Isospectral flows related to Frobenius-Stickelberger-Thiele polynomials

Xiangke Chang

LSEC, ICMSEC, Academy of Mathematics and Systems Science
Chinese Academy of Sciences

The Asia-Pacific Integrable Online Seminars
Oct.13th. 2020

(Joint work with X.B. Hu, J. Szmigielski and A. Zhedanov)

Deformation of OPs

■ Consider a family of monic polynomials $\{P_n(x)\}$ orthogonal with respect to the measure $\nu(x)dx$ satisfying

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x).$$

- If $\nu(x;t)dx = e^{xt}\nu(x;0)dx$, then
  - $\frac{\partial}{\partial t} P_n(x;t) = -u_n(t) P_{n-1}(x;t)$

    $\Rightarrow$ Toda lattice : $\dot{u}_n = u_n(b_n - b_{n-1}), \qquad \dot{b}_n = u_{n+1} - u_n.$

Deformation of OPs

- If $\nu(x;t)dx = e^{xt}\nu(x;0)dx$, then
  - Symmetric OPs $\rightarrow$ Lotka-Volterra lattice
  - Laurent BOPs $\rightarrow$ Relativistic Toda lattice
  - R_I polynomials $\rightarrow$ R_I lattice
  - Skew OPs $\rightarrow$ Pfaff lattice
  - String OPs $\rightarrow$ 2-Toda lattice
  - Researchers: Adler, van Moerbeke; Gekhtman; Miki; Nakamura; Nenciu; Peherstorfer; Spiridonov; Tsujimoto; van Assche, Vinet; Zhedanov; et al.
- How about $\nu(x;t)dx = e^{\frac{t}{x}}\nu(x;0)dx$??

Peakon=Peaked soliton

Peakons are non-smooth solitons coming from the superposition of profiles with shape $e^{-|x-x_j|}$:

$$u(x,t) = \sum_{j=1}^{N} m_j(t) e^{-|x-x_j(t)|}$$

where $m_j(t), x_j(t)$ are smooth functions of t.

- Some PDEs admit (weak) solutions of this form.
- Of course, the positions $x_k(t)$ and the amplitudes $m_k(t)$ must have the right time dependence. (Governed by ODEs)

CH equation

- CH (Camassa&Holm 1993):

$$u_t - u_{xxt} + 2\kappa u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0$$

or

$$m_t + um_x + 2mu_x = 0, \quad m = u - u_{xx} + \kappa$$

- Integrability: Lax pair, bi-Hamiltonian, infinitely many conservation laws, etc.
- Peakons ($\kappa = 0$): $u(x,t) = \sum_{j=1}^{N} m_j(t) e^{-|x - x_j(t)|}$
- Shallow water wave: u (velocity)
- Meaningful wave-breaking (Constantine&Escher 1998)
- Stability of CH peakons (Constantin&Strauss 2000)
- ⋯⋯

Remark: Fuchssteiner & Fokas 1981

Peakons as weak solutions

- Two natural derivations for Peakon ODEs
  - Weak formulation (regular distribution)
    - Sobolev space
    - Well-posedness and wave breaking
  - Distributional Lax pair (singular distribution to be regularized)
    - Preserving Lax integrability
    - Conserved quantites
    - Solvability by use of inverse spectral method

- This talk is on peakon equations preserving Lax integrability!

CH peakons by ISM

- Beals&Sattinger&Szmigielski 1999, 2000
  - Isospectral evolution

$$\begin{array}{ccc}
\{x_k(0), m_k(0)\} & \xrightarrow{\text{spectral map}} & \{\lambda_k(0), a_k(0)\}(\text{or } W(z,0)) \\
\downarrow \text{CH flow} & & \downarrow \text{evolution of spectral data} \\
 & & \{\lambda_k(t) = \lambda_k(0) \\
\{x_k(t), m_k(t)\} & \xleftarrow{\text{inverse spectral map}} & a_k(t) = a_k(0)\exp(-\frac{t}{\lambda_k})\} \\
 & & (\text{or } W(z,t))
\end{array}$$

- String problem (Liouville transformation)
- Weyl function
- Stieltjes' theorem on continued fractions
- Hankel determinant, orthogonal polynomials
- Global existence, long time asymptotics: scattering property

Remark: Pure peakon (all m_k are positive (or negative))

Finite Toda by ISM

- Moser 1975
  - Isospectral evolution

$$\begin{array}{ccc}
\{u_k(0), b_k(0)\} & \xrightarrow{\text{spectral map}} & \{\lambda_k(0), c_k(0)\}(\text{or } W(z,0)) \\
\downarrow \text{Toda flow} & & \downarrow \text{evolution of spectral data} \\
& & \{\lambda_k(t) = \lambda_k(0) \\
\{u_k(t), b_k(t)\} & \xleftarrow{\text{inverse spectral map}} & c_k(t) = c_k(0)\exp(\lambda_k t)\} \\
& & (\text{or } W(z,t))
\end{array}$$

- Jacobi matrix spectral problem (orthogonal polynomials)
- Weyl function
- Stieltjes' theorem on continued fractions
- Long time asymptotics: scattering property

