Meromorphic connections / Holonomic regular $\mathcal{D}$-modules in dim 1 – Yohan

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1) Meromorphic connections : X complex manifold, $Y \subseteq X$ a hypersurf

$\mathcal{O}_X(*Y)$ = shf of meromorphic functions on X with poles in Y.

Locally, if $Y = \{f=0\}$, $\mathcal{O}_X(*Y) = \mathcal{O}_X[\frac{1}{f}] = \mathcal{O}_X[y]/(yf-1)$

$\mathcal{O}_X(*Y)$ is a coherent shf of rings.

De Rham complex : $\Omega^p(*Y) = \mathcal{O}_X(*Y) \otimes_{\mathcal{O}_X} \Omega^p_X$

Def : A meromorphic connection is given by:

- $\mathcal{V}$ a coherent $\mathcal{O}_X(*Y)$-module
- ∇ a connection : $\mathcal{V} \to \mathcal{V} \otimes_{\mathcal{O}_X(*Y)} \Omega^1_X(*Y)$ [Leibniz rule + flatness].

$\rightsquigarrow$ De Rham complex : $\Omega^p(\mathcal{V}) = \mathcal{V} \otimes_{\mathcal{O}_X(*Y)} \Omega^p(*Y)$

(*) Meromorphic connection $(\mathcal{V}, \nabla)$ $\rightsquigarrow$ $\mathcal{D}_X$-module structure on $\mathcal{V}$

Thm : This is a holonomic $\mathcal{D}_X$-module

Eg $(\mathcal{O}_X(*Y), d)$ is a meromorphic connection.

2) Localization : $Y \subset X$ as before ; $M \in$ Holonomic (X)

Def : The localization of M along Y is defined as:

$M \otimes_{\mathcal{O}_X} \mathcal{O}_X(*Y)$ $\to$ tensor product of left $\mathcal{D}$-modules.

Thm : Localization preserves holonomicity & regularity.

Prop : M coherent $\mathcal{D}$-module on X. Suppose that

(i) $\mathrm{Sing}(M) \subset Y$, and (ii) M is localized at Y, i.e. $M \simeq M(*Y)$.

Then M is a meromorphic connection on (X, Y).

* $M \in \mathrm{Hol}(X)$. Find a hypersurface $Y \supseteq \mathrm{Sing}(M)$.

Consider $0 \to K \to M \to M(*Y) \to N \to 0$

K & N are supported on $Y \rightsquigarrow$ gives some sort of inductive breakdown of M.

<u>Proof</u>: Locally, $\exists$ a good filtration $\{M_r\}$ of M.

$\mathrm{Sing}(M) \subset Y \Rightarrow$ for $\ell \gg \ell_0$, $\mathrm{Supp}(M_\ell / M_{\ell_0}) \subseteq Y$

$\forall\, \ell \geq \ell_0$, $M_{\ell_0} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X(*Y) \xrightarrow{\sim} M_\ell \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X(*Y)$: limit

$\xrightarrow{\sim} M(*Y)$.

But M is localized along Y

$\Rightarrow M = M_{\ell_0} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X(*Y)$.

③ <u>Simple normal crossing case</u>: Y a simple normal crossings divisor $D = \cup D_i$: each D_i smooth & locally intersection of hyperplanes: $z_1 \cdots z_p = 0$; coords $z_1, \ldots, z_n$

$$\Omega^1_X(\log D) = \left\{ \text{locally generated as } \mathcal{O}_X\text{-module by } \frac{dz_1}{z_1}, \ldots, \frac{dz_p}{z_p}, dz_{p+1}, \ldots dz_n \right\}$$

Logarithmic connection on the pair $(\mathcal{V}, \nabla)$ on (X, D) is:

- $\mathcal{V}$ locally free $\mathcal{O}_X$-module
- $\nabla : \mathcal{N} \to \mathcal{N} \underset{\mathcal{O}_X}{\otimes} \Omega^1_X(\log D)$.

Rmk: From log connection $\rightsquigarrow$ canonically associated a meromorphic connection $\rightsquigarrow$ a D-module.

Defn: A meromorphic connection is regular on (X, D); D normal crossings, if it comes from a log connection.

