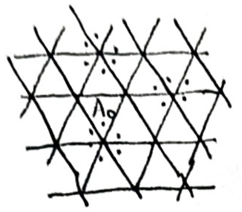
# A simple character formula $\quad p \geq h$

wts in principle block of $G_1T$-modules $\hat{T}_A$,

$\parallel$

$W \xrightarrow{\sim}$  $\mathcal{A} \xleftrightarrow{\sim} W \cdot_p 0$


$\cup \qquad\qquad \cup \qquad\qquad \cup$

${}^f W \xrightarrow{\sim}$  $\mathcal{A}^+ \xleftrightarrow{\sim}$ weights in principle block of $G$

$L_x = L_{x \cdot_p 0}$, $T_x =$ etc.

**Main theorem of this course:** ( [AR], [MR], [ARider], [AMRW], based on [ABG], [AB] )

[Elias-Losev]

conj. in [RW]

$$(T_x : \Delta_y) = {}^p n_{y,x}(1).$$

**Applications:** ① Supports ~~of~~ varieties of tilting modules (Humphreys conjecture).

Ask Simon!

② Decomposition numbers for symmetric groups.

~~(...)~~ E.g. "Billiards conjecture" of Lusztig-W, watch "billiards and tilting characters" on youtube, esp. last 1/3 !

Seems to point to a genuinely new structure in representation theory.

③ A simple formula for simple characters (now).

**Simple character formula:** (Ricke-W) Suppose $p \geq 2h-1$

$$(\hat{T}_A : \hat{\Delta}_B) = {}^p q_{B,A}(1).$$

( in world of $G_1T$-modules )

$\parallel$

$$(\hat{P}_{A\downarrow} : \hat{\Delta}_B) \underset{\uparrow}{=} [\hat{\Delta}_B : L_{A\downarrow}]$$

Braver-Humphreys reciprocity.

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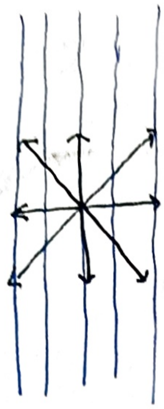
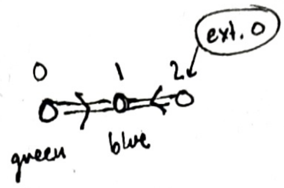
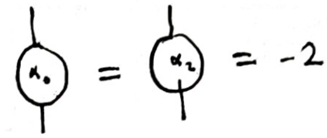
E.g.



$\tilde{C}_2$




This is practically useful!

E.g.

