# Integrable Dynamic Systems with $N$-Peakon

Bo Xue

Zhengzhou University, P.R.China

(Joint work with Xianguo Geng, et al.)

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# Outline

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1. Introduction
2. Integrable Dynamic Systems with $N$-Peakon

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# I. Introduction

# Introduction

## Background

- In the past few decades, Soliton and Integrable System, Chaos, Fractals have become three important parts of Nonlinear Science.
- In nonlinear dynamic systems, solitons are caused by a cancellation of nonlinear and dispersive effects in the medium.
- Solitons, which can be discovered in a number of important natural phenomena and practical applications, have been of great research interest.

# Introduction

Korteweg-de Vries (KdV) equation:

$$u_t + 6uu_x + u_{xxx} = 0, \quad u(0, x) = u_0(x).$$

- Integrable, soliton solution, connections to physical problems.
- When $u_0(x) \in H^1(\mathbb{R})$, the solutions are global.
- KdV equation can not describe breaking of waves.

# Introduction

"Breaking phenomena is one of the most intriguing long-standing problems of water wave theory."

——— G.B. Whitham, 《Linear and Nonlinear Waves》

It is intriguing to know what mathematical models for shallow water waves could include both the phenomena of wave breaking and soliton interaction.

# Introduction

Since 1993, Camassa-Holm (CH) equation:

(Camassa and Holm, *Phys. Rev. Lett.* **71** (1993) 1661 – 1664)

$$m_t + qm_x + 2q_x m = 0, \quad m = q - q_{xx} + \kappa$$

has become new master equation for shallow water wave theory.

◆ CH equation is the first integrable model for shallow water waves which include both the phenomena of wave breaking and soliton interaction.

◆ Different from the soliton solution of KdV, CH supports peaked soliton solution, dubbed "peakon".

# Introduction

## Definition

We say that a continuous function $u(x,t)$ has a peak at $x_0$ if $u$ is smooth locally on either side of $x_0$ and

$$0 \neq \lim_{x \uparrow x_0} u_x(x,t) = -\lim_{x \downarrow x_0} u_x(x,t) \neq \pm\infty.$$

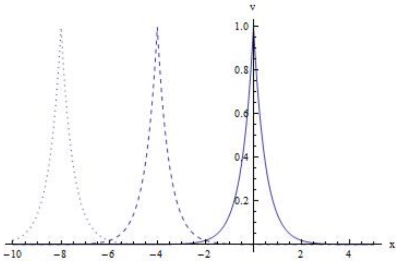
Wave profiles with peaks are called peaked waves or peakons.

## Example

Suppose $\quad u(x,t) = m(t)e^{-|x-x_0|}, \quad$ then

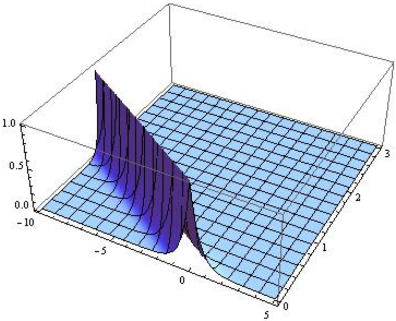
$$u_x(x,t) = \begin{cases} m(t)e^{x-x_0}, & x < x_0, \\ -m(t)e^{x_0-x}, & x > x_0. \end{cases}$$

# Introduction

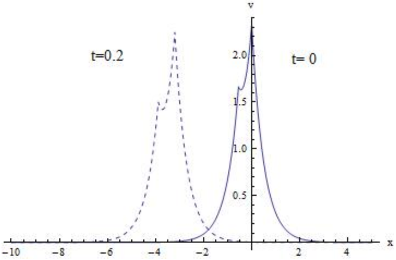



← Caption of Fig 1

Caption of Fig 2 →

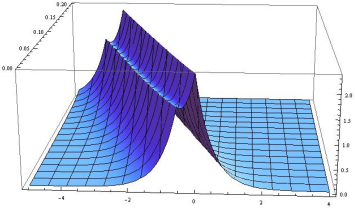


# Introduction




← Caption of Fig 3

Caption of Fig 4 →



# Introduction

Camassa and Holm proposed solutions of the form:

$$u(x,t) = \frac{1}{2}\sum_{j=1}^{N} m_j(t)e^{-2|x-x_j(t)|},$$

that are continuous but only piecewise analytic, the positions $\{x_j\}$ and momenta $\{m_j\}$ evolve according to a Hamiltonian system of equations.