Residue: $\Omega^1_X(\log D) \xrightarrow{res_{D_i}} \mathcal{O}_{D_i}$ [Let D_i be a (smooth) irred cpt.] $\{z_1 = 0\}$

an $\mathcal{O}_X$-linear map

locally, $\displaystyle\sum_{i=1}^{p} f_i \cdot \frac{dz_i}{z_i} + \sum_{j=p+1}^{n} f_j \, dz_j$

$$\Big\downarrow res$$

$f_1(0, z_2, \dots, z_n)$.

this is well-defined.

* For a log connection $(\mathcal{V}, \nabla)$ on (X, D):

$$\begin{tikzcd} \nabla : \mathcal{V} \arrow[r] \arrow[d] & \mathcal{V} \otimes_{\mathcal{O}_X} \Omega^1_X(\log D) \arrow[r, "1 \otimes res_i"] & \mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{O}_{D_i} = \mathcal{V}|_{D_i} \\ \mathcal{V}|_{D_i} \arrow[urr, dashed] & & \end{tikzcd}$$

$\Rightarrow res_{D_i}(\nabla) \in End(\mathcal{V}|_{D_i})$.

Locally: $\nabla = d + \displaystyle\sum_{i=1}^{p} N_i(z) \frac{dz_i}{z_i} + \sum_{j=p+1}^{n} N_j(z_i)\, dz_j$

So, $res_{D_i}(\nabla) = N_i(z_1, z_2, \dots, 0, z_{i+1}, \dots, z_n)$.

E.g. $(\Delta, 0)$ & $(\mathcal{V}, \nabla)$ on this.

There exists T such that: $T \in End_{\mathcal{O}_X}(\mathcal{V})$, such that $T|_{\Delta^*}$ is monodromy endomorphism, T commutes with ∇ and $T(0) = \exp(-2\pi i \, Res_0(\nabla)) \in End(\mathcal{V}_0)$.

Thm: Let $(\mathcal{V},\nabla)$ be a regular meromorphic connection on (X,D). Let τ = section of $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$.
Then $\exists!$ locally free $\mathcal{O}_X$-submodule V^{τ} of V such that
(i) ∇ has log. poles w.r.t V^{τ}
(ii) The eigenvalues of the residues of ∇ w.r.t. V^{τ} are contained in image of $\mathrm{Im}(\tau)$.

Defn: The lattice corresponding to the section $[0,1)$ is called the Deligne lattice.
[Ref: Deligne's book, Malgrange's talk in Borel's book].

Rmk: If on Δ we have $\nabla = d + M(z)\frac{dz}{z}$.

$$\downarrow$$

change $M(z)$ to a constant matrix M
$\to T = \exp(-2\pi i M)$ [?]

Analytic case: $\{Y \subseteq X$ hypersurface in X; regular meromorphic conns on $(X,Y)\}$

$$\downarrow$$

$\{$local system on $X-Y\}$

The arrow $\downarrow$ becomes $\updownarrow$ (equivalence) if we look at regular meromorphic connections.

Comparison thm: $(\mathcal{V},\nabla)$ regular meromorphic connection on (X,D). Let $\mathcal{U} = X \setminus D$. Then:

$$DR(M) \longrightarrow j_* j^! DR(M) \overset{qis}{\simeq} j_* \mathcal{L}, \text{ where}$$

$\mathcal{L} = \ker(\nabla|_{\mathcal{U}})$.
Claim: $DR(M) \underset{qis}{\xrightarrow{\sim}} j_* j^! DR(M)$.

Algebraic case: X smooth complex alg. variety

- flat connection on X "regular at ∞" [algebraic]
  - compactify to a smooth proper variety; pushforward, get meromorphic conn; the "analytification" of this should be regular. [$\bar{X} \setminus X$ a divisor, then indep of $\bar{X}$]
- $\{$flat alg conn on X "regular at ∞"$\}$ $\xleftrightarrow[\text{of cat.}]{\text{equiv}}$ $\{$local system on $X\}$

- Flat connection regular at ∞ : $(\mathcal{V}, \nabla)$; $X \hookrightarrow \bar{X}$

$$j_* DR^{alg}(M) \overset{\text{qis}}{\simeq} j_* \mathcal{L} \quad ; \quad \mathcal{L} = (M^{an})^{\nabla}$$

$$H^i(X, \mathcal{L}) = H^i(X, DR(M))$$

algebraic

[More general version of Grothendieck thm].