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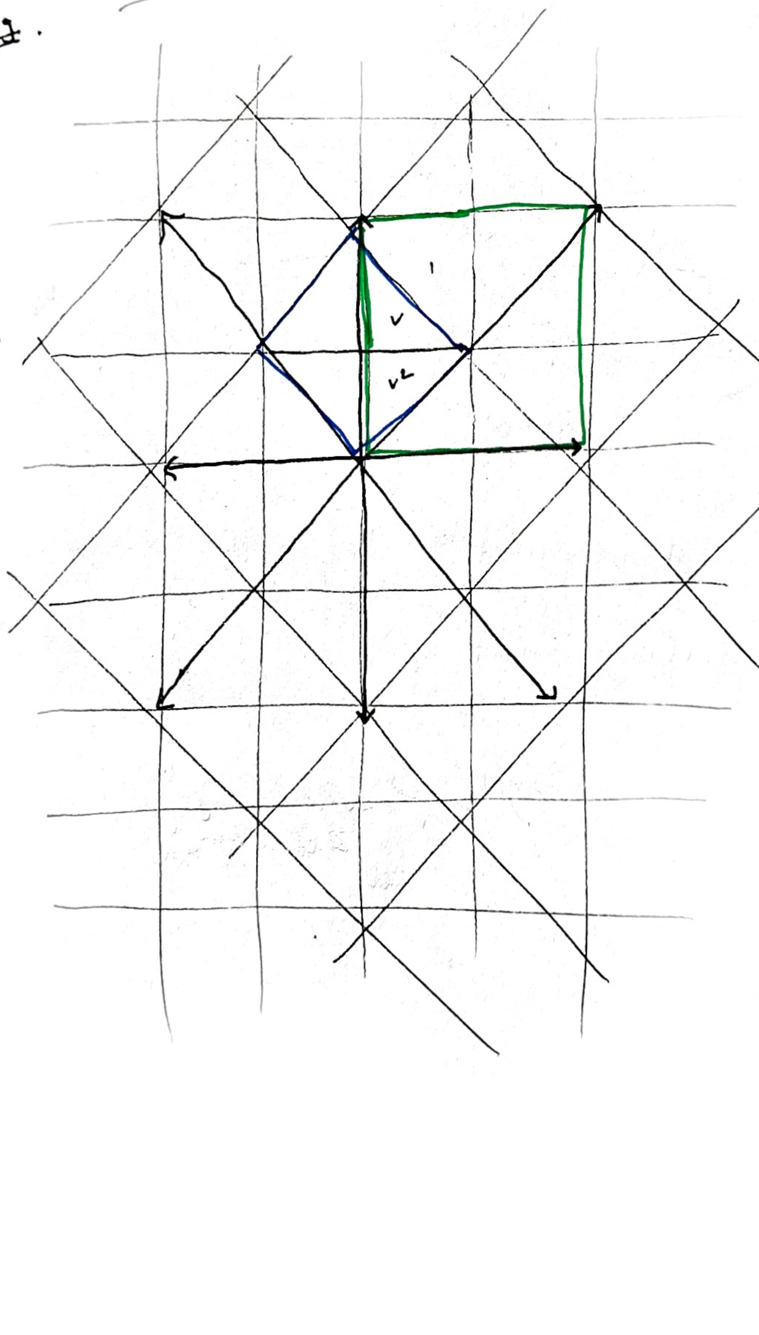
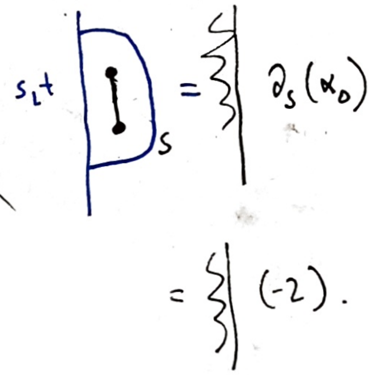
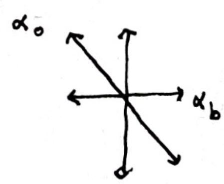



LC OK for ~~p~~ $p \geq 6$.



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Back to periodic module: $\mathcal{H}$ Hecke algebra.

$\mathcal{H}$: $\quad A \cdot \underline{H}_s = \begin{cases} As + vA & \text{if } As \geq A, \\ As + v^{-1}A & \text{if } As \leq A \end{cases}$ (right regular module)

$\mathcal{P} \circlearrowleft \mathcal{H}$, $\circlearrowleft \mathcal{X}$: $\quad A \cdot \underline{H}_s = \begin{cases} As + vA & \text{if } As \succ A \\ As + v^{-1}A & As \prec A \end{cases}$

Canonical basis $\{\underline{q}_A\}$

$$\underline{q}_A = \sum q_{B,A} B.$$

${}^f\mathcal{M} = \mathrm{triv} \otimes_{\mathcal{H}_f} \mathcal{H}$ spherical module.

$$\underline{H}_{w_f} = \sum_{w \in W_f} v^{\ell(w_f)-\ell(w)} H_w$$

$$\begin{array}{ccc} {}^f\mathcal{M} & \xrightarrow{\phi} & \mathcal{H} \\ \Psi & & \Psi \\ 1 & \longmapsto & \underline{H}_{w_f} \end{array}$$

$$\pi_f = \sum_{w \in W_f} v^{\ell(w_f) - 2\ell(w)}, \quad \text{then} \quad \underline{H}_{w_f}^2 = \pi_f \underline{H}_{w_f}.$$

Lusztig: Suppose $\underset{B=}{x(A_0)} \in \mathcal{A}_{res} = \{A \mid A \in \lambda \dots\}$. Then

$$\underline{q}_{A_0} = \sum v^{\ell(x)} x A_0$$

$$\underline{q}_{xB} = \underline{q}_{A_0} \cdot \phi(\underline{M}_x).$$

Rmk: ① Alternative combinatorial definition exists.

