# SOME EXAMPLES

W30 2013

$E$ elliptic curve. Fix a basis $\delta, \gamma$ for $H_1(E, \mathbb{Z})$ s.t. $\delta \cdot \gamma = 1$

$\delta^*, \gamma^*$ dual basis in $H^1(E, \mathbb{Z})$

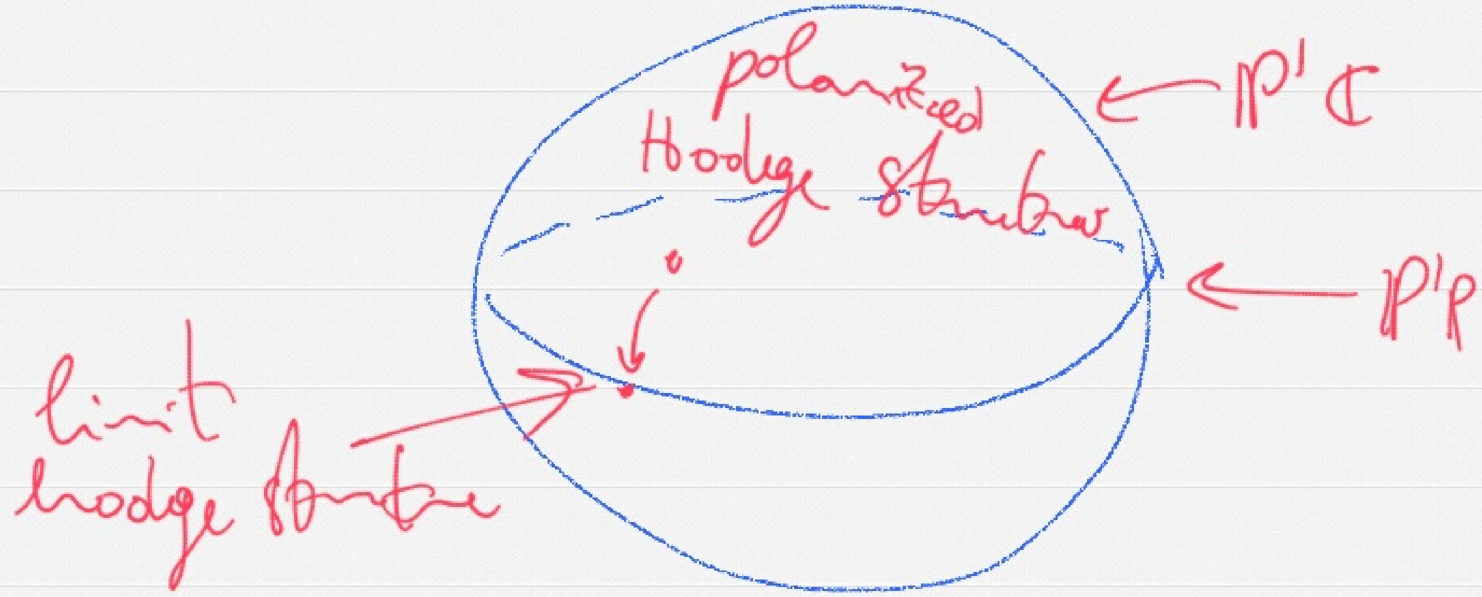
Given $j \in H^1_{DR}(E)$ $\quad j = \left(\int_\delta j\right)\delta^* + \left(\int_\gamma j\right)\gamma^*$

If $\omega$ is a hol. 1-form $\quad F^1 =$ line in $H_{\mathbb{C}}$ generated by $\left(\int_\delta \omega\right)\delta^* + \left(\int_\gamma \omega\right)\gamma^*$

Hodge structures on $H$ is determined by $F^1 \subset H_{\mathbb{C}}$ : Hodge structures on $H \hookrightarrow \mathbb{P}^1\mathbb{C}$