CH peakon and Toda

Theorem (Ragnisco&Bruschi 1996, Beals&Sattinger&Szmigielski 2001)

Given

$$A_i(0) = \sum_{p=1}^{n} \lambda_p^i c_p(0),$$

with

$$0 < \lambda_1(0) < \lambda_2(0) < \cdots < \lambda_n(0), \qquad c_p(0) > 0,$$

let $\tau_k^{(l)}(0)$ *be defined as Hankel determinants*

$$\tau_k^{(l)}(0) = \begin{vmatrix} A_l(0) & A_{l+1}(0) & \cdots & A_{l+k-1}(0) \\ A_{l+1}(0) & A_{l+2}(0) & \cdots & A_{l+k}(0) \\ \vdots & \vdots & \ddots & \vdots \\ A_l(l+k-1) & A_{l+k}(0) & \cdots & A_{l+2k-2}(0) \end{vmatrix}$$

for any nonnegative integer k, *and* $\tau_0^{(l)}(0) = 1$, $\tau_k^{(l)}(0) = 0$ *for* $k < 0$.

CH peakon and Toda

Theorem (Ragnisco&Bruschi 1996, Beals&Sattinger&Szmigielski 2001)

1. *Introduce the variables* $\{x_k(0), m_k(0)\}_{k=1}^n$ *defined by*

$$x_{k'}(0) = \ln\left(\frac{2\tau_k^{(0)}(0)}{\tau_{k-1}^{(2)}(0)}\right), \qquad m_{k'}(0) = \frac{\tau_k^{(0)}(0)\tau_{k-1}^{(2)}(0)}{\tau_k^{(1)}(0)\tau_{k-1}^{(1)}(0)},$$

where $k' = n + 1 - k$.
If $\{\lambda_p(t), c_p(t)\}_{p=1}^n$ *evolve as*

$$\dot\lambda_p = 0, \qquad \dot c_p = \frac{c_p}{\lambda_p} \qquad (\textit{implies } \dot A_k = A_{k-1}),$$

then $\{x_k(t), m_k(t)\}_{k=1}^n$ *satisfy the CH peakon ODEs:*

$$\dot x_k = \sum_{j=1}^n m_j e^{-|x_j - x_k|}, \qquad \dot m_k = \sum_{j=1}^n \mathrm{sgn}(x_k - x_j) m_j e^{-|x_j - x_k|}.$$

CH peakon and Toda

Theorem (Ragnisco&Bruschi 1996, Beals&Sattinger&Szmigielski 2001)

2. *Introduce the variables* $\{u_k(0), b_k(0)\}_{k=0}^{n-1}$ *defined by*

$$u_k(0) = \sqrt{\frac{\tau_{k+1}^{(1)}(0)\tau_{k-1}^{(1)}(0)}{(\tau_k^{(1)}(0))^2}}, \qquad b_k = \frac{\tau_{k+1}^{(1)}(0)\tau_{k-1}^{(2)}(0)}{\tau_k^{(2)}(0)\tau_k^{(1)}(0)} + \frac{\tau_{k+1}^{(2)}(0)\tau_k^{(1)}(0)}{\tau_k^{(2)}(0)\tau_{k+1}^{(1)}(0)}.$$

If $\{\lambda_p(t), c_p(t)\}_{p=1}^{n}$ *evolve as*

$$\dot{\lambda}_p = 0, \qquad \dot{c}_p = \lambda_p c_p \qquad (\textit{implies } \dot{A}_k = A_{k+1}),$$

then $\{u_k(t), b_k(t)\}_{k=0}^{n-1}$ *satisfy the finite Toda lattice with* $u_0 = u_n = 0$.

CH peakon and Toda

Theorem (Ragnisco&Bruschi 1996, Beals&Sattinger&Szmigielski 2001)

3. *There exists a mapping from* $\{x_k(0), m_k(0)\}_{k=1}^{n}$ *to* $\{u_k(0), b_k(0)\}_{k=0}^{n-1}$ *according to*

$$u_k(0) = \frac{e^{\frac{1}{2}(x_{k'-1}(0)-x_{k'}(0))}}{\sqrt{m_{k'-1}(0)m_{k'}(0)}\left(1 - e^{x_{k'-1}(0)-x_{k'}(0)}\right)},$$

$$b_k(0) = \frac{1 - e^{x_{k'-2}(0)-x_{k'}(0)}}{m_{k'-1}(0)(1 - e^{x_{k'-2}(0)-x_{k'-1}(0)})(1 - e^{x_{k'-1}(0)-x_{k'}(0)})}$$

with the convention

$$u_0(0) = 0, \quad b_0(0) = \lim_{x_0 \to -\infty} b_0(0), \quad b_{n-1}(0) = \lim_{x_{n+1} \to +\infty} b_{n-1}(0).$$

