VARIATIONS OF MHS 11/03

EXAMPLE of a MHS on the cohomology of a sing. variety

Y smooth & projective variety

$X \subset Y$ simple normal crossing divisor

locally $X = \{z_1 \cdots z_r = 0\}$

the irred. comp. of X are smooth (α not allowed)

$X = X_1 \cup \dots \cup X_r$ smooth.

Goal Understand the MHS of $H^\cdot(X, \mathbb{Q})$

For $I \subseteq \{1, \dots, r\}$, $X_I = \bigcap_{i \in I} X_i$ is a disjoint union of smooth varieties

The X_I's define a cubical variety. For $I' \subseteq I$ $X_{I'} \subseteq X_I$

The corresponding augmented semisimplicial variety

$$\cdots \coprod_{ijk} X_{ijk} \rightrightarrows \coprod_{ij} \rightrightarrows \coprod_i X_i \to X$$

$$X_r = \coprod_{|I| = r+1} X_I$$

Lemma $X_\cdot \to X$ has cohomological descent.

It is a semisimplicial hyperresolution of X

Proof $\mathcal{F}$ a sheaf on X. We have to show that we have a long exact sequence

$$0 \to \mathcal{F} \to \bigoplus_i \mathcal{F}_{X_i} \to \bigoplus_{ij} \mathcal{F}_{X_{ij}} \to \bigoplus_{ijk} \mathcal{F}_{X_{ijk}}$$

($i : S \to X$, $\mathcal{F}_S = i_* i^* \mathcal{F}$). Diff. are alternating sums of restrictions.

(EASY)

$K^m = \bigoplus_{|I|=m+1} \mathbb{Q}_{\cancel{X}_I}$, $d: K^m \to K^{m+1}$

$H^k(X, \mathbb{Q}) \cong H^k(X, K^\cdot)$

Let W the filtration $W_1 K^\cdot = K^{\geq -m}$

$(\ \ldots \subseteq W_{-2} \subseteq W_{-1} \subseteq W_0 = K)$

Poincaré lemma $\underline{\mathbb{C}}_{X_I} \xrightarrow{\sim} \Omega^\cdot_{X_I}$

This gives a quasi-is $K^\cdot \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \mathrm{Tot}\left(\bigoplus_{|I|=m+1} \Omega^\cdot_{X_I}\right) = T^\cdot$

compatible between W ~~for weight~~ and column filtration

Hodge filtration $F^p T^\cdot = \mathrm{Tot}\left(\bigoplus_{|I|=\cdot+1} \Omega^{\geq p}_{X_I}\right)$

<u>Claims</u>. It induces a MHS on $H^\cdot(X, \mathbb{Q})$.

<u>Spectral sequence for W</u>

$$E_1^{pq} = \mathbb{H}^{p+q}(X, gr^W_{-p} K^\cdot) = \bigoplus_{|I|=p+1} H^q(X_I) \Rightarrow E_\infty^{p,q} = gr^W_q H^{p+q}(X$$

This degenerates at E_2.

<u>Rmk</u> This construction is almost dual to the one for the cohomology of ~~Y\X~~ $U = Y \setminus X$.

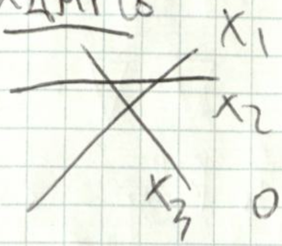
More precisely we look at $H^\cdot(Y, X) = H^\cdot(Y, j_! \underline{\mathbb{Q}}_{Y \setminus X})$

For $H^\cdot(Y, X)$ spectral sequence $E_1^{p,q} = \bigoplus_{|I|=p} H^q(X_I) \Rightarrow E_\infty^{pq}$

$H^k(Y, X) = (H^{2d-k}(Y \setminus X))^\vee \otimes \mathbb{Q}(-d)$

$d = \dim Y$ the two spectral sequences are dual to each other.

