WEDNESDAY - 1

Tensor products

Suppose for a moment that G is a finite group.

$$N \hookrightarrow G \twoheadrightarrow G/N.$$

Clifford theory: V simple G-module, then $V|_N$ is semi-simple, and all irreducible summands are G-conjugate.

Assume: All simple N-modules extend to G. Then $V^g \cong V$ for all irreducible simple N-modules.

Moreover if $V' \subset V|_N$ is an irreducible summand, then

$$\mathrm{Hom}_N(V', V) \otimes V' \xrightarrow{\sim} V$$

is an iso of G-modules, and $\mathrm{Hom}_{N}(V', V)$ is a simple G/N-module.

"All simple G-modules are $\otimes$-products"!

Back to reductive groups:

$$G_1 \hookrightarrow G \xrightarrow{Fr} G^{(1)} \cong G, \quad \text{note } L_\lambda^{(1)} = L_{p\lambda}.$$

A weight $\lambda \in X_+$ is p-restricted if $\langle \alpha^\vee, \lambda \rangle < p$ for all simple roots α. $X^{<p}$

Thm: If $\lambda \in X^{<p}$ then $L_\lambda|_{G_1}$ is simple, moreover all simple G-modules occur in this way.

(E.g. for Sl_2, $L_0, \dots, L_{p-1}$ are all simple over $\mathfrak{g}$.)

Assume G semi-simple, simply-connected, (FROM NOW ON)

Thm: All simple G-modules are of the form $L_\lambda \otimes L_\mu^{(1)}$, $\lambda \in X^{<p}$.

Steinberg $\otimes$ Thm: Write $\lambda = \lambda_0 + p\lambda_1 + \dots + p^m \lambda_m$, $\lambda_i \in X^{<p}$.

Then $L_\lambda \cong L_{\lambda_0} \otimes L_{\lambda_1}^{(1)} \otimes \dots \otimes L_{\lambda_m}^{(m)}$

Example:

This gives a complete answer for Sl_2.

Let $Fr : \mathbb{Z}[\mathcal{X}] \to \mathbb{Z}[\mathcal{X}]$ be defined via $e^\lambda \mapsto e^{p\lambda}$.

Write $n = \sum_{i \geq 0} \lambda_i p^i$ with $0 \leq \lambda_i < p$. Then

$$ch\, L_n = \prod_{i \geq 0} ch\left(L_{\lambda_i}^{(Fr)^i}\right) = \prod_{i \geq 0} \left(e^{-\lambda_i} + e^{-\lambda_i + 2} + \dots + e^{\lambda_i}\right)^{(Fr^i)}.$$

E.g. $\quad p^i - 1 = (p-1) + (p-1)p + (p-1)p^2 + \dots + (p-1)p^{i-1}$

$$\underset{ch\, L_{(p^i-1)}}{\overset{St_{p^i-1}}{\parallel}} = \left(\frac{e^p - e^{-p}}{e - e^{-1}}\right) \cdot \left(\frac{e^p - e^{-p}}{e - e^{-1}}\right)^{Fr} \cdot \dots \cdot \left(\frac{e^p - e^{-p}}{e - e^{-1}}\right)^{(Fr)^{i-1}}$$

$$= \left(\frac{e^{p^i} - e^{-p^i}}{e - e^{-1}}\right) = ch\, \nabla_{p^i - 1}$$

Rmk: This reduces character to question to <u>finite set</u> $\mathcal{X}_1^{<p}$.

However Sl_2 is the only group for which all L_λ for $\lambda \in \mathcal{X}_1^{<p}$ are irreducible.

<u>Kazhdan-Lusztig conjecture</u>

Our goal now is to state the Lusztig conjecture on the characters of L_λ. However, to do so we first revisit the Kazhdan-Lusztig conjecture, which was a major motivator for LC.

$\mathfrak{g}$ complex semi-simple Lie algebra $/ \mathbb{C}$.

$\mathfrak{h} \subset \mathfrak{b}^+ \subset \mathfrak{g}$ Cartan subalgebra, Borel subgroup, $\mathfrak{n}^+ \subset \mathfrak{b}^+$ nilpotent radical.

$\begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \subset \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \subset \mathfrak{gl}_n$

$$Hom(\mathfrak{b}^+, \mathbb{C}) = Hom(\mathfrak{b}^+/[\mathfrak{b}^+, \mathfrak{b}^+], \mathbb{C}) = \mathfrak{h}^*.$$

$\lambda \in \mathfrak{h}^* \rightsquigarrow \Delta_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} \mathbb{C}_\lambda$ free over $U(\mathfrak{n}^-)$ (easy formal character).