Peakon&Toda&OPs

- 2mCH/SQQ peakons and m-Toda (KvM)
  - C&Hu&Szmigielski 2016
  - Symmetric OPs
- Novikov peakons and B-Toda
  - C&Hu&Li&Zhao 2018
  - Partial skew OPs: C&He&Hu&Li 2018
- DP peakons and C-Toda
  - C&Hu&Li, 2018
  - Cauchy bi-OPs: Bertola&Gekhtman&Szmigielski 2010

mCH/FORQ

- **Ref:** Fokas 1995; Fuchssteiner 1996; Olver&Rosenau1996; Qiao 2006

$$m_t + \left((u^2 - u_x^2)m\right)_x = 0, \quad m = u - u_{xx}$$

- Integrability
  - bi-Hamitonian structure: tri-Hamiltonian duality to the mKdV
  - Lax pair
  - infinitely many conservation laws
- Multisolitons (Matsuno 2013; Hu&Yin&Wu 2016; Xia&Zhou&Qiao 2016)
- Algebro-geometric solutions (Hou&Fan&Qiao 2017)
- BT (Wang&Liu&Mao 2020)
- RHP (Xu&Fan 2019; Boutet de Monvel&Karpenko&Shepelsky 2020)
- Wave breaking, Stability (Gui, Liu, Liu, Olver, Qu et al. 2013 2014)
- Cauchy problem (Himonas, Holliman, Mantzavinos, 2014 2019)
- ……

mCH peakons

- Suppose

$$u(x,t) = \sum_{k=1}^{N} m_k(t) e^{-|x-x_k(t)|}$$

  - mCH peakon ODEs (C&Szmigielski 2016, 2017, 2018)

$$\dot{m}_k = 0, \qquad \dot{x}_k = u^2(x_k) - \langle u_x^2 \rangle (x_k),$$

    which is explicitly solvable by use of inverse spectral method under the assumption:
    - $x_1(0) < x_2(0) < \cdots < x_n(0)$;
    - m_k are all positive (or negative).
  - Note that $\langle f \rangle (x_k, t) = \frac{1}{2}(f(x_k+,t) + f(x_k-,t))$.

Cauchy-Stieltjes-Vandermonde matrix

Definition (C&Szmigielski 2018)

Given a (strictly) positive vector $\mathbf{e} \in \mathbf{R}^k$, a non-negative number c, an index l such that $0 \le l \le k$, another index p such that $0 \le p$, $p + l - 1 \le k - l$, and a positive measure ν with support in $\mathbf{R}_+$, a *Cauchy-Stieltjes-Vandermonde (CSV) matrix* is that of the form

$$CSV_k^{(l,p)}(\mathbf{e}, \nu, c)$$

$$= \begin{pmatrix} e_1^p \nu_c(e_1) & e_1^{p+1}\nu_c(e_1) & \cdots & e_1^{p+l-1}\nu_c(e_1) & 1 & e_1 & \cdots & e_1^{k-l-1} \\ e_2^p \nu_c(e_2) & e_2^{p+1}\nu_c(e_2) & \cdots & e_2^{p+l-1}\nu_c(e_2) & 1 & e_2 & \cdots & e_2^{k-l-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ e_k^p \nu_c(e_k) & e_k^{p+1}\nu_c(e_k) & \cdots & e_k^{p+l-1}\nu_c(e_k) & 1 & e_k & \cdots & e_k^{k-l-1} \end{pmatrix},$$

where ν_c is the (shifted) Stieltjes transform of the measure ν and is given by $\nu_c(y) = c + \int \frac{d\nu(x)}{y+x}$.

Exact-explicit-closed form

Theorem (C&Szmigielski 2018)

The mCH equation with the regularization of the singular term $u_x^2 m$ given by $\langle u_x^2 \rangle m$ admits the multipeakon solution

$$u(x,t) = \sum_{k'=1}^{n} m_{k'}(t) \exp(-|x - x_{k'}(t)|),$$

where $x_{k'}$ are given by equations

$$x_{k'} = \ln \frac{\mathbf{e}_{[1,k-1]} \mathcal{D}_k^{(\frac{k+1}{2},0)} \mathcal{D}_{k-1}^{(\frac{k-1}{2},0)}}{m_{k'} \mathcal{D}_k^{(\frac{k-1}{2},1)} \mathcal{D}_{k-1}^{(\frac{k-1}{2},1)}}, \qquad \textit{if } k \textit{ is odd,}$$

$$x_{k'} = \ln \frac{\mathbf{e}_{[1,k-1]} \mathcal{D}_k^{(\frac{k}{2},0)} \mathcal{D}_{k-1}^{(\frac{k}{2},0)}}{m_{k'} \mathcal{D}_k^{(\frac{k}{2},1)} \mathcal{D}_{k-1}^{(\frac{k}{2}-1,1)}}, \qquad \textit{if } k \textit{ is even.}$$

Exact-explicit-closed form

Theorem (C&Szmigielski 2018)

Here $\mathcal{D}_k^{(l,p)}$ is Cauchy-Stieltjes-Vandermonde determinant with the peakon spectral measure

$$d\mu = \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} b_j(t)\delta_{\zeta_j},$$

$b_j(t) = b_j(0)e^{\frac{2t}{\zeta_j}}$, $0 < b_j(0)$, *ordered eigenvalues* $0 < \zeta_1 < \cdots < \zeta_{\lfloor \frac{n}{2} \rfloor}$ *and* $c = 0$ *for even* n, $c > 0$ *for odd* n.

