Simons Centre for Geometry and Physics
Geometric and modular representation theory of algebraic groups
August 2019

Tuesday Exercises

You should do Exercise 1 at least once in your life …

1. Calculate the root data of:

a) SL_2 and PGL_2;

b) GL_n, SL_n and PGL_n;

c) Sp_{2n}, SO_{2n} and SO_{2n+1}.

Identify the Langlands dual in each case.

2. Suppose that G is SL_2, and let ϖ denote the fundamental weight, i.e. $\langle \alpha^\vee, \varpi \rangle = 1$ where α is the positive root. Check that $\mathcal{O}(n\varpi)$ (defined in lectures) agrees with the invertible sheaf $\mathcal{O}(n)$ on $\mathbb{P}^1$.

3. Recall that a group scheme U is *unipotent* if it is isomorphic to a closed subgroup of the strictly upper triangular matrices U_n in GL_n. Show that a unipotent group has the trivial module as its unique simple module. Deduce that if U is unipotent and V is a U-module, then $V^U \neq 0$ if $V \neq 0$. (*Hint:* You might like to show this first for $U = U_n$ via an analysis of $k[U_n]$.)

4. Recall (a slight variant of) the Bruhat decomposition

$$G/B = \bigsqcup_{x \in W} B^+ \cdot xB/B$$

where each $B^+ \cdot xB/B$ is isomorphic $\mathbb{A}^{\ell(w_0) - \ell(x)}$, where W is the Weyl group and $w_0 \in W$ is its longest element.

a) Use the Bruhat decomposition to determine the Picard group of G/B. (*Hint:* You only need to know that $B^+ \cdot B/B \cong \mathbb{A}^{\ell(w_0)}$ and that its complement is a divisor, with $|S|$ many components, where S denotes the set of simple reflections in W.)

b) Let $\mathcal{O}(\lambda)$ denote the equivariant line bundle on G/B considered in the lecture. Determine its class in G/B in the description you found in (a).

c) All equivariant line bundles on G/B are of the form $\mathcal{O}(\lambda)$ for some $\lambda \in \mathscr{X}$. Use this fact to determine when a line bundle on G/B admits an equivariant lift, in terms of the root datum of G.