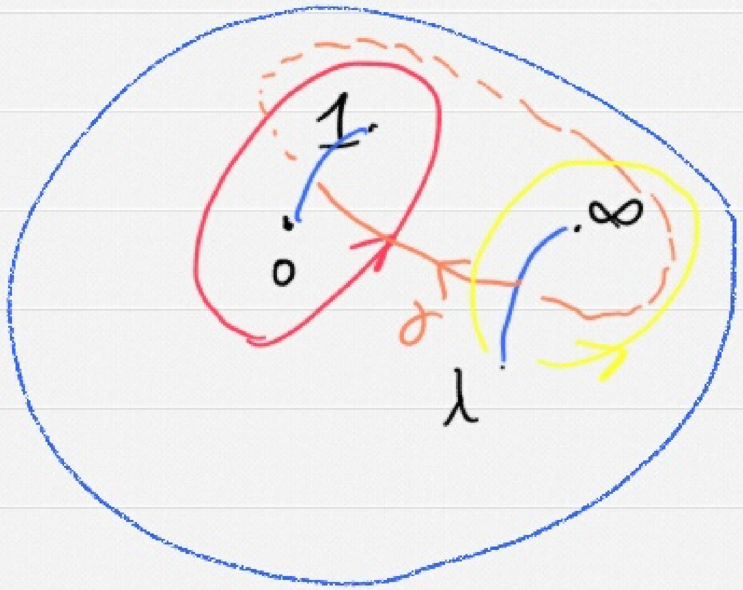
Hodge structure $= \{ F_1 \mid \overline{F_1} \neq F_1 \} = \mathbb{P}^1\mathbb{C} \setminus \mathbb{P}^1\mathbb{R}$

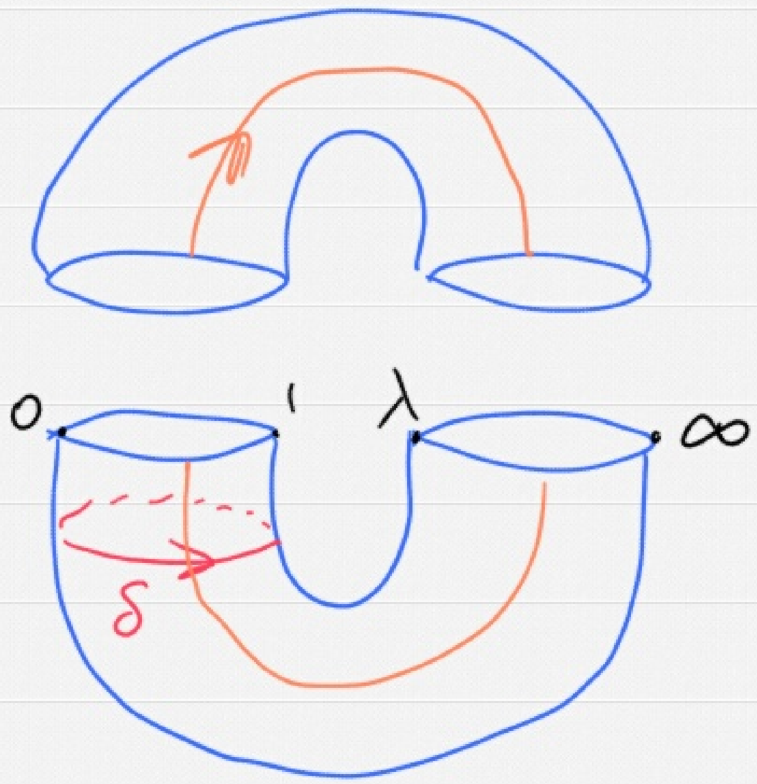
Expand $\quad i\int \omega \wedge \overline{\omega} = 2\int |f(z)|^2 \, dx \wedge dy > 0 \Rightarrow \omega = A\delta^* + B\gamma^* \Rightarrow \operatorname{Im}\left(\frac{B}{A}\right) > 0$



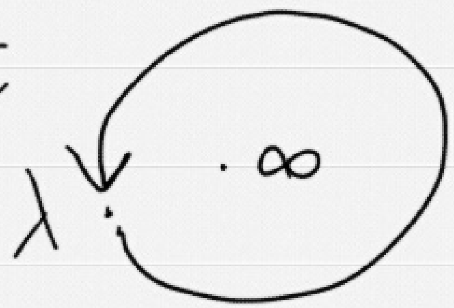
We calculate in an example. $\{y^2 = x(x-1)(x-\lambda)\}$ family of ell. curves.