Remark 1. Valid for $x_1 < x_2 < \cdots < x_n$

Remark 2. Global existence, long time asymptotics: scattering property

Frobenius-Stickelberger-Thiele polynomials

- C&Hu&Szmigielski&Zhedanov 2020

Consider a family of polynomials $\{T_k(z)\}_{k=0}^{\infty}$ given by

$$T_k(z) = \frac{1}{N_k} \det \begin{pmatrix} 1 & -z & \cdots & (-z)^{\lfloor \frac{k}{2} \rfloor} & 0 & 0 & \cdots & 0 \\ V(e_1) & e_1 V(e_1) & \cdots & e_1^{\lfloor \frac{k}{2} \rfloor} V(e_1) & 1 & e_1 & \cdots & e_1^{\lfloor \frac{k-1}{2} \rfloor} \\ V(e_2) & e_2 V(e_2) & \cdots & e_2^{\lfloor \frac{k}{2} \rfloor} V(e_2) & 1 & e_2 & \cdots & e_2^{\lfloor \frac{k-1}{2} \rfloor} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ V(e_k) & e_k V(e_k) & \cdots & e_k^{\lfloor \frac{k}{2} \rfloor} V(e_k) & 1 & e_k & \cdots & e_k^{\lfloor \frac{k-1}{2} \rfloor} \end{pmatrix},$$

with $T_0(z) = 1,\ T_{-1}(z) = 0$, where

$$N_{2p} = \det\left(C_{2p}^{(p,0)}\right), \qquad N_{2p+1} = (-1)^p \det\left(C_{2p+1}^{(p+1,0)}\right).$$

- Frobenius&Stickelberger 1880
- Thiele 1908
- Spiridonov&Tsujimoto&Zhedanov 2007

Properties of FST polynomials

Property (1)

For $p = 0, 1, \ldots,$

$$\deg(T_{2p}(z)) = \deg(T_{2p+1}(z)) = p.$$

Polynomials $T_{2p}(z)$ *are monic, while* $T_{2p+1}(z)$ *have the form*

$$T_{2p+1}(z) = (-1)^p \frac{\det\left(C_{2p+1}^{(p,0)}\right)}{\det\left(C_{2p+1}^{(p+1,0)}\right)} z^p + O(z^{p-1}).$$

In particular, $T_0(z) = 1,\ T_1(z) = \frac{1}{V(e_1)}.$

Properties of FST polynomials

Property (2)

The polynomials $\{T_k(z)\}_{k=0}^{\infty}$ *satisfy the orthogonality relation*

$$\int \frac{T_k(z)z^j}{\prod_{i=1}^{k}(z+e_i)} d\nu(z) = \beta_k \delta_{\lfloor k/2 \rfloor, j}, \qquad j = 0, 1, \ldots, \lfloor \frac{k}{2} \rfloor,$$

where

$$\beta_{2p} = \int d\nu(z) + (-1)^p \frac{\det\left(C_{2p}^{(p+1,0)}\right)}{\det\left(C_{2p}^{(p,0)}\right)}, \qquad \beta_{2p+1} = 1, \qquad p = 0, 1, \ldots,$$

with the proviso that for $p = 0$ *the second term in* β_{2p} *is absent.*

Properties of FST polynomials

Property (3)

The polynomials $\{T_k(z)\}_{k=0}^{\infty}$ satisfy the three term recurrence

$$T_{k+1}(z) = d_{k+1}T_k(z) + (z + e_k)T_{k-1}(z),$$

where

$$\begin{aligned}
d_{2p+1} &= (-1)^p \frac{\det\left(C_{2p+1}^{(p,0)}\right)}{\det\left(C_{2p+1}^{(p+1,0)}\right)} - (-1)^{p-1} \frac{\det\left(C_{2p-1}^{(p-1,0)}\right)}{\det\left(C_{2p-1}^{(p,0)}\right)}, \\
d_{2p+2} &= (-1)^{p+1} \frac{\det\left(C_{2p+2}^{(p+2,0)}\right)}{\det\left(C_{2p+2}^{(p+1,0)}\right)} - (-1)^p \frac{\det\left(C_{2p}^{(p+1,0)}\right)}{\det\left(C_{2p}^{(p,0)}\right)}
\end{aligned}$$

for $p = 0, 1, \ldots,$ with the proviso that the second terms above are absent for $p = 0$.

