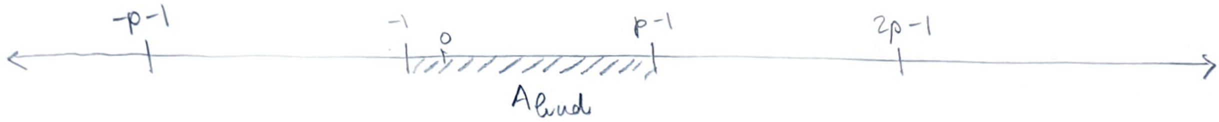
G semi-simple, simply-connected

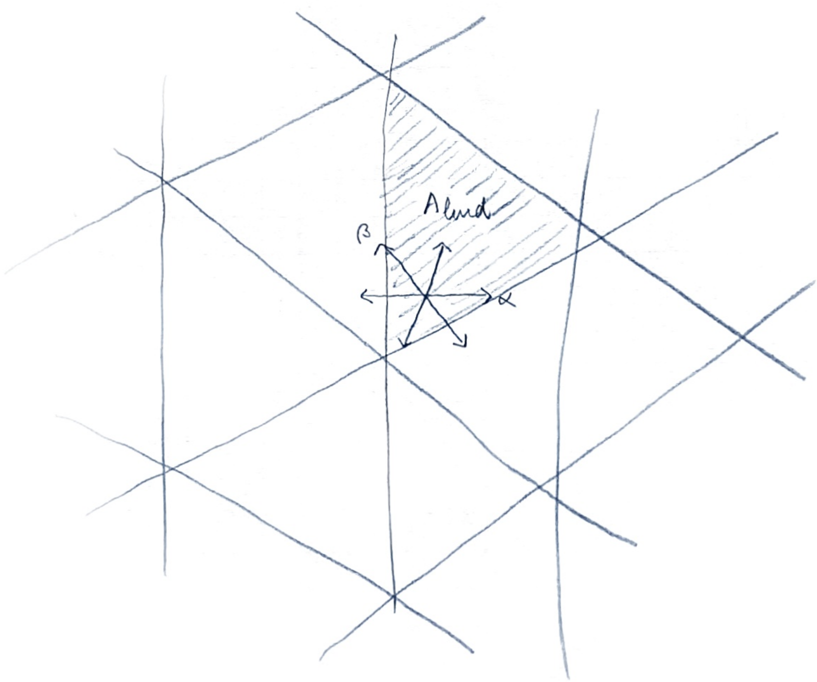
$W = W_f \ltimes \mathbb{Z}R$. affine Weyl group (of dual root system)

THURSDAY-1

$W = \left\langle \begin{array}{c} \text{reflections in} \\ \langle \alpha^\vee, \lambda+\rho \rangle = mp \\ \text{for } \alpha \in R,\ m\mathbb{Z} \end{array} \right\rangle$.

$A_{fund} = \{ \lambda \mid 0 \leq \langle \alpha^\vee, \lambda+\rho \rangle \leq p,\ \forall \alpha \in R^+ \}$.





W is a Coxeter system w/ $S =$ reflections in walls of A_{fund}.

Linkage principal

$$\operatorname{Rep} G = \bigoplus_{\lambda \in \mathfrak{X}/(W_{\dot p})} \operatorname{Rep}_\lambda$$

$\operatorname{Rep}_\lambda = \langle L_\gamma \mid \gamma \in (W \cdot_p \lambda) \cap \mathfrak{X}_+ \rangle$.

$\operatorname{Rep}_0 =$ "principal block."

Translation functors: most Q's about $\operatorname{Rep} G$ reduce to Rep_0.

Assume $p \geq h$ Coxeter number $\overset{\text{Exercises}}{\Longleftrightarrow}$ $0 \in A_{fund}$ and $\operatorname{Stab}_{(W\cdot)}(0) = \{1\}$.

E.g. $h = n$ for SL_n

$L_x := L_{x \cdot_p 0}$, $\nabla_y := \nabla_{y \cdot_p 0}$, $x \in {}^f W$

${}^f W$ = $\boxed{\text{minimal coset reps for } W_f \backslash W.}$

Lusztig Conjecture:

Suppose $p \geq h$ and $x \cdot_p 0$ has ≤ 2 p-adic digits. Then

$$\text{(LCF)} \quad [L_x] = \sum (-1)^{\ell(y)+\ell(x)} P_{w_0 y, w_0 x}(1) \, [\nabla_y].$$

Key point: "independent of p".

$\boxed{\text{Gives complete answer for all } \lambda \text{ in principal block by St} \otimes \text{Thm}}$

One p-adic digit would be enough.

History: Conjectured 1980, proved in mid 90's for $p \geq N$ (root system)

Kashiwara-Tanisaki, Kazhdan-Lusztig, Lusztig, Andersen-Jantzen-Soergel.

N: non-explicit, only depends on root system.

Mid 2000s: new proof by Bezrukavnikov et al.

Late 2000s: explicit but enormous N by Fiebig, $N = 10^{100}$ for GL_{10}.

2013: W help from He, Elias, Kontorovich, McNamara.

Not true for many $p \sim$ exponential function of n.

Example: SL_2.

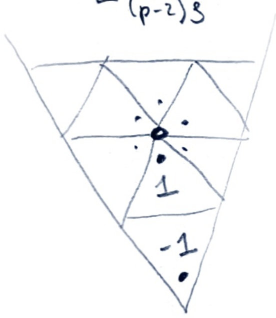
	0	8	10	18	20	28	30
L_0	1						
L_8	-1	1					
L_{10}	1	-1	1				
L_{18}	-1	1	-1	1			
L_{20}	1	-1	1	-1	1		
L_{28}	0	0	0	0	-1	1	
L_{30}				-1	1	-1	1

Region of validity of Lusztig's conjecture.

