NEARBY AND VANISHING CYCLES

(18/02)

Formalism of nearby and vanishing cycles

MOTIVATION 1) "taking a limit of sheaves"

$$\begin{tikzcd} & X \arrow[d, "f"] \\ c \arrow[r] & C \end{tikzcd}$$

curve

family of constructible sheaves over $C \setminus \{c\}$

i.e. a constructible sheaf **on** $f^{-1}(C\setminus\{c\})$

Nearby cycles: allows one to "take the limit" and obtain a constructible sheaf on ~~$f^{-1}(\cdot)$~~ $f^{-1}(c)$

2) (Original of [illegible])

Compare cohomology of special and general fibres

$$\begin{tikzcd} X \arrow[d, "f"] \\ C \end{tikzcd}$$

Some possible singular fiber $f^{-1}(c)$, smooth nearby fiber

Q: How does the cohomology change?

$$H^*(f^{-1}(c)) \text{ vs } H^*(f^{-1}(\text{nearby}))$$

Rmk We can go both way.

FIRST DEFINITION

~~$X \xrightarrow{f} D$ (unit disc)~~ $\subseteq \mathbb{C}$

$$\begin{tikzcd} X_0 \arrow[r, hook, "i"] \arrow[d] & X \arrow[d, "f"] & X^* \arrow[l, "k"'] \arrow[d, "f"] & X_\infty \arrow[l, "j"'] \arrow[d, "f"] \arrow[ll, bend right, "j^!"'] \\ \{0\} \arrow[r, hook] & D \arrow[d, phantom, sloped, "\subseteq"] & D^* \arrow[l, hook'] & \mathbb{H} = \{\mathrm{Im}\, z > 0\} \arrow[l, "e"'] \\ & \mathbb{C} & e^{2\pi i z} & z \arrow[l] \end{tikzcd}$$

(often assume f is smooth over $D^* = D \setminus \{0\}$)

$$\boxed{\psi_f K = i^{*} j^{!}_{*} j^{*} K}$$

Useful to consider $f = id$. $X = D$

$K \in D^b_c(D^\times)$ consider $k_* K$

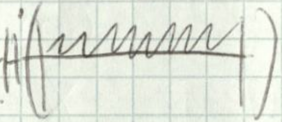
$$H^*(k_* K)_0 = \lim_{U \ni 0} H^*(U \setminus \{0\}, K)$$

Eg K is a local system with stalk V and monodromy μ

$$H^n(k_* K)_0 = \begin{cases} V^\mu & n=0 \\ V_\mu & n=1 \\ 0 & \text{otherwise} \end{cases}$$

(Vogliamo usare il rivestimento universale $\mathbb{H}$ perchè altrimenti entra in gioco la coomologia di D^* che non ci fa ottenere ciò che vogliamo).

lim



EXERCISE K local system on $D^\times$

$(\psi_f K)_0 = K_{\varepsilon}$ (ε small).

Remarks $\psi_f : D^b_c(X) \to D^b_c(D)$ $\leftarrow X_0$?

(constructibility is not obvious since exp is not algebraic).

2) The automorphism $z \mapsto z+1$ of exp induces an automorphism.

auto $\mu : \psi_f \to \psi_f$

We can also define $\psi_f(K) := \psi_f(k^* K)$ for $X \in D^b_c(X)$

The adjunction morphism

$$K \longrightarrow \overset{!}{j}_* j^* k^* K \rightsquigarrow i^* K \xrightarrow{can} \psi_f K$$

Taking cone over gives a ~~dt~~ dist. triang "vanishing cycle".

$$i^* K \xrightarrow{can} \psi_f K \longrightarrow \phi_f K \xrightarrow{+1}$$

MORE WORK 1) ϕ_f is a functor

2) ϕ_f has a monodromy automorphism μ

3)

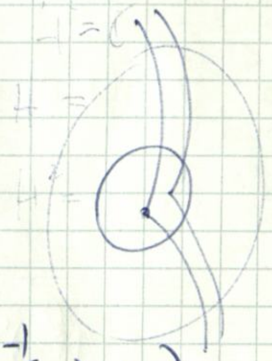
$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "f'"'] & & Y \arrow[dl, "f"] \\ & D & \\ X_0 \arrow[ur] \arrow[rr, "\pi"'] \arrow[uu, bend left] & & Y_0 \arrow[ul] \arrow[uu, bend right] \end{tikzcd}$$

π ~~prope~~ proper base change

$\Longrightarrow \pi_* \psi_{f'} K = \psi_f \pi_* K$

4) We have $x \in X_0$ for $0 < \eta \ll \varepsilon$

$$H^*(\psi_f K)_x = H^*(B(x,\varepsilon) \cap f^{-1}(\eta), K)$$



Also $H^*(\phi_f K)_x = H^*(B(0,\varepsilon), B(x,\varepsilon) \cap f^{-1}(\eta), K)$

↑ relative cohomology

5) ψ_f depends on f in general.

6) $\phi_f K$ is zero → "no change in cohomology from general to special fiber".

Important (but harder).

1) The definition is not ideal (e.g. does not make sense for D-mod).

2) ${}^p\psi_f := \psi_f[-1]$, ${}^p\phi_f := \phi_f[-1]$

Then ${}^p\psi_f$, ${}^p\phi_f$ are t-exact (perverse t-structure) and commute with duality

3) Gabber's theorem: monodromy filtration = wt. filtration (étale setting).

(it's deep semisimplicity statement, like hard Lefschetz and DT)

"GENERIC ISOLATED SINGULARITY"

$$f : \sum_{i=0}^{N} z_i^2 : \mathbb{C}^{N+1} \longrightarrow \mathbb{C}$$

$\varepsilon \neq 0$

$C = f^{-1}(0)$ singular quadric cone $\qquad C_\varepsilon = f^{-1}(\varepsilon)$ smooth affine quadric

$$C \leftsquigarrow T^* S^n$$

"vanishing cycle" $\Longleftrightarrow$ contraction of $S^n \subset T^* S^n$ to a point.

