Regular meromorphic connections / quiver description on a disc – Yohan   27 May

<u>Thm</u>: Let M be a regular meromorphic connection on a disc with singularities $\subseteq \{0\}$ [= regular $\mathcal{D}$-module M on disk + sing M satisfying $M \xrightarrow{\cong} M(*0) = M \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta(*0)$]

For any section τ of $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$, there is a unique lattice $V^\tau \subset M$ [ie a coherent $\mathcal{O}_\Delta$-submodule of M such that $V^\tau \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta(*0) = M$], s.t.:

(i) V^τ is stable by $t\partial_t$.

(ii) The eigenvalues of $t\partial_t$ acting on $V^\tau/(t)V^\tau$ are in $\operatorname{im} \tau$.

<u>E.g.</u> If we take τ such that $\operatorname{im}\tau = \{z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z < 1\}$: get "Deligne lattice".

<u>Cor</u>: Let $\nabla = d + N(z_1) \cdot \frac{dz_1}{z_1}$ on $\mathcal{O}_\Delta^n$

$N(z_1)$: $n \times n$ matrix of holomorphic fns on Δ

$(\mathcal{O}_\Delta^n, \nabla)$ is equivalent to $\left(\mathcal{O}_\Delta^n, \nabla' = d + N(0) \cdot \frac{dz_1}{z_1}\right)$ as long as: two eigenvalues of $N(0)$ do not differ by a non-zero integer

– In this case, in particular, monodromy $T = \exp(-2\pi i N(0))$.

Counterexample: $\nabla = d + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{dz}{z_1}$ on $\mathcal{O}_\Delta^2 = \langle e_1, e_2 \rangle$

$n \geq 1$; $f_1 = z_1^{-n} e_1$, $f_2 = e_2$; $\nabla = \begin{pmatrix} n & z_1^n \\ 0 & 0 \end{pmatrix} \frac{dz_1}{z_1}$ in this basis.

First case,

$T = \begin{pmatrix} 1 & -2\pi i \\ 0 & 1 \end{pmatrix}$ &

Second case:

$T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Solutions: $e^{-N \log(z_1)}$.

② Regular holonomic $\mathcal{D}$-modules on the disc:

$X = \Delta$; I = ideal of $\{0\}$. Let $V^k(\mathcal{D}_X) = \{P \in \mathcal{D}_X \mid \forall i,\ P(I^i) \subset I^{i+k}\}$

Decreasing exhaustive filtration by $\mathcal{O}_\Delta$-coherent submodules.

It is compatible with the ring structure. In particular, $V^0\mathcal{D}_X$ is a ring.

$gr^0_V(\mathcal{D}_X) = \mathbb{C}[E]$; E = image of $t\partial_t$ (indep. of coordinate t)

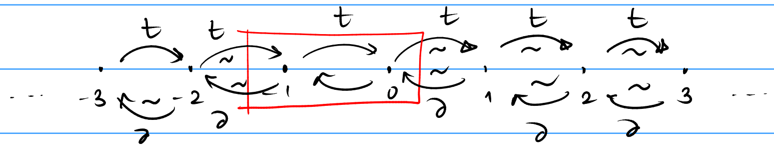
Defn-thm: Let M be a [regular] holonomic $\mathcal{D}$-module on Δ. There exists a unique exhaustive filtration (decreasing), satisfying:

(1) $(V^k\mathcal{D}_X)(V^\ell M) \subset V^{k+\ell}M$ [← regularity here?]

(2) Equality if $\begin{cases} k \geq 0, & \ell \gg 0 \\ k \leq 0, & \ell \ll 0 \end{cases}$

(3) The eigenvalues of $E = t\partial_t$ acting on $Gr^k_V M$ have real part in $[k, k+1)$.

Rmk: τ a section of $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$; $\tau(0 \bmod \mathbb{Z}) = 0$, or: replace ③ with eigenvalues of $E \subset \mathrm{Im}(\tau) + k$.



$\psi M := Gr^0_V M$

$\varphi M := Gr^{-1}_V M$.

E.g. M regular meromorphic connection

$\forall k$; $V^{Del} \subset M$; $V^k M = (z^k) V^{Del}$.

Defn: Let $\mathcal{C}$ = abelian category of $\mathcal{D}_X$-modules $[X = \Delta]$: holonomic & regular with singularities $\subset \{0\}$.

Thm: $\exists$ equivalence of categories

$\mathcal{C} \to \{ E \underset{v}{\overset{u}{\rightleftarrows}} F \mid E, F \in \mathrm{Vect}_{fin}$, eigenvalues of uv & vu

have $\mathrm{Re}(-)$ in $[0,1)\}$.

Functor is : $M \mapsto \{ \psi M \underset{t}{\overset{\partial}{\rightleftarrows}} \varphi M \}$; $\partial t = 1 + t\partial$.

* Fix z_1 coordinate ; ∂ such that $[\partial, z_1] = 1$ & $E := z_1 \partial$.

$\overline{M} = \{ m \in M \mid \exists\, P(x) \in \mathbb{C}[x] \setminus \{0\},\ P(E).m = 0 \}$.
$= \bigcup \{$finite-diml $\mathbb{C}$-subspace of M stable by $E\}$

$\overline{M}$ is a $\mathbb{C}[z_1, \partial]$-module.