Exercise: Check that for $p \leq 20$ this is as predicted by Lusztig conjecture.

Example: SL_3

$L_{(p-2)s}$

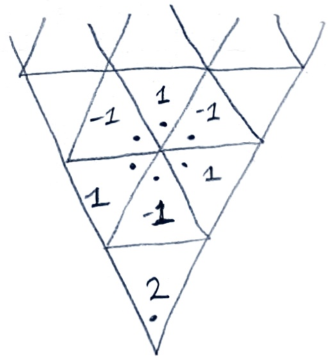


$\Updownarrow$

enough to conclude.

L_{ps}

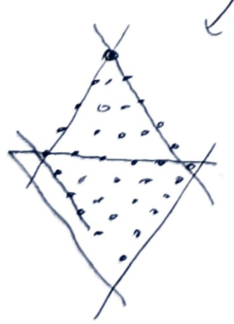
L_{ps}



(It seemed worthwhile to give the first example where 2's occur!)

Can be checked by Steinberg tensor product theorem.

Exercise!



discuss which cases follow etc.

Geometry:

G semi-simple, simply connected $\twoheadrightarrow G_{ad}$ adjoint group.

$\boxed{\begin{array}{ll} \text{Note:} & G \xrightarrow{Fr} G \to G_{ad} \\ \rightsquigarrow & \mathrm{Rep}\, G_{ad} \xrightarrow{(\,)^{Fr}} \mathrm{Rep}_0 \end{array}}$

$G^\vee$ dual group to over $\mathbb{C}$.

$\mathcal{K} = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$ $\qquad G^\vee(\mathcal{K}) \supset K := G^\vee(\mathcal{O})$ "maximal compact"

$$\begin{tikzcd} K \arrow[d, "ev"', "t=0"] & \supset & Iw := ev^{-1}(B^\vee) \arrow[d] \\ G^\vee & \supset & B^\vee \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{F}\ell = G^\vee(\mathcal{K})/Iw = \bigsqcup_{w \in W_a} Iw \cdot x\, Iw/Iw \arrow[d, "K/Iw = G^\vee/B^\vee \text{ bundle}"] & \text{affine flag variety} \\ \mathcal{G}r = G^\vee(\mathcal{K})/K = \bigsqcup_{w \in W/W_f} Iw \cdot x K/K & \text{affine Grassmannian.} \end{tikzcd}$$

Eg: For $G = SL_2$, $G^\vee = SL_2$, $\mathcal{G}r = \mathbb{C}^0 \sqcup \mathbb{C} \sqcup \mathbb{C}^2 \sqcup \mathbb{C}^3 \sqcup \dots$ but glueing maps are very complicated!

Finkelberg-Mirković conjecture: There exists an equivalence of abelian categories:

$$\begin{tikzcd} \mathrm{Rep}_0^{ext} \arrow[r, leftrightarrow, "\sim"] & \mathrm{Perv}_{(Iw)}(\mathcal{G}r; k) \\ \mathrm{Rep}\, G_{ad} \arrow[u, "(-)^{Fr}"] \arrow[r, "Sat", "\sim"'] & \mathrm{Perv}_{(G^\vee(\mathcal{O}))}(\mathcal{G}r; k) \arrow[u] \end{tikzcd}$$

$L_x \longleftrightarrow IC^k_{x^{-1}}$

$\nabla_x \longleftrightarrow \nabla^{geom}_{x^{-1}}$

Proof recently announced by Bezrukavnikov-Riche ... we assume the conjecture from now on.

$\boxed{\text{Assume FM conjecture from now on:}}$

We want to write:

$$[L_x] = \sum a_{y,x}[\nabla_y]$$

FM $\Downarrow$

$$[IC^k_{x^{-1}}] = \sum a_{y,x}[\nabla^{geom}_{y^{-1}}]$$

Taking Euler's characteristics of costalks at $y^{-1}\cdot Iw/Iw$:

$$\chi\left((IC^k_{x^{-1}})^!_{y^{-1}}\right) = (-1)^y a_{y,x}.$$

$$IC^{\mathbb{Z}}_{x^{-1}} \xrightarrow{(\ )\otimes_{\mathbb{Z}}\mathbb{Q}} IC^{\mathbb{Q}}_{x^{-1}}$$

There exists a complex of sheaves $IC^{\mathbb{Z}}_{x^{-1}}$ ("integral IC complex")

$$IC^{\mathbb{Z}}_{x^{-1}} \xrightarrow{(\ )\overset{L}{\otimes}_{\mathbb{Z}} k} \text{a perverse sheaf,}$$

ASSUME THIS HOLDS $\to$ $\cong IC^k_{x^{-1}}$ if certain tors vanish

???

$$(-1)^y a_{y,x} = \chi\left((IC^k_{x^{-1}})^!_{y^{-1}}\right) = \chi\left((IC^{\mathbb{Z}}_{x^{-1}}\overset{L}{\otimes}\mathbb{Q})^!_{y^{-1}}\right)$$

$$= (-1)^{\ell(x)} P_{y^{-1}w_0,\, x^{-1}w_0}(1)$$

$$\rightsquigarrow \quad a_{y,x} = (-1)^{\ell(x)+\ell(y)} P_{w_0y,\, w_0x}(1).$$

__Conclusion:__ Lusztig conjecture holds $\Longleftarrow$ $IC^{\mathbb{Z}}_{x^{-1}}\overset{L}{\otimes} k$ stays simple, for all $x \in {}^fW$ s.t. $x\cdot_p 0$ has 2-p-adic digits.

Induction implies that this is $\Longleftrightarrow$.