Properties of FST polynomials

Property (4)

Introduce the associated FST polynomials $\left\{T_k^{(1)}(z)\right\}_{k=0}^{\infty}$, *defined by the same recurrence relation*

$$T_{k+1}^{(1)}(z) = d_{k+1}T_k^{(1)}(z) + (z+e_k)T_{k-1}^{(1)}(z),$$

but with different initial conditions, namely, $T_0^{(1)}(z) = 0,\ T_1^{(1)}(z) = 1.$

The ratio $\frac{T_k^{(1)}(z)}{T_k(z)}$ *has the continued fraction expansion*

$$\frac{T_k^{(1)}(z)}{T_k(z)} = \cfrac{1}{d_1 + \cfrac{z+e_1}{d_2 + \cfrac{z+e_2}{d_3 + \cdots + \cfrac{z+e_{k-2}}{d_{k-1} + \cfrac{z+e_{k-1}}{d_k}}}}}$$

Infinite FST lattice

- C&Hu&Szmigielski&Zhedanov 2020
  - Introduce a “time” evolution of the measure $d\nu(z)$ given by

$$d\nu(z;t) = e^{zt} d\nu(z;0)$$

and define the *FST fractions* as

$$\left\{ \psi_k(z;t) \stackrel{def}{=} \frac{T_k(z;t)}{\prod_{i=1}^{k}(z+e_i)} \right\}_{k=0}^{\infty}.$$

Infinite FST lattice

Theorem (C&Hu&Szmigielski&Zhedanov 2020)

The FST fractions $\{\psi_k(z;t)\}_{k=0}^{\infty}$ *corresponding to the time-dependent measure* $d\nu(z;t) = e^{zt} d\nu(z;0)$ *undergo the time evolution*

$$\dot{\psi}_{2p+1}(z) = \left(-A_p B_p + e_{2p+1}\right)\psi_{2p+1}(z) + A_{p-1}\psi_{2p}(z) - \psi_{2p-1}(z),$$

$$\dot{\psi}_{2p}(z) = A_{p-1}B_p\psi_{2p}(z) - B_p\psi_{2p-1}(z),$$

where $A_p = \sum_{i=0}^{p} d_{2i+1}, B_p = \sum_{i=0}^{p} d_{2i}$ *and the dot means the derivative with respect to* t*.*

Infinite FST lattice

Let $\Psi = (\psi_0(z;t), \psi_1(z;t), \cdots)^\top$, then

$$L\Psi = zE\Psi, \qquad \dot{\Psi} = FR\Psi,$$

where

$$L = \begin{pmatrix} d_1 & -e_1 & & & \\ 1 & d_2 & -e_2 & & \\ & 1 & d_3 & -e_3 & \\ & & \ddots & \ddots & \ddots \\ & & & & \ddots \end{pmatrix}, \; E = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \ddots \\ & & & & \ddots \end{pmatrix}, \; F = E^\top,$$

$$R = \begin{pmatrix} 0 & -d_0d_1 + e_1 & 0 & & \\ 0 & -d_0 - d_2 & d_1(d_0 + d_2) & & \\ & -1 & d_1 & -(d_0 + d_2)(d_1 + d_3) + e_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Infinite FST lattice

- Matrix form

$$\dot{L} = RFL - LFR.$$

$$\Longleftrightarrow$$

$$\dot{d}_{2k-1} = -d_{2k-1}\left(\sum_{j=0}^{k-1} d_{2j}\right)\left(2\sum_{j=0}^{k-2} d_{2j+1} + d_{2k-1}\right) - e_{2k-2}\left(\sum_{j=0}^{k-2} d_{2j+1}\right) + e_{2k-1}\left(\sum_{j=0}^{k-1} d_{2j+1}\right),$$

$$\dot{d}_{2k} = d_{2k}\left(\sum_{j=0}^{k-1} d_{2j+1}\right)\left(2\sum_{j=0}^{k-1} d_{2j} + d_{2k}\right) + e_{2k-1}\left(\sum_{j=0}^{k-1} d_{2j}\right) - e_{2k}\left(\sum_{j=0}^{k} d_{2j}\right),$$

valid for $k = 1, 2 \ldots$. Again, note that $d_0 = \int d\nu(z)$ and any void sum is taken to be zero.

Finite FST lattice

- Consider the finite discrete measure

$$d\nu(z;t) = \sum_{i=1}^{K} b_i(t)\delta_{\zeta_i} dz$$

where

$$b_i(t) = b_i(0)e^{\zeta_i t}, \qquad 0 < \zeta_1 < \zeta_2 < \cdots < \zeta_K.$$

Lemma

Suppose $d\nu(z)$ is a finite discrete measure. Then

$$d_0 = \int d\nu(z) = (-1)^{K+1} \frac{\det\left(C_{2K}^{(K+1,0)}\right)}{\det\left(C_{2K}^{(K,0)}\right)} = -\sum_{j=1}^{K} d_{2j}.$$