EXAMPLE



$3\,\mathbb{P}^1 \subseteq \mathbb{P}^2$

$$0 \to \bigoplus H^2(X_i) \to 0$$

$$0$$

$$0 \to \bigoplus H^0(X_i) \to \bigoplus H^0(X_{ij}) \to 0$$

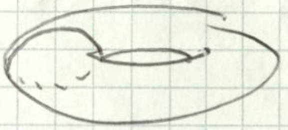
$$H^2(X) \cong \bigoplus_i H^2(X_i)$$

$$d_0 : \begin{cases} x_1 \mapsto x_{12} + x_{13} \\ x_2 \mapsto -x_{12} + x_{23} \\ x_3 \mapsto x_{13} - x_{23} \end{cases} \qquad \begin{array}{l} \ker = \langle x_1 + x_2 + x_3 \rangle \\ \operatorname{coker} \; x_{12} = -x_{13} = x_{23} \end{array}$$

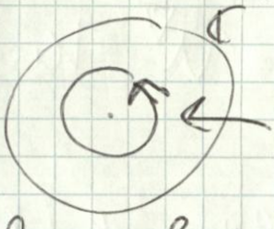
$H^1(X) \cong \mathbb{Q}(0)$ ~~H^0~~ $H^0(X) \cong \mathbb{Q}(0)$

is of weight 0

LOOP of weight 1

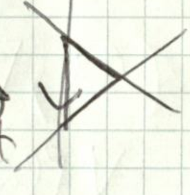


$\mathbb{C}^*$



loop of w 2

loop of weight 0



These are different from an algebraic geometry point of view.

GAUSS – MANIN CONNECTION

<u>Equivalence</u>: S a smooth complex manifold

$$\left\{ \begin{array}{c} \text{loc. systems} \\ \text{of } \mathbb{C}\text{-vs on } S \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{hol. fibre bundles on } S \text{ equipped} \\ \text{with a flat connection} \end{array} \right\}$$

V a holomorphic vector bundle on S, a connection on V is

$$\nabla : V \to \Omega^1_S \otimes_{\mathcal{O}_S} V \quad \text{s.t. } \nabla(fs) = df \otimes s + f \nabla s$$

$$f \in \mathcal{O}_S,\ s \in V$$

One can extend to $\nabla : \Omega^p_S \otimes_{\mathcal{O}_S} V \to \Omega^{p+1}_S \otimes_{\mathcal{O}_S} V$

$\nabla(\omega\otimes s) = d\omega\otimes s + (-1)^p \omega\wedge\nabla s$

Def ∇ is flat (or integrable) if $\nabla\circ\nabla = 0$

$\rightsquigarrow (\Omega^{\cdot}_S\otimes\mathcal{V}, \nabla)$ is called the de Rham complex of ∇

In loc. coordinates $\mathcal{V}\cong\mathcal{O}_S^{\oplus m}$ is given by

$$\nabla\begin{pmatrix}\omega_1\\ \vdots\\ \omega_m\end{pmatrix} = \begin{pmatrix}d\omega_1\\ \vdots\\ d\omega_m\end{pmatrix} + \Omega\wedge\begin{pmatrix}\omega_1\\ \vdots\\ \omega_m\end{pmatrix}$$

Ω is a matrix of 1-forms.

∇ is flat $\Longleftrightarrow d\Omega + \Omega\wedge\Omega = 0$

$\mathbb{V}$ loc system on S $\rightsquigarrow \mathcal{V}\cong\mathbb{V}\otimes_{\underline{\mathbb{C}}_S}\mathcal{O}_S$

The connection ∇ is obtained by tensoring

$\mathcal{O}_S\xrightarrow{d}\Omega^1_S$ by $\mathbb{V}$

$(\mathcal{U},\nabla)\rightsquigarrow\mathbb{V} = \mathcal{U}^{\nabla} = \ker(\nabla: \mathcal{U}\to\Omega^1_S\otimes\mathcal{V})$

∇ flat.

This amounts to solving of diff eqs

$$d\begin{pmatrix}f_1\\ \vdots\\ f_m\end{pmatrix} + \Omega\begin{pmatrix}f_1\\ \vdots\\ f_m\end{pmatrix} = 0$$

DATUM $\begin{array}{c}X\\ \downarrow f\\ S\end{array}$ a proper & submersive morphism b/w complex manifold

Ehresmann's theorem Locally on S, we have ~~a~~ C^∞-trivialization

$$\begin{tikzcd} f^{-1}(\mathcal{U}) \arrow[rr, "\simeq"] \arrow[dr, "f"'] & & X\times\mathcal{U} \arrow[dl] \\ & \mathcal{U} & \end{tikzcd}$$

$\mathcal{U}$ small disk contained in O

<u>CONSEQUENCE</u> for $0 \in U' \subset U$

$$H^{\cdot}(f^{-1}(U)) \xrightarrow{\sim} H^{\cdot}(f^{-1}(U')) \xrightarrow{\sim} H^{\cdot}(X_0)$$

Let us write $\mathbb{V}^n = R^n f_* \underline{\mathbb{C}}_X$ then $\mathbb{V}^n$ is a loc. sys. on S

