PS 5

Geometric Hecke category

Origin of Hecke algebra: $\mathrm{End}(\mathbb{C}[G(\mathbb{F}_q)/B(\mathbb{F}_q)])$ or $\overline{\mathbb{Q}_\ell}$

$$H_{\mathbb{F}_q} = \mathrm{Fun}_{B(\mathbb{F}_q)\times B(\mathbb{F}_q)}(G(\mathbb{F}_q), \mathbb{C})$$

$$= \mathrm{Fun}\left(B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/B(\mathbb{F}_q), \mathbb{C}\right)$$

Basis $\left(\mathbb{1}_{B(\mathbb{F}_q)wB(\mathbb{F}_q)}\right)_{w\in W}$ $\underset{t_w}{''}$

Convolution product

$$f_1 * f_2(g) = \frac{1}{|B(\mathbb{F}_q)|}\sum_{g_1g_2=g} f_1(g_1)f_2(g_2)$$

Exercise: check relations $\begin{cases} t_s^2 = (q-1)t_s + q \\ t_st_u\cdots = t_ut_s\cdots \end{cases}$ Iwahori

$\boxed{\text{depend on } q \text{ in uniform way, make sense for any Coxeter group}}$

$\rightsquigarrow$ define H over $\mathbb{Z}[q^{\pm 1}]$ by gen. and rel.
↳ formal variable.

$v := q^{-1/2}$, $H_s = vt_s$ $\begin{cases} H_s^2 = (v^{-1}-v)H_s + 1 \\ H_sH_u\cdots = H_uH_s\cdots \end{cases}$

Grothendieck's function-sheaf correspondence (PS6)

$\rightsquigarrow$ $D_{B\times B}(G)$ (to get $H_{\mathbb{F}_q}$: trace of Frob at rat. points $G(\mathbb{F}_q)$)

or $D([B\backslash G/B])$ From now on, G over $\mathbb{C}$.

or $D_B(G/B)$

<u>Idea:</u>

$$\begin{tikzcd} & {[B\backslash G\times^B G/B]} \arrow[dl, "p"'] \arrow[dr, "m"] & \\ {[B\backslash G/B]\times[B\backslash G/B]} & & {[B\backslash G/B]} \end{tikzcd}$$

$\mathcal{F} \boxtimes \mathcal{G}$ $\mathcal{F}*\mathcal{G} := m_* p^*(\mathcal{F}\boxtimes\mathcal{G})$

<u>Or:</u>

$$\begin{tikzcd} G\times G/B \arrow[r, "q_2"] \arrow[d, "q_1"] & G\times^B G/B \arrow[d, "m"] \\ G/B\times G/B & G/B \end{tikzcd}$$

$\mathcal{F}\boxtimes_B\mathcal{G}$ s.t. $q_2^*\,\mathcal{F}\boxtimes_B\mathcal{G} \simeq q_1^*\,\mathcal{F}\boxtimes\mathcal{G}$

$\mathcal{F}\boxtimes\mathcal{G}$ $\mathcal{F}*\mathcal{G} := m_*(\mathcal{F}\boxtimes_B\mathcal{G})$

Get monoidal category $D_B(G/B,\underline{k})$, $\Delta_w = i_{w!}\underline{k}[\ell(w)] \to \nabla_w = i_{w*}\underline{k}[\ell(w)]$

$D_B(G/P,k)$ module. $\twoheadrightarrow IC_w \hookrightarrow$

For $s\in S$, $P_s = \overline{BsB} = BsB\cup B \subset G$

$\overline{BsB}/B = P_s/B \simeq \mathbb{P}^1$, $IC_s = \mathcal{E}_s = \underline{k}_{\mathbb{P}^1_s}[1]$. $\mathcal{E}_w$ parity

$\mathcal{H}^k_{geom} = \langle \mathcal{E}_s \mid s\in S\rangle_{*,\oplus,(1),\mathrm{Kar}}$.

<u>Thm:</u> $\mathcal{H} \xrightarrow{\sim} [\mathcal{H}^k_{geom}]_\oplus$ split Grothendieck group

$H_s \longmapsto \mathcal{E}_s := \underline{k}_{P_s/B}[1]$

Inverse?

Define $ch: \mathbb{D}_B(G/B) \longrightarrow H$

by $ch(\mathcal{F}) = \sum_{i,x} \mathrm{rk}\, \mathcal{H}^{-\ell(x)-i}(j_x^* \mathcal{F})\, v^i H_x$

Note: As it stands, does not pass to Groth. group of $D_B(G/B)$ (need weights to make sense of v)

$0 \to \mathcal{F} \to \mathcal{F} \xrightarrow{+1}$ $\quad \mathcal{F} \to \mathcal{F} \to 0 \xrightarrow{+1}$ $\quad \mathcal{F} \to 0 \to \mathcal{F}[1]$

hence $[1]$ acts by -1, not v...

$[D_B(G/B)] = \mathbb{Z}W \ldots$

But ok if restrict to $\mathcal{H}^k$ geom and take split Grothendieck gp

Lemma: If $\mathcal{F} \in D_B(G/B)$ is $*$-even/odd

then $\mathcal{F} * \mathcal{E}(s) = \pi_s^* \pi_{s*} \mathcal{F}[1]$ is $*$-~~odd~~/even

and $ch(\mathcal{F} * \mathcal{E}(s)) = ch(\mathcal{F})\, \underline{H}_s$

$\mathbb{D}(\pi_s^* \pi_{s*}[1]) \simeq (\pi_s^* \pi_{s*}[-1])\mathbb{D}$

Here $\pi_s: G/B \longrightarrow G/P_s$

$\mathbb{D}\pi_s^* \pi_{s*} = \pi_s^! \pi_{s!} \mathbb{D} \cong \pi_s^* \pi_{s*}[2] \circ \mathbb{D}$
so both $\pi_s^* \pi_{s*}$ and $\pi_s^! \pi_{s!}$ preserve parity.

Note: $P_s/B = \overline{BsB/B}$ is smooth $\simeq \mathbb{P}^1$

$IC_s = \mathcal{E}_s = \underline{k}_{\mathbb{P}^1_s}[1]$

$ch(\mathcal{E}_s) = H_s + v = \underline{H}_s$.

	-1	0
BsB	k	
BeB	k	