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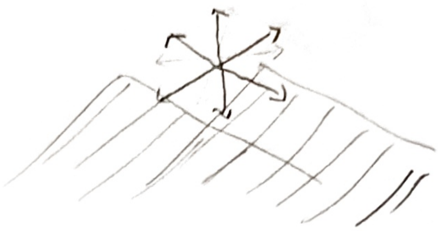


$$(\text{BGG})\ \mathcal{O} = \left\{ \begin{array}{l} \mathfrak{h}\text{-diagonalisable} \\ \mathfrak{b}^+\text{-locally-finite} \\ \mathfrak{g}\text{-finitely generated} \end{array} \right\} \subset \mathfrak{g}\text{-mod}.$$

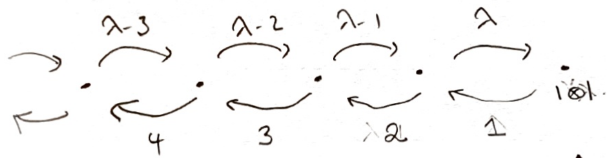
E.g. Δ_λ and finite dim. simples lie in $\mathcal{O}$.

Δ_λ has a unique simple quotient L_λ.

$$\{\text{simples in } \mathcal{O}\}_{/\cong} \longleftrightarrow \mathfrak{h}^*$$

$$L_\lambda \longleftrightarrow \lambda$$

Example $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$.



$\lambda \notin \mathbb{Z}_{\geq 0}$, Δ_λ is simple.

$\lambda \in \mathbb{Z}_{\geq 0}$

$$\Delta_{-\lambda-2} = L_{-\lambda-2} \hookrightarrow \Delta_\lambda \twoheadrightarrow L_\lambda$$

Dot action: $x \cdot \lambda = x(\lambda + \rho) - \rho$.

Harish-Chandra isomorphism: $Z(U(\mathfrak{g})) \xrightarrow{\sim} S(\mathfrak{h})^{(W\cdot)} = \mathcal{O}(\mathfrak{h}^*/(W\cdot))$.

$$\rightsquigarrow \quad \mathcal{O} = \bigoplus_{\lambda \in \mathfrak{h}^*/(W\cdot)} \mathcal{O}_\lambda$$

subcategory of modules w/ generalised central char. λ

Enough to understand $\mathcal{O}_0$ ("principal block")

$L_x := L_{xw_0 \cdot 0}$

$\Delta_x := \Delta_{xw_0 \cdot 0}$

Kazhdan-Lusztig conjecture:

$$[L_x] = \sum (-1)^{\ell(x)-\ell(y)} P_{y,x}(1)\,[\Delta_y].$$

Kazhdan-Lusztig polynomial

Compare w/ the last mat, on G/B :

$$[IC_x] = \sum (-1)^{\ell(x)-\ell(y)} P_{y,x}(1)\,[\Delta_y^{geom}]$$

$$\Delta_y^{geom} = j_{y!}\,\underline{\mathbb{C}}_{ByB/B}(\ell(y))$$

Thm (Beilinson-Bernstein)

$$\begin{tikzcd}
U(\mathfrak{g})/(Z^+)\text{-mod} & \mathcal{D}_{G/B}\text{-mod} \arrow[l, "\sim"'] \arrow[r, "RH"] & {} \\
& \mathcal{D}_{G/B}\text{-mod}^{hol,\,r.s.} \arrow[u, phantom, "\cup"] \arrow[r, leftrightarrow, "\overset{\text{Riemann-}}{\sim}"', "\text{Hilbert}"] & \mathrm{Perv}(G/B,\mathbb{C})
\end{tikzcd}$$

$\Delta_y^{geom} \in U(\mathfrak{g})/(Z^+)\text{-mod}$, $\Delta_y \in$; $L_x \in U(\mathfrak{g})/(Z^+)\text{-mod}$, $IC_x \in \mathrm{Perv}(G/B,\mathbb{C})$

reg. holonomic

$\Rightarrow$ KL conj.

Distributions

X affine scheme.

$X \in \mathrm{End}(k[X])$ is a differential operator of degree $\leq n$ if $\mathrm{Diff}_{\leq n}(k[X])$

1. $X = f\cdot(\ )$ for $n=0$ and some $f \in k[X]$
2. $[X, f]$ is a diff. operator of degree $\leq (n-1)$ for all $f \in k[X]$.

E.g. 1. $\mathrm{Diff}\,\mathbb{A}^1_{\mathbb{Z}}$ = has a $k[z]$-basis given by $\frac{1}{n!}\frac{\partial^n}{\partial z^n}$.

Dist G = left-invariant differential operators on G.

Eg. 2. $z\frac{\partial}{\partial z} \in \mathrm{Dist}\,\mathbb{G}_m$, has a basis $\binom{z\frac{\partial}{\partial z}}{p}$, $p \geq 0$.

G_k semi-simple $\rightsquigarrow G_{\mathbb{Z}} \rightsquigarrow G_{\mathbb{C}} \rightsquigarrow \mathfrak{g}_{\mathbb{C}} = \langle f_\alpha, h_\alpha, e_\alpha \rangle$

Kostant $\mathbb{Z}$-form: $\left\langle \frac{f_\alpha^p}{p!}, \binom{h_\alpha}{q}, \frac{e_\alpha^r}{r!} \right\rangle \subset U(\mathfrak{g}_{\mathbb{C}})$.

$\mathrm{Dist}\,G_{\mathbb{Z}} = U_{\mathbb{Z}}$, $\quad \mathrm{Dist}\,G_k = U_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$.