# Introduction

Since the derivation of peakon, it has attracted much attention in the fields of Mathematics and Physics:

- Fokas and Liu, *Phys. Rev. Lett.* **77** (1996) 2347 – 2351.
- Constantin and Escher, *Acta Math.* **181** (1998) 229-243.
- Beals, et al., *Inverse Problems* **15** (1999) L1 – L4.
- Li and Olver, *J. Differ. Equations* **162** (2000) 27-63.
- Beals, et al., *Comm. Pure Appl. Math.* **54** (2001) 91-106.
- Constantin, *Invent. Math.* **166** (2006) 523 – 535.
- Qu, Liu and Liu, *Comm. Math. Phys.* **322** (2013) 967 – 997.

# Introduction

- Why peakons are interesting?

(i) Peakons provide many subtle and challenging problems concerning existence, uniqueness, stability and breakdown of solutions for PDEs.

(ii) The mathematics of peakons is very beautiful in itself, with connection to classical topics such as Sturm-Liouville theory, orthogonal polynomials and generalizations thereof.

# II. Integrable Dynamic Systems with $N$-Peakon

# Integrable Dynamic Systems with $N$-Peakon

- Camassa-Holm equation

  (Camassa and Holm, *Phys. Rev. Lett.* **71** (1993) 1661-1664)

$$m_t + qm_x + 2q_x m = 0, \quad m = q - q_{xx} + \kappa. \tag{1}$$

- ◇ B. Fuchssteiner, A. Fokas (1981): derivation of CH equation;
- ◇ R. Camassa, D.D. Holm (1993): peaked soliton solutions;
- ◇ F. Cooper, H. Shepard (1994): solitons in CH equation;
- ◇ R. Beals, D.H. Satinger, J. Szmigielski (1999): multipeakons;
- ◇ A. Constantin and W.A. Strauss (2000): stability of peakons;
- ◇ J. Lenells (2005): traveling wave solutions of CH, etc.

# Integrable Dynamic Systems with $N$-Peakon

- Degasperis-Procesi equation

(Degasperis, Holm and Hone, *Theoret. Math. Phys.* **133** (2002) 1463 – 1474)

$$m_t + qm_x + 3q_x m = 0, \quad m = q - q_{xx}. \tag{2}$$

◇ A. Degasperis, M. Procesi (1999,2002): derivation of DP;
◇ H. Lundmark, J. Szmigielski (2005): multipeakons;
◇ G.M. Coclite, K.H. Karlsen (2005): well-posedness of DP;
◇ J. Lenells (2005): traveling wave solutions of DP;
◇ J. Escher, Y. Liu, Z.Y. Yin (2006): global weak solutions and blow-up structure for DP;
◇ Z.W. Lin, Y. Liu (2009): stability of peakons for DP, etc.

# Integrable Dynamic Systems with $N$-Peakon

- Novikov equation
  (Novikov, *J. Phys. A* **42** (2009) 342002)

$$m_t + q^2 m_x + 3qq_x m = 0, \quad m = q - q_{xx}. \tag{3}$$

◇ V. Novikov (2009): derivation of Novikov;
◇ A.N.W. Hone, J.P. Wang (2008): peakons;
◇ A. Hone, H. Lundmark, J. Szmigielski (2009): multipeakons;
◇ A.A. Himonas, C. Holliman (2012): Cauchy problem;
◇ X.C. Liu, Y. Liu, C.Z. Qu (2014): stability of peakons, etc.

# Integrable Dynamic Systems with $N$-Peakon

In contrast with the traditional soliton equations, there are only a few number of dynamic systems with $N-$peakon. Therefore, it is a important and innovative work to search for new dynamic models with $N-$peakon.

Since 2009, we have constructed some new peakon equations:

- Coupled peakon equation with cubic nonlinearity;
- 3-component generalization of CH equation;
- Super CH equation with N-peakon;
- Generalized CH equation with N-peakon;
- Coupled CH type equation with cubic nonlinearity.