Finite FST lattice

Lemma

Let $\Psi_{[2K]} = (\psi_0(z;t), \psi_1(z;t) \cdots, \psi_{2K-1}(z,t))^\top, \quad F_{[2K]} = E_{[2K]}^\top$, *then*

$$L_{[2K]}\Psi_{[2K]} = zE_{[2K]}\Psi_{[2K]}, \qquad \dot{\Psi}_{[2K]} = F_{[2K]}R_{[2K]}\Psi_{[2K]},$$

$$L_{[2K]} = \begin{pmatrix} d_1 & -e_1 & & & \\ 1 & d_2 & -e_2 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & d_{2K-1} & -e_{2K-1} \\ & & & 1 & d_{2K} \end{pmatrix}, \quad E_{[2K]} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix},$$

$$R_{[2K]} = \begin{pmatrix} 0 & d_1 \sum\limits_{j=1}^{K} d_{2j} + e_1 & & & \\ 0 & \sum\limits_{j=2}^{K} d_{2j} & -d_1 \sum\limits_{j=2}^{K} d_{2j} & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & \sum_{j=0}^{K-2} d_{2j+1} & d_{2K} \sum_{j=0}^{K-1} d_{2j+1} + e_{2K-1} \\ & & & 0 & 0 \end{pmatrix}.$$

Finite FST lattice

Corollary

The time evolution of $L_{[2K]}$ *takes the form*

$$\dot{L}_{[2K]} = R_{[2K]}F_{[2K]}L_{[2K]} - L_{[2K]}F_{[2K]}R_{[2K]},$$

or, equivalently,

$$\dot{d}_{2k-1} = d_{2k-1}\left(\sum_{j=k}^{K} d_{2j}\right)\left(2\sum_{j=0}^{k-2} d_{2j+1} + d_{2k-1}\right) - e_{2k-2}\left(\sum_{j=0}^{k-2} d_{2j+1}\right) + e_{2k-1}\left(\sum_{j=0}^{k-1} d_{2j+1}\right),$$

$$\dot{d}_{2k} = -d_{2k}\left(\sum_{j=0}^{k-1} d_{2j+1}\right)\left(d_{2k} + 2\sum_{j=k+1}^{K} d_{2j}\right) - e_{2k-1}\left(\sum_{j=k}^{K} d_{2j}\right) + e_{2k}\left(\sum_{j=k+1}^{K} d_{2j}\right)$$

for $k = 1, 2, \ldots, K$.

mCH peakon spectral problem

- Left boundary value problem:
  - Let $q_k = \Phi_1(x_k+)$, $p_k = \Phi_2(x_k+)$, then

$$\begin{aligned} q_k - q_{k-1} &= h_k p_{k-1}, & 1 \le k \le n, \\ p_k - p_{k-1} &= -z g_k q_{k-1}, & 1 \le k \le n, \end{aligned}$$
$$q_0 = 0, \quad p_0 = 1, \quad p_n = 0.$$

    - Eigenvalues: zeroes of $p_n(z)$.
    - $q_k(z)$ is a polynomial of degree $\lfloor \frac{k-1}{2} \rfloor$ in z, and $p_k(z)$ is a polynomial of degree $\lfloor \frac{k}{2} \rfloor$, respectively.
    - The spectrum is of finite number ($\lfloor \frac{n}{2} \rfloor$), simple, positive.

mCH peakon spectral problem

- Right boundary value problem:

$$\begin{aligned} \hat{q}_j - \hat{q}_{j-1} &= -h_{j'}\hat{p}_{j-1}, && 1 \le j \le 2K, \\ \hat{p}_j - \hat{p}_{j-1} &= z g_{j'}\hat{q}_{j-1}, && 1 \le j \le 2K, \\ \hat{p}_0 = 0, &\quad \hat{q}_0 = 1, \end{aligned}$$

where ˆ over qs or ps indicates that we are moving from right to left while the prime over j reflects the counting from left to right, thus $j' = 2K - j + 1$.

mCH peakon spectral problem and FST

Theorem

1. $\left\{\hat{Q}_k(z)\right\}_{k=1}^{2K}$ *form a finite family of FST polynomials associated to the measure* $d\mu$*.*
2. $\{Q_k(z)\}_{k=1}^{2K}$ *form a finite family of associated FST polynomials.*

Here

$$q_{2k} = \frac{(-1)^k \prod_{i=1}^{K} h_{2i-1}g_{2i}}{\prod_{i=k+1}^{K} g_{2i-1}h_{2i}} Q_{2k}, \quad q_{2k+1} = \frac{(-1)^k \prod_{i=1}^{K} h_{2i-1}g_{2i}}{g_{2K}\prod_{i=k+1}^{K-1} g_{2i}h_{2i+1}} Q_{2k+1},$$

$$\hat{q}_{2p} = ((-1)^p \prod_{i=1}^{p} g_{(2i-1)'}h_{(2i)'})\hat{Q}_{2p}, \quad \hat{q}_{2p+1} = ((-1)^p h_{1'} \prod_{i=1}^{p} g_{(2i)'}h_{(2i+1)'})\hat{Q}_{2p+1}$$

mCH peakon spectral problem and FST

Theorem

Let $e_1, e_2, \dots e_{2K}$ be $2K$ positive and distinct constants. Given positive constants $\{h_k\}_{k=1}^{2K}$ and $\{g_k\}_{k=1}^{2K}$ satisfying $g_k h_k = \frac{1}{e_{k'}}$, then there exists a mapping from the mCH left boundary value problem with $n = 2K$ to the generalized eigenvalue problem