Affine Weyl group and p-dilated dot action

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$W = W_f \ltimes \mathbb{Z}R$ affine Weyl group (of dual root system).

p-dilated dot action:

$W \underset{\dot{p}}{\curvearrowright} X_{\mathbb{R}}$

$x \in W_f$: $x \cdot_p \lambda = x(\lambda+\rho) - \rho$.

$t_\gamma \in \mathbb{Z}R$: $t_\gamma \cdot_p \lambda = \lambda + p\gamma$.

Exercise: $W \underset{\dot{p}}{\curvearrowright} X_{\mathbb{R}} = \left\langle \begin{array}{c} \text{reflections in the} \\ \text{hyperplanes } \langle \lambda+\rho, \alpha^\vee \rangle = pm \\ \text{for all } \alpha, m \end{array} \right\rangle \subset \mathrm{Aff}(X_{\mathbb{R}})$

SL_2:

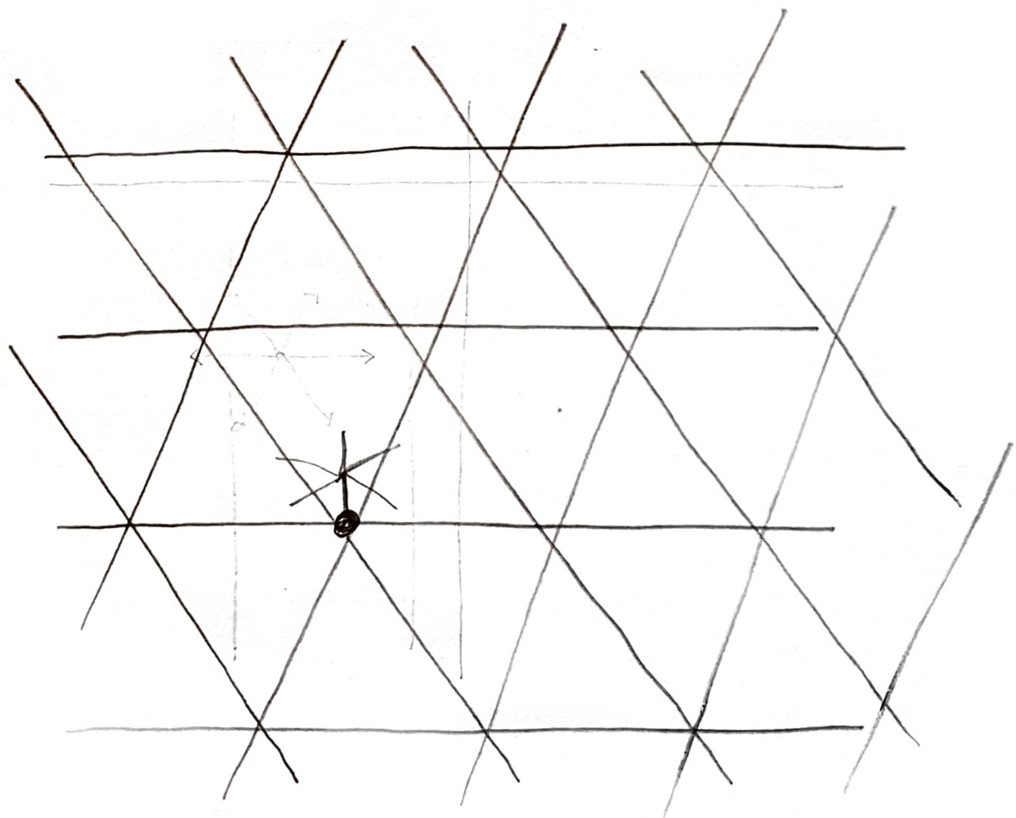
$-\rho = -1$

$p: 5$

SL_3:

$-\rho = -\alpha-\beta$

$p = 5$



Linkage principle: $\mathrm{Rep}\, G = \bigoplus_{\lambda \in X/(W\cdot_p)} \mathrm{Rep}_\lambda$

$\mathrm{Rep}_\lambda = \langle L_\gamma \mid \gamma \in (W \cdot_p \lambda) \cap X_+ \rangle.$

WEDNESDAY-6

Remark: Can be proved if $Z(G)$ is reduced and if p is good via consideration of $Z(\mathrm{Dist}_G)$. Proof in general is (Andersen) rather different.

reference? write to Simon

$$A_{fund} = \{ \lambda \in \mathfrak{X}_{\mathbb{R}} \mid 0 \le \langle \alpha^\vee, \lambda + \rho \rangle \le p \ \ \forall \alpha \in R_+ \} \quad (\text{fundamental alcove}).$$

is a fundamental domain for W_p° action.

If $p \geq h$ (Coxeter number), then $0 \in A_{fund}$ and is regular.

$$L_x := L_{x \cdot_p 0} \in \mathrm{Rep}_0 \quad \text{"principal block".}$$

Lusztig conjecture: Under certain assumptions on p (and ...)

$$[L_x] = \sum (-1)^{\ell(x)+\ell(y)} P_{w_0 y, w_0 x} [\Delta_y].$$

Key point: "independence of p".