## Coupled peakon equation with cubic nonlinearity

- Coupled peakon equation with cubic nonlinearity
  [*Nonlinearity*, **22** (2009) 1847-1856.]

$$\begin{aligned} & m_t + qrm_x + 3rq_x m = 0, \\ & n_t + qrn_x + 3qr_x n = 0, \\ & \text{with} \quad m = q - q_{xx}, \ n = r - r_{xx}. \end{aligned} \tag{4}$$

In Eq. (4),

$$\text{when} \quad r = 1, \quad (4) \Longrightarrow \text{DP equation (2);}$$

$$\text{when} \quad r = q, \quad (4) \Longrightarrow \text{Novikov equation (3).}$$

# Coupled peakon equation with cubic nonlinearity

Eq. (4) admits the following $N-$peakon:

$$q(x,t)=\sum_{j=1}^{N} q_j(t)e^{-|x-x_j(t)|}, \quad r(x,t)=\sum_{j=1}^{N} r_j(t)e^{-|x-x_j(t)|}.$$

If the derivative of $e^{-|x|}$ at $x=0$ is interpreted as being zero, then

$$m=2\sum_{j=1}^{N} q_j(t)\delta(x-x_j(t)), \quad n=2\sum_{j=1}^{N} r_j(t)\delta(x-x_j(t)).$$

Integrating against test functions at $x=x_j$, functions $q_j$, $r_j$ and $x_j$ evolve according to the following dynamic system:

$$\dot{q}_j = q_j\left(q(x_j)r_x(x_j)-2r(x_j)q_x(x_j)\right),$$

$$\dot{r}_j = r_j\left(r(x_j)q_x(x_j)-2q(x_j)r_x(x_j)\right),$$

$$\dot{x}_j = q(x_j)r(x_j),$$

## Coupled peakon equation with cubic nonlinearity

which is equivalent to

$$\dot{q}_j = q_j \sum_{k,\ l=1}^{N} q_k r_l \left(2\mathsf{sgn}(x_j - x_k) - \mathsf{sgn}(x_j - x_l)\right) e^{-|x_j - x_k| - |x_j - x_l|},$$

$$\dot{r}_j = r_j \sum_{k,\ l=1}^{N} q_k r_l \left(2\mathsf{sgn}(x_j - x_l) - \mathsf{sgn}(x_j - x_k)\right) e^{-|x_j - x_k| - |x_j - x_l|},$$

$$\dot{x}_j = \sum_{k,\ l=1}^{N} q_k r_l e^{-|x_j - x_k| - |x_j - x_l|}.$$

# Coupled peakon equation with cubic nonlinearity

- In 2011, Prof. Jacek Szmigielski gave a talk about their study on the coupled peakon equation (4) on "The 7th IMACS International Conference on Nonlinear Evolution Equations and Wave Phenomena" and named Eq. (4) as Geng-Xue equation. Based on the inverse spectral problem, Prof. Szmigielski derived the explicit expression of the multipeakon solution to Eq. (4).

- H. Lundmark and J. Szmigielski, An Inverse Spectral Problem Related to the Geng-Xue Two-Component Peakon Equation, *Memoirs of the American Mathematical Society* 244 (2016) No. 1155, X+87pp.

# Coupled peakon equation with cubic nonlinearity

Studies on the Geng-Xue equation (4):

- Bi-Hamiltonian Structure of Geng-Xue equation;
  (Q.P. Liu, et al., *Phys. Lett. A* **377** (2013) 257 – 261)
- The Cauchy problem for Geng-Xue equation in the critical Besov space;
  (Z.R. Liu, et al., *J. Math. Anal. Appl.* **423** (2015) 120 – 135)
- A reciprocal transformation for the Geng-Xue equation.
  (N.H. Li, et al., *J. Math. Phys.* **55** (2014) 053505)

# 3-component generalization of CH equation

- 3-component generalization of CH equation
  [*Advances in Mathematics*, **226** (2011) 827-839.]

$$\begin{aligned} u_t &= -vp_x + u_x q + \frac{3}{2}uq_x - \frac{3}{2}u(p_x r_x - pr), \\ v_t &= 2vq_x + v_x q, \\ w_t &= vr_x + w_x q + \frac{3}{2}wq_x + \frac{3}{2}w(p_x r_x - pr) \end{aligned} \tag{5}$$

$$\text{with} \quad u = p - p_{xx}, \quad w = r_{xx} - r,$$

$$v = \frac{1}{2}(q_{xx} - 4q + p_{xx}r_x - r_{xx}p_x + 3p_x r - 3pr_x).$$

In Eq. (5),

$$\text{when} \quad u = w = 0, \quad (5) \Longrightarrow \text{CH equation (1)}.$$

