BERNSTEIN-SATO POLYNOMIAL & V-FILTRATION

History: Gelfand's question from '54

If \( f(x) \) polynomial

\[ f_+(x) = \begin{cases} f(x) & x \geq 0 \\ 0 & x < 0 \end{cases} \]

If \( q \in C_0^\infty(\mathbb{R}^n) \)

\[ s \mapsto \int_{\mathbb{R}^n} f_+^s(x)q(x)dx \]

is analytic in \( s \) when \( \Re(s) > 0 \)

\( \Box \). Does it extend to other values of \( s \)?

BS - POLYNOMIAL

For any polynomial \( f \):

Let \( s \) be a formal parameter.

For some \( P \), diff. operator depending polynomially in \( s \) and some polynomial \( b(s) \) in \( s \)

\[ P \cdot f^{s+1} = b(s)f^s \]

Use this equation to continue

\[ s \mapsto \int_{\mathbb{R}^n} f_+^s(x)q(x)dx = \int \frac{P \cdot f^{s+1}}{b(s)} q(x)dx \]
By integrating by parts: extends meromorph. (in s) to Re(s) > -1. By induction extends anyul.

Motivation #2: The existence of $b(s)$ gives us the existence of $V$-filtrations.

Proof of existence of $P, v$

1) $X = A^r \checkmark$

2) More gen. case

Recall that $D_X$ has an $\omega$-filtration, but $D_X = D_{A^r}$ has another one $\Rightarrow$ Bernstein filt.

$D_{A^m} = \mathbb{C} < x_1, \ldots, x_n, e_1, \ldots, e_m \rangle / (\lambda x_1 = 12x_1 = 1)$ Graded pieces are finite dim.

FACT: If $X = A^m$, then $M \in D_X\text{-mod}$ is holonomic iff it is holonomic w.r.t. Bernstein filtration $\mathfrak{f} \in \mathbb{C}[x]$. Consider $D_X(s) = D_X \otimes_{\mathbb{C}} \mathbb{C}(s)$

$M \leq O_x(s)[\mathfrak{f}^{-1}] \cdot \mathfrak{f}^s$ generated by the symbol $\mathfrak{f}^s$.

If $g \in O_x$, $g \mathfrak{f}^s = g \mathfrak{f}^s$. If $z \in T_x$, $z \cdot \mathfrak{f}^s = s \cdot f(z) \mathfrak{f}^s$. $M$ is a $D_X(s)$-module.

Claim: $M$ is holonomic as a $D_X(s)$-module; $D_X$ has a filtration.
Consequence of claim Holonomic module $\Rightarrow$ finite length

So $M$ has fin. length

$$M \not\subseteq M \cdot f \not\subseteq \ldots \not\subseteq M \cdot f^n \not\subseteq \ldots$$

This stabilizes $\Rightarrow$ $f^{s+k} \in M \cdot f^{k+1} = D_x(s)$-submod. gen. by $f^{s+k+1}$

Then there is some $L \in D_x(s)$ s.t. $L \cdot f^{s+k+1} = f^{s+k}$ clear denominators in $S$

$$p \cdot f^{s+k+1} = b'(s) \cdot f^{s+k}$$

Claim $M$ is holonomic, in fact $N:C_x(s)[f^{-1}]$ $f^s$ is also hol. as $D_x(s)$-modules

Consider a filtration on $N$: $F_\alpha N = \{ p \cdot f^{s-\alpha} \mid p \in C(s)[x], \deg p \leq i(1+\deg f) \}$

Compare with Bernstein filtration $x_j \cdot p \cdot f^{s-\alpha} = x_j \cdot p \cdot f \cdot f^{s-\alpha-1}$

Cont degrees: $\deg (x_j \cdot p \cdot f) \leq (\alpha + 1)(1 + \deg f)$

Rank Graded pieces of $F_\alpha N$ are fin. dim. In fact $G_\alpha N$ has $\dim = \dim$ of polynomials of degree $

\alpha (1 + \deg f)$
Lemma If $\mathcal{F}$ is a filtration on $\mathcal{N}$ such that \( \dim \mathcal{G}_x \mathcal{N} \leq \frac{x^{m-1}}{(m-1)!} + o(x^{m-2}) \), then $\mathcal{N}$ is holonomic.

In our case we have \( \binom{x(1+\deg f)+m-1}{m-1} = \frac{x^{m-1}(1+\deg f)^{m-1}}{(m-1)!} + o(x^{m-2}) \)

$\Rightarrow$ $\mathcal{N}$ is holonomic $\Rightarrow$ So is $\mathcal{M}$ since $\mathcal{M} \subseteq \mathcal{N}$

**V-Filtration**

$\mathcal{O}_X$

$\iota : X \hookrightarrow X \times \mathbb{C}$

$\iota_+ \mathcal{O}_X \cong \mathcal{O}_X \otimes \mathbb{C}[\mathbb{D}_t]$ as $\mathcal{O}_X$-mod.

As a $\mathcal{D}$-mod, action depends on the embedding

$\iota_+ \mathcal{O}_X = \mathcal{O}_X \times \mathcal{D}$-module generated by $\partial_t \partial_f$; i.e.

$(t-f) \partial_{t-f} = 0$; freely generated as an $\mathcal{O}_X$-mod by $\partial_t \partial_f, \partial_t \partial_{t-f}, \partial_f \partial_{t-f}, ...$
$ \mathcal{N} = \mathcal{L} \subset \mathcal{L}_x$

$M = D_x[\mathcal{L}_x] \delta_{t-f} \text{ a } D_x-\text{mod. } \delta_{t-f} \cdot \delta_{t-f} = \delta_{t-f} \cdot \delta_{t-f}$

$S \subset D_x(s) \text{-mod. gen. by the symbol } f^s \text{ s.t. } f^s f^s = f^s \cdot f^s$

$f(s)$ can be thought as the minimal polynomial of $s$ acting on the quotient module $D_x[s]. \frac{f^s}{D_x[s]. f}$

On the other side $f(s)$ is the min. pol. of $-\delta_{t-f}$ on $M/\mathcal{L}M$

Let $A$ any hol. $D$-mod. on sing $x$. We can construct $M = A \otimes_{D_x} D_x(s) f$

It is enough to show that $M$ is hol.

Idea: Find some $M' \subset M$ holonomic s.t. $M'/W = M/W$ where $U = X \setminus \{f = 0\}$