2:1 ramified cover of $\mathbb{P}^1\mathbb{C}$



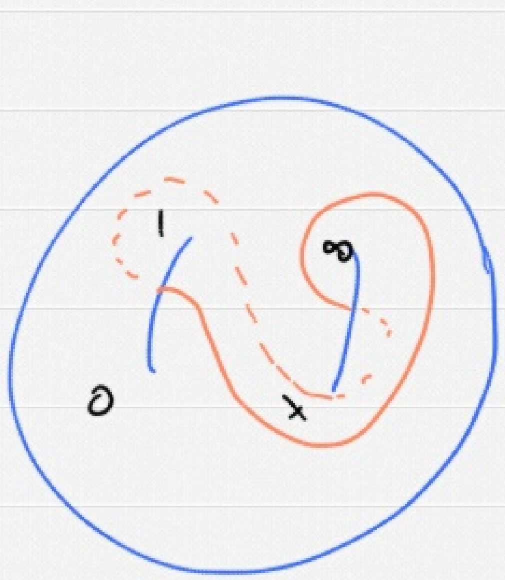



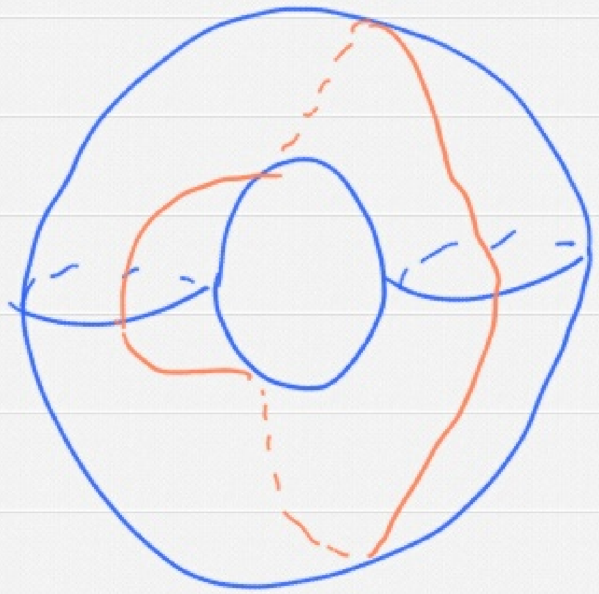

Purely topological fact




induces $\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$ on $H_1(E,\mathbb{Z})$

EXPLANATION



DOUBLE DEHN TWIST

**Nilpotent orbit thm** Asymptotic behaviour of period map is determined by local (topological) monodromy

As. of period map $\qquad \omega = \frac{dx}{y}$ spans $F^1 \subset H^1_{\mathbb{C}}$

we want to understand the behaviour of the line $F^1$ as $\lambda \to \infty$

$$\int_\gamma \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} \underset{\text{for } \lambda \text{ large}}{\sim} \int \frac{dx}{x\sqrt{-\lambda}} = \frac{2\pi i}{i\sqrt{\lambda}} = \frac{2\pi}{\sqrt{\lambda}}$$

**Trickier** $\displaystyle \int_\gamma \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} = \frac{2i}{\sqrt{\lambda}} \log \lambda +$ terms hol. in $\lambda^{-1}$

Hence $\underset{\underset{\text{space of Hodge structure}}{\mathbb{P}^1\mathbb{C}\ \cap}}{\tau(\lambda)} = \dfrac{\frac{2i}{\lambda}\log(\lambda)}{\frac{2\pi}{\sqrt{\lambda}}} + \ldots = \underbrace{\frac{i}{\pi}\log\lambda}_{\text{"nilpotent orbit"}} + \left(\begin{array}{c}\text{terms hol.}\\ \text{at } \infty\end{array}\right)$