# 3-component generalization of CH equation

According to the structure of (5), we can deduce the $N$-peakon of this equation as

$$p(x,t)=\sum_{j=1}^{N}p_j(t)e^{-|x-x_j(t)|},\ q(x,t)=\sum_{j=1}^{N}q_j(t)e^{-2|x-x_j(t)|},$$

$$r(x,t)=\sum_{j=1}^{N}r_j(t)e^{-|x-x_j(t)|}. \tag{6}$$

Then the expressions of $u$, $v$ and $w$ can be written as:

$$\begin{aligned} u&=2\sum_{j=1}^{N}p_j(t)\delta(x-x_j(t)),\quad w=-2\sum_{j=1}^{N}r_j(t)\delta(x-x_j(t)),\\ v&=-2\sum_{j=1}^{N}q_j\delta(x-x_j)+\sum_{\substack{j<k\\k=2}}^{N}(p_jr_k-p_kr_j)e^{x_j-x_k}(x_k-x_j)\delta(x-x_j). \end{aligned} \tag{7}$$

# 3-component generalization of CH equation

Substituting (6) and (7) into (5) and integrating against test functions at $x = x_j(t)$, we can obtain that $p_j,\ q_j,\ r_j$ and $x_j$ evolve according to the following system:

$$\begin{aligned}
\dot{p}_j = & -\frac{1}{2}p_x(x_j)\sum_{\substack{j<k\\k=2}}^{N}(p_j r_k - p_k r_j)e^{x_j - x_k}(x_k - x_j) + q_j p_x(x_j) \\
& +\frac{3}{2}p_j\left(p(x_j)r(x_j) - p_x(x_j)r_x(x_j) + \frac{1}{3}q_x(x_j)\right), \\
\dot{r}_j = & -\frac{1}{2}r_x(x_j)\sum_{\substack{j<k\\k=2}}^{N}(p_j r_k - p_k r_j)e^{x_j - x_k}(x_k - x_j) + q_j r_x(x_j) \\
& +\frac{3}{2}r_j\left(p_x(x_j)r_x(x_j) - p(x_j)r(x_j) + \frac{1}{3}q_x(x_j)\right), \\
\dot{x}_j = & -q(x_j), \quad 1 \le j \le N,
\end{aligned}$$

# 3-component generalization of CH equation

$$
\begin{aligned}
\dot{q}_j = \quad & q_j q_x(x_j) - \frac{1}{2} q_x(x_j) \sum_{\substack{j<k \\ k=2}}^{N} (p_j r_k - p_k r_j) e^{x_j - x_k} (x_k - x_j) \\
& + \frac{1}{2} \sum_{\substack{j<k \\ k=2}}^{N} (\dot{p}_j r_k + p_j \dot{r}_k - \dot{p}_k r_j - p_k \dot{r}_j) e^{x_j - x_k} (x_k - x_j), \\
& + \frac{1}{2} \sum_{\substack{j<k \\ k=2}}^{N} (p_j r_k - p_k r_j) e^{x_j - x_k} (\dot{x}_j - \dot{x}_k)(x_k - x_j) \\
& + \frac{1}{2} \sum_{\substack{j<k \\ k=2}}^{N} (p_j r_k - p_k r_j) e^{x_j - x_k} (\dot{x}_k - \dot{x}_j).
\end{aligned}
$$

# Super CH equation with N-peakon

- Super CH equation with N-peakon
  [*Studies in Applied Mathematics*, **130** (2013) 1-16.]

$$\begin{aligned}
&u_t = -(\partial u + u\partial)w,\\
&2\alpha_t = u\sigma_x - (\alpha\partial + 2\partial\alpha)w,\\
&\text{with}\quad u = w_{xx} - w + \frac{3}{4}\sigma\sigma_x + \sigma_x\sigma_{xx},\quad \alpha = \frac{1}{4}\sigma - \sigma_{xx}.
\end{aligned}\tag{8}$$

where $u$, $w$ are the commuting variables which can be indicated by the degree (mod 2) $p$ as $p(u) = p(w) = 0$; $\alpha$ and $\sigma$ are the anticommuting variables which can be indicated by $p$ as $p(\alpha) = p(\sigma) = 1$.