$$\hat{L}_{[2K]}\hat{\Psi}_{[2K]} = zE_{[2K]}\hat{\Psi}_{[2K]}, \tag{0.4}$$

where

$$\hat{\Psi}_{[2K]} = (\hat{\psi}_0(z;t), \hat{\psi}_1(z;t)\cdots, \hat{\psi}_{2K-1}(z;t))^\top, \qquad \hat{\psi}_k(z,t) = \frac{\hat{Q}_k(z;t)}{\prod_{i=1}^{k}(z+e_i)}$$

$$\hat{L}_{[2K]} = \begin{pmatrix} \hat{d}_1 & -e_1 & & & \\ 1 & \hat{d}_2 & -e_2 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \hat{d}_{2K-1} & -e_{2K-1} \\ & & & 1 & \hat{d}_{2K} \end{pmatrix}, \qquad E_{[2K]} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}.$$

mCH peakon spectral problem and FST

Theorem

The mapping from $\{h_k\}_{k=1}^{2K}$ (or eqivalently $\{g_k\}_{k=1}^{2K}$) to $\{\hat{d}_k\}_{k=1}^{2K}$ is given by

$$\hat{d}_{2p} = -\left(\frac{1}{h_{(2p)'}} + \frac{1}{h_{(2p-1)'}}\right)\frac{h_{1'}\prod_{i=1}^{p-1} g_{(2i)'}h_{(2i+1)'}}{g_{1'}\prod_{i=1}^{p-1} h_{(2i)'}g_{(2i+1)'}}, \quad \hat{d}_{2p+1} = \left(\frac{1}{h_{(2p+1)'}} + \frac{1}{h_{(2p)'}}\right)\frac{\prod_{i=1}^{p} g_{(2i-1)'}h_{(2i)'}}{\prod_{i=1}^{p} h_{(2i-1)'}g_{(2i)'}},$$

under which, the Weyl function defined by

$$W(z) = \frac{q_{2K}(z)}{p_{2K}(z)}$$

for the mCH left boundary value problem is equivalent to the element in the first row and first column of $\left(zE_{[2K]} - \hat{L}_{[2K]}\right)^{-1}$, i.e.

$$W(z) =< 1|(zE_{[2K]} - \hat{L}_{[2K]})^{-1}|1 > .$$

mCH peakon, FST lattice and FST polynomials

Theorem

Given positive and distinct constants $e_k, 1 \leq k \leq 2K$*, let*

$$\beta_j(0) = \sum_{i=1}^{K} \zeta_i(0)^j b_i(0), \qquad V_k(0) = \sum_{i=1}^{K} \frac{b_i(0)}{\zeta_i(0) + e_k},$$

with

$$0 < \zeta_1(0) < \zeta_2(0) < \cdots < \zeta_K(0), \qquad b_i(0) > 0,$$

For any positive integer k*, index* p*, and* l *such that* $0 \leq l \leq k$*, define* $\tau_k^{(l,p)}(0)$ *as*

$$\tau_k^{(l,p)}(0) = \det \begin{pmatrix} e_1^p V(e_1) & e_1^{p+1} V(e_1) & \cdots & e_1^{p+l-1} V(e_1) & 1 & e_1 & \cdots & e_1^{k-l-1} \\ e_2^p V(e_2) & e_2^{p+1} V(e_2) & \cdots & e_2^{p+l-1} V(e_2) & 1 & e_2 & \cdots & e_2^{k-l-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ e_k^p V(e_k) & e_k^{p+1} V(e_k) & \cdots & e_k^{p+l-1} V(e_k) & 1 & e_k & \cdots & e_k^{k-l-1} \end{pmatrix},$$

as well as $\tau_0^{(l,p)}(0) = 1$, $\tau_k^{(l,p)}(0) = 0$ *for* $k < 0$ *or* $l > k$*.*

mCH peakon, FST lattice and FST polynomials

Theorem

1 *Let the variables* $\{x_k(0), m_k(0)\}_{k=1}^{2K}$ *be defined by*

$$x_{k'}(0) = \ln \frac{(-1)^{\lfloor \frac{k}{2} \rfloor} \mathbf{e}_{[1,k]} \tau_k^{(\lfloor \frac{k+1}{2} \rfloor, 0)}(0) \tau_{k-1}^{(\lfloor \frac{k}{2} \rfloor, 0)}(0)}{m_{k'} \tau_k^{(\lfloor \frac{k}{2} \rfloor, 1)}(0) \tau_{k-1}^{(\lfloor \frac{k-1}{2} \rfloor, 1)}(0)}, \qquad m_{k'}(0) = \frac{1}{\sqrt{e_k}},$$

where $k' = 2K + 1 - k$, $\mathbf{e}_{[1,k]} = \prod_{i=1}^{k} e_i$.
If $\{\zeta_i(t), b_i(t)\}_{i=1}^{K}$ *evolve as*

$$\dot{\zeta}_i = 0, \qquad \dot{b}_i = \frac{2b_i}{\zeta_i} \qquad \Rightarrow \qquad \dot{\beta}_j = 2\beta_{j-1}$$

then $\{x_k(t), m_k(t)\}_{k=1}^{2K}$ *satisfy the mCH peakon ODEs with* $n = 2K$.