Rewrite with $\mu^{-1} = \lambda$ $\quad \tau(\mu) = \frac{1}{\pi i}\log\mu + \underset{\text{in } \mu}{\text{holomorphic}}$

On $H^1$ $\quad T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad N = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{\pi i}\log z & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{\pi i}\log z + \alpha \end{pmatrix}$$

$\underline{\text{Schmid's N.O.T.}}$ $\tau(\mu)$ is asymptotic to $z \mapsto \exp\left(\frac{1}{2\pi i}(\log z)N\right)\cdot\begin{pmatrix} 1 \\ \alpha \end{pmatrix} \in \mathbb{P}^1\mathbb{R}$

(or motivated hol. fcts) $\qquad$ (or multivalued hol. fct)

What is the limit Hodge structure?

The weight filt. is determined by $N$

$$0 = W_{-2} \subseteq W_{-1} = \ker N = \mathbb{Q}\gamma^* \qquad \text{Here } \overline{F_1} = F_1$$

$$W_{-1} = W_0 \subseteq W_1 = H^1 = \mathbb{Q}\delta^* \oplus \mathbb{Q}\gamma^* \qquad F_1 = \delta^* + \alpha\gamma^*$$

# DIRECT IMAGE THM FOR CURVES

KAISER 24/06

$\overline{S}$ compact RS. $S \overset{j}{\hookrightarrow} \overline{S}$, $\# \overline{S} \setminus S < \infty$

$\mathcal{H}_{\mathbb{Z}}$ $\mathbb{Z}$-local system on $S$ + PVHS$(\mathcal{V}, \nabla, \dots)$ of wt $m$ on $\mathcal{H} = \mathcal{H}_{\mathbb{Z}} \otimes \mathbb{C}$

Thm $H^1(\overline{S}, j_* \mathcal{H})$ has a hodge structure pure of wt $m+1$

Local theory $z \in \overline{S} \setminus S$: $0 \in \mathbb{D} = \{ t \in \mathbb{C} \mid |t| < 1 \}$ $(\mathcal{V}, \nabla, \dots)$ PVHS on $\dot{\mathbb{D}}$ of wt $m$.

Meromorphic Deligne ext: $\widetilde{\mathcal{V}} \subseteq j_* \mathcal{V}$

finite free $\mathcal{O}_{\mathbb{D}}[t^{-1}]$-module, reg. hol. $\mathcal{D}_{\mathbb{D}}$-mod with $DR(\widetilde{\mathcal{V}}) \cong Rj_* \mathcal{H}$

Note: $j_* \mathcal{H} \subseteq Rj_* \mathcal{H}$ is the smallest perverse subsheaf wich is $\cong$ over $\mathbb{D}^*$

$\rightsquigarrow j_* \mathcal{H} = DR(\mathcal{D}_{\mathbb{D}} \widetilde{\mathcal{V}}^{>-1}) =: DR(\widetilde{\mathcal{V}}_{min})$

Rmk Borel $\rightsquigarrow$ Monodromy on $\mathcal{H}$ is quasi-unipotent $\rightsquigarrow \exists$ refined KM-filt. $\widetilde{\mathcal{V}}$ with $\cdot \in \mathbb{Q}$

$$\begin{array}{c} \psi_t = \widetilde{\mathcal{V}}^0 / t \widetilde{\mathcal{V}}^0 \\ \partial_t \downarrow \; \uparrow t \\ \phi_t = \widetilde{\mathcal{V}}^1 / t \widetilde{\mathcal{V}}^1 \end{array} \rightsquigarrow j_* \mathcal{H} = DR\left( \left[ \begin{array}{c} \psi_t \\ \downarrow \uparrow \\ \mathrm{im}\, \partial_t \end{array} \right] \right)$$