathcal{B}^\vee \backslash \mathcal{G}^\vee / \mathcal{B}_h^\vee), \tilde{*})$$

By killing deformation parameters and passing to k^b we obtain

$$\tilde{\mathcal{K}}: D_{\mathcal{B}}^{\text{mix}}(\mathcal{G} / \mathcal{B}) \xrightarrow{\sim} D_{(\mathcal{B})}^{\text{mix}}(\mathcal{G}^\vee / \mathcal{B}^\vee)$$

$$\begin{array}{ccc} \Delta_w & \xrightarrow{\quad} & \Delta_w^\vee \\ \nabla_w & \xrightarrow{\quad} & \nabla_w^\vee \\ \mathcal{E}_w & \xrightarrow{\quad} & \mathcal{T}_w^\vee \\ \mathcal{T}_w & \xrightarrow{\quad} & \mathcal{E}_w^\vee \end{array}$$

$$\mathcal{K} \circ \langle 1 \rangle = \langle -1 \rangle [-1] \circ \mathcal{K}$$

Remark on the proof: (I haven't yet said what both sides are!)

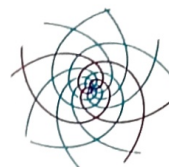
Bezrukhavnikov-Yon's approach:

$$\text{Parity} \xrightarrow{\text{IH}} \text{SBim} \xleftarrow{\text{log of monodromy}} \text{Tilt}$$

Our proof uses generators and relations, and reverses the arrows:

$$\begin{array}{ccc} \text{Parity} & \xleftarrow{\quad} & \text{diagrammatic SBim} \xrightarrow{\quad} \text{Tilt} \\ & \nearrow \text{"easy", already done in RW} & \uparrow \text{tilting, uses head} \end{array}$$

Both arrows use that relⁿs are limited to finite parabolic subgroups.



Instead we seek an algebraic model:

"geometry is exchanged for homological algebra!"

$$D^{\text{mix}}(B; G/B)$$

left-monodromic category.
(complicated)

$$D^{\text{mix}}(B; G/B)$$

free-monodromic category.
(very complicated!)

$$D_{(B)}^{\text{mix}}(G/B) \xleftarrow{\sim} D_U^{\text{mix}}(G/B) \xrightarrow{\text{isomorphism}} D^b(B; G/B)$$

$$D_{(B)}^{\text{mix}}(G/B) = K^b(\text{Soergel modules})$$

Where has the monodromy gone?

Beautiful idea (Achar, Makiwami independently)

$$\forall M \in K^b(\text{Soergel modules}) \rightarrow M^i \rightarrow M^{i+1} \rightarrow$$

$$\downarrow$$

$$\widetilde{M} \rightarrow \dots \rightarrow \widetilde{M}^i \rightarrow \widetilde{M}^{i+1} \rightarrow \dots$$

sequence of Soergel ^{bimodules} ~~modules~~.

failure of $\widetilde{M}$ to be a complex gives monodromy action!

Eg $0 \rightarrow D_{id}(-1) \rightarrow B_S \rightarrow D_{id}(1) \rightarrow 0$

$$\{ \}$$

$$B_{id}(-1) \xrightarrow{\downarrow} B_S \xrightarrow{\uparrow} B_{id}(1)$$

Revising

Theorem: [AMRW] There exist categories

$$D^{\text{comb}}(B; G/B)$$

objects "complexes" of parity sheaves,
w/ differential
morphisms $\rho \in \Lambda \otimes \bigoplus \text{Hom}^i(F^i, F^j)$
satisfying a "Koszul condition".

$$V = X_*(T) \otimes_{\mathbb{Z}} k$$

$$\Lambda = S(V^*)$$

$$S = S(V)$$

$$D^{\text{comb}}(B; G/B)$$

objects "
with differential in $\Lambda \otimes \text{Hom}(F^i, F^j) \otimes S$

Such that

- ① $D^{\text{comb}}(B; G/B)$ has a left-monodromy action,
- ② $D^{\text{comb}}(B; G/B)$ has a $\mathbb{Z}$ monodromy action,
- ③ $D^{\text{comb}}(B; G/B)$ contains a subcategory of nonvanishing objects +
nice morphisms.

$\Rightarrow$ Koszul duality.