$$\text{When } \sigma = 0, \quad (8) \Longrightarrow \text{CH equation } (1),$$

# Super CH equation with N-peakon

Based on the structure of the super CH equation (8), a direct calculation shows that it possesses $N$-peakon:

$$\begin{aligned}\sigma(x,t) &= \sum_{j=1}^{N} \sigma_j(t) e^{-\frac{1}{2}|x-x_j(t)|},\\ w(x,t) &= \sum_{j=1}^{N} w_j(t) e^{-|x-x_j(t)|} + \frac{1}{2}\sigma(x,t)\sigma_x(x,t).\end{aligned} \tag{9}$$

Therefore, the expressions of $\alpha$ and $u$ are

$$\begin{aligned}\alpha &= \sum_{j=1}^{N} \sigma_j \delta(x-x_j),\\ u &= -2\sum_{j=1}^{N} w_j \delta(x-x_j) + \frac{1}{4}\sum_{j=1}^{N}\sum_{\substack{k=2\\ j<k}}^{N} \sigma_j\sigma_k e^{\frac{1}{2}(x_j-x_k)}(x_j-x_k)\delta(x-x_k).\end{aligned} \tag{10}$$

# Super CH equation with N-peakon

$\sigma_j,\ w_j$ and $x_j$ evolve according to the following super system

$$
\begin{aligned}
\dot{\sigma}_j = & -w_j\sigma_x(x_j) + \frac{1}{8}\sigma_x(x_j)\sum_{k=2}^{N}\sigma_j\sigma_k e^{\frac{1}{2}(x_j-x_k)}(x_j-x_k) - \frac{1}{2}\sigma_j w_x(x_j) \\
\dot{x}_j = & \ w(x_j), \\
\dot{w}_j = & \ \frac{1}{8}\sum_{\substack{k=2\\ j<k}}^{N}\dot{\sigma}_j\sigma_k e^{\frac{1}{2}(x_j-x_k)}(x_j-x_k) + \frac{1}{8}\sum_{\substack{k=2\\ j<k}}^{N}\sigma_j\dot{\sigma}_k e^{\frac{1}{2}(x_j-x_k)}(x_j-x_k) \\
& + \frac{1}{16}\sum_{\substack{k=2\\ j<k}}^{N}\sigma_j\sigma_k e^{\frac{1}{2}(x_j-x_k)}(\dot{x}_j-\dot{x}_k)(x_j-x_k) + \frac{1}{8}\sum_{\substack{k=2\\ j<k}}^{N}\sigma_j\sigma_k e^{\frac{1}{2}(x_j-x_k} \\
& + \frac{1}{8}\sum_{\substack{k=2\\ j<k}}^{N}\sigma_j\sigma_k e^{\frac{1}{2}(x_j-x_k)}(x_j-x_k)w_x(x_j) - w_j\cdot w_x(x_j).
\end{aligned}
$$

# Super CH equation with N-peakon

Assume $N = 1$, then

$$\sigma(x,t) = \sigma_1(t)e^{-\frac{1}{2}|x-x_1(t)|}, \quad w(x,t) = w_1(t)e^{-|x-x_1(t)|},$$

then the super dynamical system is reduced to

$$\dot{w_1} = 0, \quad \dot{\sigma_1} = 0, \quad \dot{x_1} = w_1.$$

Therefore, the super CH equation (8) admits exact 1-peakon of the form

$$\sigma(x,t) = \xi e^{-\frac{1}{2}|x-l_1t-l_2|}, \quad w(x,t) = l_1e^{-|x-l_1t-l_2|}, \tag{11}$$

where $l_1, l_2$ and $\xi$ are constants, $p(l_1) = p(l_2) = 0$, $p(\xi) = 1$.

# Generalized CH equation with N-peakon

- Generalized CH equation with $N$-peakon
  [*J. Math. Anal. Appl.* **403** (2013) 262 – 271.]

$$\begin{aligned} v_t &= 2(\partial v + v\partial)q + 2vr, \\ \text{with} \quad v &= r_x, \quad v = (\partial^2 - 4\alpha_0)q. \end{aligned} \tag{12}$$

