<u>Vanishing cycles</u> <u>29 Apr 2015</u>

1) V.C. for constructible sheaves: $D^b_c(X,\mathbb{Q})$

X is a cplx mfld; $f: X \to \Delta := \{z \in \mathbb{C} \mid |z|<1\}$

$X^* \xrightarrow{sm.} \Delta^* = \Delta \setminus \{0\}$

$$\begin{tikzcd} \tilde{X} \arrow[r, "k"] \arrow[d] & X \arrow[d] & X_0 \arrow[l, hook', "i"'] \arrow[d] \\ \mathbb{H} \arrow[r] & \Delta & 0 \arrow[l, hook'] \end{tikzcd}$$

$z \mapsto e^{2\pi i z}$

If $K \in D^b_c(X,\mathbb{Q})$

Then $\psi_f(K) := i^{-1}Rk_*k^{-1}K$.

is the complex of nearby cycles.

Vanishing cycles are $\varphi_f(K) := \mathrm{Cone}\left(i^{-1}K \xrightarrow{can} \psi_f(K)\right)$

$\exists$ a monodromy operator $T: \psi_f(K) \to \psi_f(K)$ &

$T: \varphi_f(K) \to \varphi_f(K)$

<u>Rmk</u>: φ_f is actually functorial.

Can : $\psi_f(K) \longrightarrow \varphi_f(K)$

Var : $\varphi_f(K) \longrightarrow \psi_f(K)(-1)$ Tate twist.

$\hookrightarrow$ "ignore for constructible sheaves"

$$\begin{tikzcd} \arrow[r] & i^{-1}K \arrow[r] \arrow[d] & \psi_f(K) \arrow[r] \arrow[d, "\frac{1}{2\pi i}\log T"] & \varphi_f(K) \arrow[r, "+1"] \arrow[d, dashed, "var"] & {} \\ \arrow[r] & 0 \arrow[r] & \psi_f(K)(-1) \arrow[r] & \psi_f(K)(-1) \arrow[r, "+1"] & {} \end{tikzcd}$$

<u>Rmk</u>: Sometimes, instead of $\frac{1}{2\pi i}\log T$, people use $(1-T)$, which sends $i^{-1}K$ to 0.

<u>And</u>: $\log T = (1-T) + \frac{1}{2}(1-T)^2 + \cdots$, so $\log T$ is well-defined if $(1-T)^m = 0$ for $m \gg 0$ [i.e. unipotent monodromy]

In general, one has to make choices on the branch of log.

(b/c eigenvalues need to be "logged").

Gabber: If K is perverse, then $\left.\begin{array}{l} {}^p\psi_f(K) := \psi_f(K)[-1] \\ {}^p\varphi_f(K) := \varphi_f(K)[-1] \end{array}\right\}$ are perverse.

Because Perv is abelian, we can write

$${}^p\psi_f(K) = \bigoplus_{\lambda\in\mathbb{C}^*} {}^p\psi_{f,\lambda}(K) \quad \& \quad \psi_{f,\lambda}(K) = \ker(T-\lambda\cdot Id)^m \; ; \; m \gg 0$$

$$\begin{array}{c} \wr \\ \mathrm{Perv}(X,\mathbb{C}) \longrightarrow \text{finite length} \\ \wr \end{array}$$

$${}^p\varphi_f(K) = \bigoplus_{\lambda\in\mathbb{C}^*} {}^p\varphi_{f,\lambda}(K)$$

If $\lambda \neq 1$, then $\mathrm{can}: {}^p\psi_f(K) \longrightarrow {}^p\varphi_f(K)$ is an isomorphism.
because $i^{-1}K$ is trivial.