($\mathbb{V}^*$ is the sheaf ass. to the presheaf $U \to H^{\cdot}(f^{-1}(U))$)

$\mathbb{V}^n_s \cong H^n(X_s) \qquad X_s = f^{-1}(\{s\}) \quad (s \in S)$

$$\rightsquigarrow \left(\mathcal{V}^n = \mathbb{V}^n \otimes_{\underline{\mathbb{C}}_S} \mathcal{O}_S, \underset{\text{flat}}{\nabla}\right)$$

<u>Def</u> ∇ is called the Gauss-Manin connection of the family $\begin{array}{c} X \\ \downarrow f \\ S \end{array}$

$$\mathcal{V}^n \underset{\uparrow}{=} R f_*(f^{-1}\mathcal{O}_S)$$

Projection formula

Relative diff. forms: let $\Omega^1_{X/S}$ be the quotient

$$0 \to f^*\Omega^1_S \to \Omega^1_X \to \Omega^1_{X/S} \to 0$$

$$\Omega^k_{X/S} = \Lambda^k \Omega^1_{X/S} \quad (\Omega^0_{X/S} = \mathcal{O}_X)$$

Relative (holomorphic) Poincaré Lemma $\quad f^{-1}\mathcal{O}_S \xrightarrow{\sim} \Omega^{\cdot}_{X/S}$

$$\Rightarrow \mathcal{V}^n = R^n f_*(\Omega^{\cdot}_{X/S})$$

<u>Description of the Gauß-Manin connection</u> (using $\mathcal{C}^\infty$-diff. form)

$\mathcal{C}^\infty$ vector bundle, $\mathcal{V}^n_\infty = R^n f_* \underline{\mathbb{C}}_X \otimes_{\underline{\mathbb{C}}_S} \mathcal{C}^\infty_S \cong$

$$\cong R^n f_*(f^{-1}\mathcal{C}^\infty_S) \cong R^n f_*(\underbrace{A^{\cdot}_{X/S}}_{\text{rel. diff. forms}})$$

$$\Rightarrow \mathcal{V}^n_\infty = \mathcal{H}^n(f_* A^{\cdot}_{X/S})$$

Locally, a section of $\mathcal{V}^*_\infty$ is represented by a $\mathcal{C}^\infty$ diff. form.

ω on X such that $\forall s \in S$ $d\omega|_{X_s} = 0$

Corresponds to the section $\sigma : s \mapsto [\omega|_{X_s}]$

Let us fix a $\mathcal{C}^\infty$-vector field u on S. We want to understand $\nabla_u : \mathcal{V}^m_\infty \to \mathcal{V}^m_\infty$

Let us choose a $\mathcal{C}^\infty$-vector field v on X, such that $f_*(v) = u$

<u>Prop</u> $\nabla_u(\sigma)$ is the section $s \mapsto [\mathcal{L}_v(\omega)|_{X_s}] = [i_v(d\omega)|_{X_s}]$

$\mathcal{L}_v$: Lie derivative along v

i_v : contraction by v

(Cartan's formula: $\mathcal{L}_v = i_v d + d i_v$)

<u>Proof</u>

$$\begin{array}{ccccc} \mathcal{C}^\infty_S \otimes_{\underline{\mathbb{C}}_S} R^m f_* \underline{\mathbb{C}}_X & \cong & R^m f_*(f^{-1}\mathcal{C}^\infty_S) & \cong & R^m f_*(\mathcal{A}^\bullet_{X/S}) \\ \circlearrowleft & & \circlearrowleft & & \circlearrowleft \\ \nabla_u & \longleftrightarrow & \underset{\mathcal{L}_v}{\overset{\shortparallel}{\partial_u}} & \overset{\circledast}{\longleftrightarrow} & \mathcal{L}_v \end{array}$$

$$\circledast \quad \begin{array}{ccccccccc} 0 \to f^{-1}\mathcal{C}^\infty_S & \to & \mathcal{C}^\infty_X & \to & \mathcal{A}^1_{X/S} & \to & \mathcal{A}^2_{X/S} & \to & \dots \\ \downarrow \partial_u & & \downarrow \mathcal{L}_v = \partial_v & & \downarrow \mathcal{L}_v & & \downarrow \mathcal{L}_v & & (\mathcal{L}_v d = d\mathcal{L}_v) \\ 0 \to f^{-1}\mathcal{C}^\infty_S & \to & \mathcal{C}^\infty_X & \to & \mathcal{A}^1_{X/S} & \to & \mathcal{A}^2_{X/S} & & \end{array}$$