In Eq. (12), when $\alpha_0 = 0$,

$$(12) \Longrightarrow \text{generalized Hunter} - \text{Saxton equation.}$$

## Generalized CH equation with N-peakon

The Lax pairs of Eq. (12) are

$$\phi_x = U\phi, \quad \phi_t = V^{(-1)}\phi, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix},$$

where

$$U = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha_0 & \lambda v & 0 & 0 \\ \lambda w & \alpha_0 & 0 & 0 \end{pmatrix},$$

$$V^{(-1)} = \begin{pmatrix} -q_x + r & 0 & 2q & 0 \\ 0 & -q_x - r & 2\lambda^{-1} & 2q \\ -q_{xx} + 2\alpha_0 q + v & 2\lambda v q & q_x + r & 0 \\ 2\lambda w q + 2\alpha_0 \lambda^{-1} & -q_{xx} + 2\alpha_0 q + v & 0 & q_x - r \end{pmatrix}.$$

# Generalized CH equation with N-peakon

According to the structure of (12), we can deduce the *N*-peakon of this equation as

$$q(x,t)=\sum_{i=1}^{N} q_i(t)e^{-|s-x_i(t)|},$$

where $s=2\alpha_0^{\frac{1}{2}}x$. Then the expression of $r$ can be written as:

$$r(x,t)=\begin{cases} -4\alpha_0^{\frac{1}{2}}\sum_{j=1}^{l} q_j(t), & s\in[x_l,x_{l+1}),1\leq l\leq N,\\ 0, & s<x_1.\end{cases}$$

# Generalized CH equation with N-peakon

Moreover, we can derive the following expressions

$$q_x = -2\alpha_0^{\frac{1}{2}} \sum_{i=1}^{N} q_i(t)\mathrm{sgn}(s - x_i(t))e^{-|s-x_i(t)|},$$

$$v = (\partial^2 - 4\alpha_0)q = 2\sum_{i=1}^{N} q_i(t)\delta(s - x_i(t)),$$

$$v_t = 2\sum_{i=1}^{N} \dot{q}_i(t)\delta(s - x_i(t)) - 2\sum_{i=1}^{N} q_i(t)\dot{x}_i(t)\delta_s(s - x_i(t)),$$

$$v_x = 2\alpha_0^{\frac{1}{2}}(2\sum_{i=1}^{N} q_i(t)\delta_s(s - x_i(t))).$$

where $q_i(t)$ and $x_i(t)$ evolve according to the following dynamical system:

$$\dot{q}_i = -4\alpha_0^{\frac{1}{2}} q_i(2\sum_{j=1}^{i} q_j(t) + \sum_{j=1}^{N} q_j(t)\mathrm{sgn}(x_i - x_j)e^{-|x_i-x_j|}),$$

$$\dot{x}_i = -4\alpha_0^{\frac{1}{2}} \sum_{j=1}^{N} q_j(t)e^{-|x_i-x_j|}.$$

# Generalized CH equation with N-peakon

Assume $N = 1$ in this dynamical system, then

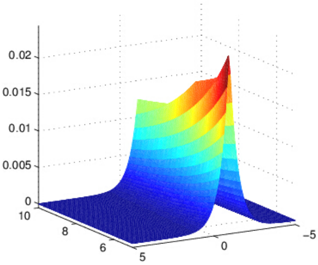
$$q_1(t) = \frac{1}{8\alpha_0^{\frac{1}{2}} t + \beta_2},$$

$$x_1(t) = -\frac{1}{2} \ln(8\alpha_0^{\frac{1}{2}} t + \beta_2) + \hat{\beta}_2,$$

where $\forall \beta_2, \hat{\beta}_2 \in \mathbb{R}$. Therefore, 1-peakon for the generalized CH equation (12) is

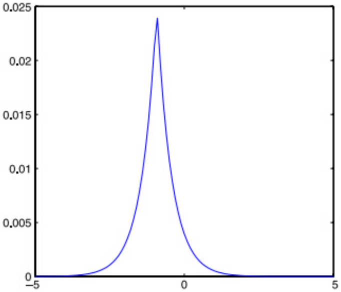
$$q(x,t) = \frac{1}{8\alpha_0^{\frac{1}{2}} t + \beta_2} e^{-|2\alpha_0^{\frac{1}{2}} x + \frac{1}{2}\ln(8\alpha_0^{\frac{1}{2}} t + \beta_2) - \hat{\beta}_2|}.$$

# Generalized CH equation with N-peakon




$\longleftarrow$ 1-peakon with $\alpha_0 = 1, \beta_2 = \hat{\beta}_2 = 0$

1-peakon with $\alpha_0 = 1, \beta_2 = 0, t = 5 \longrightarrow$



# Coupled CH type equation with cubic nonlinearity

- Coupled CH type equation with cubic nonlinearity
  [*J. Math. Anal. Appl.* **403** (2013) 262 – 271.]