BASICS ABOUT PERVERSE SHEAVES ON C

[illegible]

$$H^*(C \setminus \{0\}) = H^*(T^* S^N \setminus S^N) \underset{\text{Gysin}}{=} \begin{cases} \mathbb{Q}\,0\,0 \ldots 0\,0\,\underset{N-1}{\mathbb{Q}}\,\underset{N}{\mathbb{Q}}\,0\,0 \ldots 0\,0\,\mathbb{Q} & (N \text{ odd}) \\ \mathbb{Q}\,0 \ldots \ldots\,0\,\overset{2N-1}{\mathbb{Q}} & \chi(S^n) = 2 \ (N \text{ even}) \end{cases}$$

Deligne Construction

stalks of IC_C (simple perverse sheaves) are as follows

N odd

	$-N$			-2	-1	0
C_{reg}	$\mathbb{Q}$	0	$\ldots$	0	0	0
$\{0\}$	$\mathbb{Q}$	0	$\ldots$	0	$\mathbb{Q}$	0

N even

$$IC_C = \mathbb{Q}_C[N]$$

N odd $\quad \mathrm{Ext}^1_{\mathcal{M}_C}(IC_C, IC_0) = \mathrm{Ext}^1_{\mathcal{M}_C}(IC_0, IC_C) = \mathbb{Q}$

N even No extension (in fact $\mathcal{M}_{C,\{C_{reg}, 0\}}$ is semisimple)

$$H^*(\psi_f\, \underline{\mathbb{C}}_{\mathbb{C}^{N+1}}[N+1])_0 = H^{*-N+1}(B(0,\varepsilon) \cap f^{-1}(\eta))$$

$=$

$-N-1$	$-N$	$\ldots$	-1	0
$\mathbb{Q}$	0	$\ldots 0$	$\mathbb{Q}$	0

$H^{*-N+1}(S^N)$

Hence $^p\psi_f \underline{\mathbb{Q}}_{\mathbb{C}^{N+1}}[N+1]$ has stalks

	$-N$				0
$x \in C_{reg}$	$\mathbb{Q}$				0
$\{0\}$	$\mathbb{Q}$	0	$\dots$	0	$\mathbb{Q}$

EXERCISE $^p\psi_f IC_{\mathbb{C}^{N+1}}$ has the following structu

N even $\quad ^p\psi_f IC_{\mathbb{C}^{N+1}} \cong IC_C \oplus IC_{\cancel{pt}0}$

N odd $\quad ^p\psi_f IC_{\mathbb{C}^{N+1}} \cong \begin{pmatrix} IC_0 \\ IC_C \\ IC_0 \end{pmatrix}$

MORE DIFFICULT Monodromy $+1$ on IC_C $\quad -1$ on I
(N e

N odd monodromy is unipotent ($\mu - 1$ is nilp

$$\begin{tikzcd}
\begin{pmatrix} IC_0 \\ IC_C \\ IC_{pt} \end{pmatrix} \arrow[r, two heads] \arrow[rr, bend left, "\mu-1"] & IC_0 \arrow[r, hook] & \begin{pmatrix} IC_0 \\ IC_C \\ IC_0 \end{pmatrix}
\end{tikzcd}$$

IC_0 submodule

Gabber's theorem A an obj of an abelian category. $N: A \to A$ nilpotent end. Then there exist a unique finite increasing filtration

$$0 \subset \dots \subset W_{-1}A \subset W_0 A \subset W_1 A \subset \dots \subset A$$

s.t. 1) $N: W_\ell A \to W_{\ell-2} A$

2) N^ℓ induces an iso. $gr_\ell A \xrightarrow{\sim} gr_{-\ell} A$

Picture $\quad gr A = \begin{array}{c|ccc} \vdots & & & \\ 2 & & & P^2 \\ 1 & & P^1 & \downarrow s \\ 0 & P_0 & \downarrow s & P^2 \\ -1 & & P^1 & \downarrow s \\ -2 & & & P^2 \\ \vdots & & & \end{array}$

Because $\mathcal{M}_{X_0}$ is abelian, finite length

$$^p\psi_f \cong \bigoplus {}^p\psi_{f,\lambda} \qquad {}^p\psi_{f,\lambda} \text{ gen. } \lambda \text{ eigenspace for } \mu$$

Fact: $^p\psi_{f,\lambda}$ is "easy" if $\lambda \neq 1$

Hence: focus on $^p\psi_{f,1}$ "unipotent nearby cycles".

Gabber's thm.

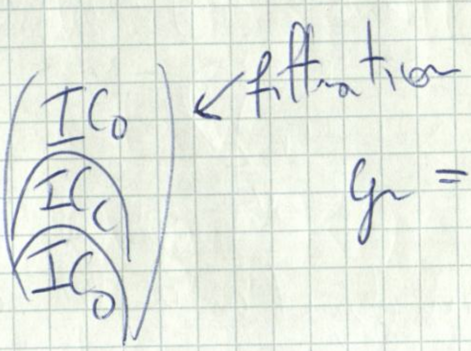
$$\mu \circlearrowright {}^p\psi_{f,1} IC(X, \mathcal{L})$$

$\Longrightarrow \mu - 1$ is a nilpotent endomorphism.

Then the filtration associated to $(\mu - 1)$ above agrees with the weight filtration.

(in particular gr is semisimple).

EXAMPLES



$$\begin{tikzcd} & & IC_0 \arrow[d, "s"] \\ gr = IC_C & & IC_0 \end{tikzcd}$$