DRAGOS

$$\begin{array}{l} X \\ \downarrow f \\ S \end{array} \quad \text{proper, smooth.}$$

<u>HODGE FILTRATION</u>

$$\mathcal{V}^m = R^m f_*(\Omega^\bullet_{X/S})$$

<u>Def</u> $F^p\mathcal{V}^m = \mathrm{Im}(R^m f_* \Omega^{\geq p}_{X/S} \to R^m f_* \Omega^\bullet_{X/S}) \subseteq \mathcal{V}^m$

<u>Def</u> same thing for $F^p \mathcal{V}^m_\infty \subseteq \mathcal{V}^m_\infty$ ⊛

Rmk ~~Eg~~ $(F^p \mathcal{V}^m)_s = \operatorname{Im}\left(\mathbb{H}(\Omega^{\geq p}_{X_s}) \to \mathbb{H}^m(\Omega^\bullet_{X_s})\right)$

$\subseteq \mathbb{H}^m(\Omega^\bullet_{X_s}) = H^m(X_s)$

⊛ $A^\bullet_{X/S} = \mathrm{Tot}(A^{\bullet,\bullet}_{X/S})$

$F^p A^\bullet_{X/S} = \mathrm{Tot}(A^{\geq p, \bullet}_{X/S})$

Thm (Griffiths' transversality) $\mu \in TS$

$\nabla_\mu : \mathcal{V}^m \to \mathcal{V}^m$ has the property

$\nabla_\mu : F^p \mathcal{V}^m \to F^{p-1} \mathcal{V}^m$

Proof (We'll do the $\mathcal{C}^\infty$ version)

$\sigma \in F^p \mathcal{V}^m_\infty = R^m f_*(\mathrm{Tot}(A^{\geq p, \bullet}_{X/S}))$

$= \mathcal{H}^m(f_*(\mathrm{Tot}(A^{\geq p, \bullet}_{X/S})))$

$\sigma \rightsquigarrow \omega \in A^m_X$ s.t. ω is ~~of type~~ a sum of forms of type (r,s) w/ $r \geq p$ and such $d\omega|_{X_s} = 0$

$\nabla_\mu(\sigma) : \gamma \to [i_V d\omega|_{X_\gamma}]$

$\omega = \sum_{\substack{r \geq p \\ r+s=m}} (r,s) \qquad d\omega = \sum_{\substack{r \geq p \\ r+s=m+1}} (r,s)$

$i_V d\omega = \sum_{\substack{r \geq p-1 \\ r+s \geq m}} (r,s) \in F^{p-1} A^\bullet_X \qquad \square$

THM SEMICONTINUITY (GRAUERT)

Thm $f : X \to S$ proper, $\mathcal{E}^{\cdot}$ coh. sh. on X
flat over S.

Then the function $s \xrightarrow{d_p} \dim H^p(X_s, \mathcal{E}^{\cdot}|_{X_s})$ is upper semicontinuous

TFAE

1) d_p is constant
2) $R^p f_* \mathcal{E}^{\cdot}$ is a vector bundle

Def $h^{p,q}(s) := \dim H^q(X_s, \Omega^p_{X_s}) = \dim H^{p,q}(X_s)$

Prop $h^{p,q}(s)$ are constant.

Proof Apply Grauert's thm to $\Omega^p_{X/S}$

Fix $s_0 \in S$. Look around s_0

$$\sum_{p+q=m} h^{p,q}(s_0) = \dim H^m(X_0) \geq \sum_{p+q=m} h^{p,q}(s) = \dim H^m(X_s)$$

(s in a nbhd of s_0)

$$\Rightarrow h^{p,q}(s) = h^{p,q}(s_0) \; \forall p,q.$$

Prop $R^m f_* \Omega^{\geq p}_{X/S}$ are vector bundles

and $F^p V^m \hookrightarrow V^m$ is an injective map of v. bundles.

Proof $\dim(R^m f_* \Omega^{\geq p}_{X/S}) = \sum_{\substack{r \geq p \\ r+t=m}} h^{r,t}(s)$ is constant.

Injectivity $R^m f_* \Omega^{\geq p}_{X/S} \hookrightarrow R^m f_* \Omega^{\cdot}_{X/S}$

because on the fibers we get $F^p H^m(X_s) \hookrightarrow H^m(X_s)$

$\rightsquigarrow$ Injectivity as coh. sheaves.

The coker is still a vector bundle because the dim is constant