② Later interpreted by Braverman-Finkelberg as stalks on $\infty/2$-flag manifold.

Periodicity $\rightsquigarrow$ $\underline{q}_A$ for all $A$.

We simply define:

$${}^P\underline{q}_B = \underline{q}_{A_0} \cdot \phi({}^P\underline{M}_x).$$

③ No intrinsic def. of P available yet. Should be very interesting...

$\boxed{\text{Hope: This is much simpler than } \mathcal{F}l^\vee \text{ on } \mathcal{G}r^\vee.}$

Where does the bound come from?

Conjecture (Donkin) injective hull of $\hat{L}_A$ as a $G_1T$-module.

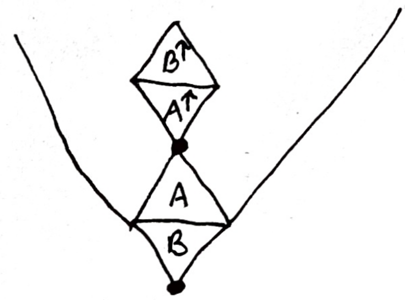
$$\hat{Q}_A = (\dot{T}_{A^\uparrow})|_{G_1T}$$

$$\mathcal{A}_{res} \underset{(\ )_\downarrow}{\overset{(\ )^\uparrow}{\rightleftarrows}} \mathcal{A}_{res}$$

$A \longmapsto A\, w_0 A$ extend via periodicity.

$A_0 \longleftarrow A$

I.e. injective hull of $\hat{L}_A$ as a $G_1T$-module extends to a $G$-module (Humphreys-Verma conj.)
Key pt is the indecomposability of RHS.




Jantzen: Conj. is OK for $p \geq 2h-2$.

Rmk: Bendel-Nakano-Pillen-Sobaje: conj. is false for $G_2$, $p=2$!

$\boxed{\text{Not quite sure what to expect!}}$

Rmk: That $(T_{A^\uparrow})|_{G_1T}$ is injective is easy to see. The key point is the indecomposability.

Upshot: For simple characters relevant tiltings are those of the form St + $\mathcal{A}_{res}$. These are not very accessible via the [AMRW] formalism.

Observation: (Assume $\varsigma \in \mathbb{Z}R$) for simplicity

① $\underline{N}_{\varsigma+A_0} = \sum_{x \in W_f} v^{\ell(x)} (\varsigma + A_0)$

② Action on $\underline{N}_{\varsigma+A_0}$ gives an embedding

$${}^f\mathcal{M} \overset{\varphi}{\hookrightarrow} {}^f\mathcal{N}$$

spherical, anti-spherical.

$\boxed{\text{gana 53-6}}$

Key commutative diagram:

$$\begin{tikzcd}
\mathcal{M}^{sph} \arrow[r, "\Upsilon"] \arrow[d, "\wr"'] \arrow[dd, bend right, "\mathrm{sum}"'] & \mathcal{M}^{sph} \arrow[r, "\sim"] & \mathrm{Rep}_0\, G \arrow[d] \arrow[r] & T_{s+A_0} \arrow[d] \\
\mathbb{Z}[W_{ext}] \arrow[rr, "\sim"] & & \mathrm{Rep}_0\, G,T \arrow[r] & (T_{s+A_0})_{G,T} \\
\sum_{w \in W_f} w & & &
\end{tikzcd}$$

diagram commutes
w/ translation
functors.