2) The coherence of semistable families:

Def: (1) A degeneration is a proper flat holomorphic map $\pi: \mathcal{X} \to \Delta$; smooth above Δ^*, such that

- $X_t := \pi^{-1}(t)$ is a smooth complex (projective) variety for all $t \neq 0$.
- $\mathcal{X}$ is Kähler

(2) A degeneration is semi-stable if: $X_0 \in \mathcal{X}$ is a reduced simple normal crossings divisor, ie: $X_0 = \bigcup X_0^{(i)}$; $X_0^{(i)}$ reduced & smooth, locally $X_0 = \{z_1 \cdots z_m = 0\}$

Semistable reduction theorem (Mumford)

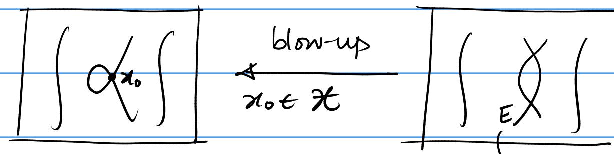
Given a degeneration $\mathcal{X} \to \Delta$, there is a finite map $b: \Delta \to \Delta$ such that:

$$\begin{array}{ccccc} \mathcal{X}^{ss} & \overset{\text{birational}}{\dashrightarrow} & \mathcal{X}' & \longrightarrow & \mathcal{X} \\ & \searrow & \downarrow & & \downarrow \\ & & \Delta & \xrightarrow{b} & \Delta \end{array}$$

(looks like $z_i \mapsto z_i^n$)

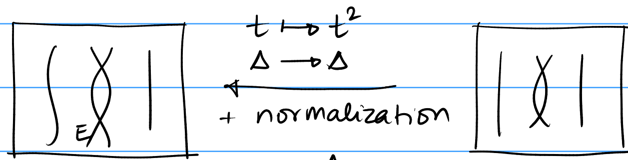
And $\mathcal{X}^{ss}|_{\Delta^*} = \mathcal{X}|_{\Delta^*}$

<u>Example</u> :



multiplicity 2, not allowed.

So :



semi-stable

is just the 2:1 branched cover of $Bl_{x_0}\mathcal{X}$, branched along $\overline{X}_0$.

<u>Monodromy thm</u> : $\pi : \mathcal{X} \to \Delta$ a degeneration, $t \in \Delta^*$
$T : H^k(X_t) \to H^k(X_t)$
$[K = R\pi_* \mathbb{Q}]$
(1) $(T^r - 1)^{k+1} = 0$ for some $r, m \geq 0$ [quasi-unipotent]
(2) If π is semistable, then $(T-1)^{k+1} = 0$ [unipotent].

<u>Retraction thm</u> : $\mathcal{X} \to \Delta$ ss. $\exists$ retraction $r : \mathcal{X} \to X_0$, such that $r^* : H^*(X_0) \xrightarrow{\cong} H^*(\mathcal{X})$ &
$r_* : H_*(\mathcal{X}) \xrightarrow{\cong} H_*(X_0)$

<u>Defn</u> : $N := \log T$ acts on cohomology groups.

The Clemens–Schmid exact sequence :
$\pi : \mathcal{X} \to \Delta$ ss ; $n = \dim X_t$, $t \neq 0$. We have LES:

$$\cdots \to \underset{\text{“}H_{2n+2-k}(X_0)\text{”}}{H_{2n+2-k}(\mathcal{X})} \xrightarrow{\alpha} \underset{\text{“}H^k(X_0)\text{”}}{H^k(\mathcal{X})} \xrightarrow{r^*} H^k(X_t) \xrightarrow{N} H^k(X_t) \xrightarrow{\beta} H_{2n-k}(\mathcal{X}) \to$$

α is given by Poincaré duality map : $H_{2n+2-k}(\mathcal{X}) \xrightarrow[\sim]{PD} H^k(\mathcal{X}, \partial\mathcal{X})$

$$\downarrow$$
$$H^k(\mathcal{X})$$

$$\beta : H^k(X_t) \xrightarrow[\sim]{PD} H_{2n-k}(X_t) \longrightarrow H_{2n-k}(\mathcal{X})$$

$[\partial\mathcal{X} := \pi^{-1}(\partial\bar{\Delta})]$

Let $K = R\pi_* \mathbb{Q}$; $\pi : X \to \Delta$ semistable.

$H^k(\psi_f(K)) = H^k(X_t, \mathbb{Q})$, $t \in \Delta^*$ [check this]. &

$H^k(i^{-1}K) \cong H^k(X_0)$

$$i^{-1}K \to \psi_f(K) \to \varphi_f(K) \xrightarrow{+1} \quad [f = \text{identity map}]$$

$$\Rightarrow H^k(\varphi(K)) \cong \underbrace{H^k(X_t)/\ker N}\ \oplus\ \mathrm{im}(H_{2n+1-k}(\mathcal{X}) \to H^{k+1}(X_0))$$

I am confused here.