<u>Key lemma</u>:

(1) $M = \mathcal{O}_X \otimes_{\mathbb{C}[z_1]} \overline{M}$

(2) $\forall \lambda \in \mathbb{C}$, $\overline{M} \supset \overline{M}_\lambda = \bigcup_N \ker(E - \lambda\,\mathrm{id})^N$ is a finite-diml $\mathbb{C}$-vector space.

<u>Inverse functor</u>: $M \in \mathcal{C}$

(u, v as before)

$$0 \to (\mathcal{D} \otimes_{\mathbb{C}} E) \oplus (\mathcal{D} \otimes_{\mathbb{C}} F) \xrightarrow{P} (\mathcal{D} \otimes_{\mathbb{C}} E) \oplus (\mathcal{D} \otimes_{\mathbb{C}} F) \to M \to 0 \ ; \quad P = \begin{pmatrix} \partial \otimes \mathrm{id}_E & -(1 \otimes v) \\ -(1 \otimes u) & z_1 \otimes \mathrm{id}_F \end{pmatrix}$$

$E = \bigoplus_{0 \le \mathrm{Re}\lambda < 1} \overline{M}_\lambda$, $F = \bigoplus_{-1 \le \mathrm{Re}\lambda < 0} \overline{M}_\lambda$.

$\overline{M} = \bigoplus_{\lambda \in \mathbb{C}} \overline{M}_\lambda$; $V^k \overline{M} = \bigoplus_{\mathrm{Re}\lambda \ge k} \overline{M}_\lambda$.

* $\overline{M}$ is the $\mathbb{C}[z_1, \partial]$ - module generated by E & F with relations

$u = \partial \times$

$v = t \times$

Thm: Let $M \in \mathcal{C}$; $DR(M)$ = perverse shf on Δ w/ singularities at 0.

$$\begin{tikzcd}
\Psi_{z_1}(DR(M)) = Gr_V^{0} M = E \arrow[d, bend right, "can"'] \arrow[d, bend left, "u"] \\
\Phi_{z_1}(DR(M)) = Gr_V^{-1} M = F \arrow[u, bend right, "var"'] \arrow[u, bend right, "v"']
\end{tikzcd}$$

Then: $can = u$,

$var = \varphi(vu)\,v$; $\varphi(z_1) = \dfrac{e^{-2\pi i z_1} - 1}{z_1}$.

Rmk: $u\,\varphi(vu)\,v = (uv)\,\varphi(vu) = \varphi(uv)\cdot uv$.

$$\underset{(T - id)|_{\Psi}}{\underset{\shortparallel}{var \circ can}} = \varphi(vu)\,vu = \exp(-2\pi i\, vu) - I$$

$$T|_{\Phi(DR(M))} = \exp(-2\pi i\, uv).$$

Prop: Let $M \in \mathcal{C}$.

(1) u is onto iff M has no quotient supported at 0.

(2) v is injective iff M has no submodule supported at 0.

(3) $F = \mathrm{Im}(u) \oplus \ker(v)$ iff M is support-decomposable.

i.e.: $M = M' \oplus M''$

M' has no quotient/sub supported at 0.

M'' supported at 0

$\mathcal{C}$ abelian category of finite length. [$\mathcal{C}^* = \mathcal{D}$-module on Δ^* w/o singularities]

* Simple objects:

$\mathcal{C}^*$: $\mathcal{D}_U / \mathcal{D}_U(t\partial - \lambda)$ on $U = \Delta^*$

$$\updownarrow$$

$\mathbb{C}_\lambda := (\mathbb{C}, \exp(-2\pi i \lambda))$

$\forall \lambda \notin \mathbb{Z}$

$\mathcal{C}$: $\left[\mathcal{D}_X / \mathcal{D}_X(t\partial - 1)\right] \longleftrightarrow j_! \mathbb{C}_\lambda = j_* \mathbb{C}_\lambda = j_{!*} \mathbb{C}_\lambda$

& $\delta = \mathcal{D}_X / \mathcal{D}_X t \longleftrightarrow i_* \underline{\mathbb{C}}_0$

& $\mathcal{D}_X / \mathcal{D}_X \partial \longleftrightarrow \underline{\mathbb{C}}_\Delta$

Blocks of $\mathcal{C}$:

Equivalence relation on simple objects:

$M \sim N$ if $\mathrm{Ext}^1_{\mathcal{C}}(M,N) \neq 0$ or $\mathrm{Ext}^1_{\mathcal{C}}(N,M) \neq 0$, or $\exists$ chain of such.

R = equivalence classes. $\forall\, r \in R$, $\mathcal{C}_r :=$ {subcategory consisting of $M \in \mathcal{C}$ whose simple composition factors $\in r$}

Claim: $\mathcal{C} = \bigoplus \mathcal{C}_r$

→ In our case, $\mathcal{D}_X/\mathcal{D}_X t$ & $\mathcal{D}_X/\mathcal{D}_X \partial$ lie in the same block.

$0 \to \mathcal{O}_\Delta \to \mathcal{O}_\Delta(*0) \to \delta \to 0$. Also $\exists$ $0 \to \delta \to (\ \ ) \to \mathcal{O}_\Delta \to 0$.

$\mathcal{O}_\Delta(*0) \wr\ j_*(\mathbb{C}_{\Delta^*})$ $\qquad$ $(\ \ ) \wr\ j_!(\mathbb{C}_{\Delta^*})$