$$
\begin{aligned}
v_t &= 2\partial v(r_x q - r q_x) + 2v(2r_{xx}q - r_x q_x - 4\alpha_0 r q),\\
w_t &= 2\partial w(r_x q - r q_x) + 2w(-2r q_{xx} + r_x q_x + 4\alpha_0 r q),
\end{aligned}
\tag{13}
$$

$$
\text{with} \quad v = (\partial^3 - 4\alpha_0\partial)r, \quad w = (\partial^3 - 4\alpha_0\partial)q.
$$

In Eq. (13), when $\alpha_0 = 0$,

$$
(13) \Longrightarrow \text{coupled Hunter} - \text{Saxton type equation.}
$$

# Coupled CH type equation with cubic nonlinearity

Based on the structure of Eq. (13), a direct calculation shows that it possesses $N$-peakon:

$$r_x = \sum_{i=1}^{N} r_i(t) e^{-|s-x_i(t)|}, \quad q_x = \sum_{i=1}^{N} q_i(t) e^{-|s-x_i(t)|},$$

where $s = 2\alpha_0^{\frac{1}{2}} x$. Therefore, the expressions of $r$ and $q$ are

$$r(s,t) = \begin{cases} \frac{\alpha_0^{-\frac{1}{2}}}{2} \sum_{j=1}^{N} r_j(t) e^{s-x_j(t)}, \quad s < x_1, \\ \frac{\alpha_0^{-\frac{1}{2}}}{2} (2\sum_{j=1}^{i_0} r_j(t) - \sum_{j=1}^{i_0} r_j(t) e^{-s+x_j(t)} + \sum_{j=i_0+1}^{N} r_j(t) e^{s-x_j(t)}), \\ \frac{\alpha_0^{-\frac{1}{2}}}{2} (2\sum_{j=1}^{N} r_j(t) - \sum_{j=1}^{N} r_j(t) e^{-s+x_j(t)}), \quad s \geq x_N. \end{cases}$$

# Coupled CH type equation with cubic nonlinearity

$$q(s,t)=\begin{cases}\dfrac{\alpha_0^{-\frac{1}{2}}}{2}\sum_{j=1}^{N}q_j(t)e^{s-x_j(t)}, \quad s<x_1,\\ \dfrac{\alpha_0^{-\frac{1}{2}}}{2}\Big(2\sum_{j=1}^{i_0}q_j(t)-\sum_{j=1}^{i_0}q_j(t)e^{-s+x_j(t)}+\sum_{j=i_0+1}^{N}q_j(t)e^{s-x_j(t)}\Big),\\ \dfrac{\alpha_0^{-\frac{1}{2}}}{2}\Big(2\sum_{j=1}^{N}q_j(t)-\sum_{j=1}^{N}q_j(t)e^{-s+x_j(t)}\Big), \quad s\geq x_N.\end{cases}$$

$r_i$, $q_i$ and $x_i$ evolve according to the following system

$$\begin{aligned}\dot{r}_i&=2r_i(2r_{xx}(x_i)q(x_i)-r_x(x_i)q_x(x_i)-4\alpha_0 r(x_i)q(x_i)),\\ \dot{q}_i&=2q_i(-2r(x_i)q_{xx}(x_i)+r_x(x_i)q_x(x_i)+4\alpha_0 r(x_i)q(x_i)),\\ \dot{x}_i&=-4\alpha_0^{\frac{1}{2}}(r_x(x_i)q(x_i)-r(x_i)q_x(x_i)).\end{aligned}$$

# III. Researches associated with Peakon Equations

# Researches associated with peakon equations

- Searching for new dynamic models with $N-$peakon;
- Inverse spectral problem and explicit multipeakon solution;
- Stability and existence of peakon;
- Integrability, Wave breaking, Blow up, Shock waves, etc.

# Integrable Dynamic Systems with $N$-Peakon

- Modified Camassa-Holm equation
- Two-component Camassa-Holm equation
- Complex Camassa-Holm equation
- Three-component Camassa-Holm equation
- Multi-component system with cubic nonlinear terms

# IV. References

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# THANKS
# FOR
# YOUR ATTENTION !