E.g. $(k=2)$:

$$H_{2n}(X_0) \longrightarrow H^2(X_0)$$

$\quad \| \qquad\qquad\qquad \Psi$

$\oplus [X_0^{(i)}] \cdot \mathbb{Q}$

& $[X_0^{(i)}]\mathbb{Q} \longmapsto [\mathcal{O}_{\mathcal{X}}(X_0^{(i)})|_{X_0}]$

$$\begin{tikzcd}
H^k(\psi(K)) \arrow[r] \arrow[d, equal] & H^k(\varphi(K)) \arrow[r] \arrow[d, equal] & H^{k+1}(i^{-1}K) \arrow[d, equal] & \\
H^k(X_t) \arrow[r] & H^k(X_t)/\ker N \oplus \mathrm{im}(H_{2n+2-k-1}(X_0) \to H^{k+1}(X_0)) \arrow[r] & H^{k+1}(X_0) \arrow[d] \\
& & H^{k+1}(X_t)
\end{tikzcd} \quad *$$

3) The Kashiwara-Malgrange filtration:

Aim: Define nearby & vanishing cycles for $\mathcal{D}$-modules.

Motivations

(1) $D^b_{\text{reg hol}}(\text{mod-}\mathcal{D}_X) \xrightarrow{\sim} D^b_{\text{const}}(X, \mathbb{C})$

$\mathcal{M} \longmapsto DR_X(\mathcal{M}) := R\mathcal{H}om(\omega_X, \mathcal{M})$ $\quad \Lambda^{top}\Omega_X$

(2) A polarized Hodge module $\mathcal{M} \in HM^{p}(X)$ is a tuple $(\mathcal{M}, F_\cdot\mathcal{M}, K)$, where $(\mathcal{M}, F_\cdot\mathcal{M})$ is a filtered $\mathcal{D}$-module; $K \in Perv(X, \mathbb{Q})$ such that $DR_X(\mathcal{M}) \simeq K \otimes_{\mathbb{Q}} \mathbb{C}$ + additional conditions.

If $\mathcal{M}$ has strict support $Z \subseteq X$, then $\exists$ Zariski-open $U \subseteq Z$ such that $\mathcal{M}|_U = VHS$

<u>Thm</u>: X a complex manifold, $f: X \to \mathbb{C}$ holomorphic fn $\neq$ const. $Z = f^{-1}(0)$. Then $M \in HM^{p}(X)$ is uniquely determined by $(M|_{X-Z}, \varphi_{f,t}(M), can, var)$

[can : $\psi_{f,1}(M) \longrightarrow \varphi_{f,1}(M)$ & var: $\varphi_{f,1}(M) \longrightarrow \psi_{f,1}(M)$]
This hints that we can use induction on dimension.

X is a complex manifold, $t: X \to \mathbb{C}$ <u>smooth</u> holomorphic fn.
∂_t a vector field with $[\partial_t, t] = 1$.

<u>Defn</u>: Kashiwara-Malgrange filtration on a right $\mathcal{D}_X$-module $\mathcal{M}$ is an increasing filtration $V_\cdot M$,

(1) $V_k \in \mathcal{M}$ coherent over $V_0\mathcal{D}_X = \{P \in \mathcal{D}_X \mid P \cdot I_{X_0} \subset I_{X_0}\}$
(2) $V_k\mathcal{M} \cdot t \subseteq V_{k-1}\mathcal{M}$, $V_k M \cdot \partial_t \subseteq V_{k+1}\mathcal{M}$
(3) $V_k\mathcal{M} = V_{k-1}\mathcal{M}$ for $k \ll 0$
(4) Each eigenvalue λ of $E = t\partial_t$ on graded pieces $gr_k^V\mathcal{M}$ has real part in $(k-1, k]$.

<u>Thm</u> (Kashiwara): (i) If $\mathcal{M}$ is holonomic, then K-M filtration exists & is unique.
(ii) If $\mathcal{M}$ regular holonomic, then $gr_k^V M$ again regular hol. with supp $\subset t^{-1}(0)$. for $k \leq -1$.

get $\mathrm{gr}_k^{D} \mathcal{M}$