mCH peakon, FST lattice and FST polynomials

Theorem

2 *Let the variables* $\{d_k(0)\}_{k=1}^{2K}$ *be defined by*

$$d_{2p+1}(0) = (-1)^p \frac{\tau_{2p+1}^{(p,0)}(0)}{\tau_{2p+1}^{(p+1,0)}(0)} - (-1)^{p-1} \frac{\tau_{2p-1}^{(p-1,0)}(0)}{\tau_{2p-1}^{(p,0)}(0)},$$

$$d_{2p+2}(0) = (-1)^{p+1} \frac{\tau_{2p+2}^{(p+2,0)}(0)}{\tau_{2p+2}^{(p+1,0)}(0)} - (-1)^p \frac{\tau_{2p}^{(p+1,0)}(0)}{\tau_{2p}^{(p,0)}(0)}.$$

If $\{\zeta_i(t), b_i(t)\}_{i=1}^{K}$ *evolve as*

$$\dot{\zeta}_i = 0, \qquad \dot{b}_i = \zeta_i b_i \qquad \Rightarrow \qquad \dot{\beta}_j = \beta_{j+1},$$

then $\{d_k(t)\}_{k=1}^{2K}$ *satisfy the finite FST lattice.*

mCH peakon, FST lattice and FST polynomials

Theorem

3 *The initial data of the mCH peakon problem* $\{x_k(0), m_k(0)\}_{k=1}^{2K}$ *is mapped to the initial data of the FST lattice* $\{d_k(0)\}_{k=1}^{2K}$ *as follows*

$$d_{2p}(0) = -\left(\frac{1}{h_{(2p)'}(0)} + \frac{1}{h_{(2p-1)'}(0)}\right)\frac{h_{1'}(0)\prod_{i=1}^{p-1} g_{(2i)'}(0)h_{(2i+1)'}(0)}{g_{1'}(0)\prod_{i=1}^{p-1} h_{(2i)'}(0)g_{(2i+1)'}(0)},$$

$$d_{2p+1}(0) = \left(\frac{1}{h_{(2p+1)'}(0)} + \frac{1}{h_{(2p)'}(0)}\right)\frac{\prod_{i=1}^{p} g_{(2i-1)'}(0)h_{(2i)'}(0)}{\prod_{i=1}^{p} h_{(2i-1)'}(0)g_{(2i)'}(0)},$$

where $g_j(0) = m_j(0)e^{-x_j(0)}$, $h_j(0) = m_j(0)e^{x_j(0)}$.

Extreme degenerate case of the FST lattice

- $e_k = 0 \Longrightarrow$

$$\dot{d}_{2k-1} = d_{2k-1}\left(\sum_{j=k}^{K} d_{2j}\right)\left(2\sum_{j=0}^{k-2} d_{2j+1} + d_{2k-1}\right),$$

$$\dot{d}_{2k} = -d_{2k}\left(\sum_{j=0}^{k-1} d_{2j+1}\right)\left(d_{2k} + 2\sum_{j=k+1}^{K} d_{2j}\right).$$

If we let

$$p_k = \ln\frac{d_{2k-1}}{m_{2k-1}}, \qquad q_k = \ln\frac{2n_{2k}}{d_{2k}},$$

where m_k, n_k are some constants, then

$$\dot{p}_k = 2\sum_{i=k}^{K} n_{2i}e^{p_k-q_i}\left(2\sum_{i=1}^{k} m_{2i-1}e^{p_i-p_k} - m_{2k-1}\right),$$

$$\dot{q}_k = -2\sum_{i=1}^{k} m_{2i-1}e^{p_i-q_k}\left(2\sum_{i=k}^{K} n_{2i}e^{q_k-q_i} - n_{2k}\right).$$

2mCH interlacing peakon ODE

- Ref: C&Hu&Szmigielski 2016

Set

$$u(x,t) = \sum_{k=1}^{K} m_{2k-1} e^{-|x-p_k(t)|}, \qquad v(x,t) = \sum_{k=1}^{K} n_{2k} e^{-|x-q_k(t)|},$$

and assume p_k, q_k are ordered according to

$$p_1 < q_1 < p_2 < \cdots < p_K < q_K,$$

then the above ODE system can be rewritten as

$$\dot{p}_j = \left(u(p_j) - \langle u_x\rangle(p_j)\right)\left(v(p_j) + v_x(p_j)\right),$$
$$\dot{q}_j = \left(u(q_j) - u_x(q_j)\right)\left(v(q_j) + \langle v_x\rangle(q_j)\right),$$

which is the interlacing peakon ODE system for the 2-mCH eq.

$$m_t + [(u-u_x)(v+v_x)m]_x = 0,$$
$$n_t + [(u-u_x)(v+v_x)n]_x = 0,$$
$$m = u - u_{xx}, \quad n = v - v_{xx}.$$

An interpretation in terms of ORF

- Ref: C&Hu&Szmigielski&Zhedanov 2020

The orthogonality, after taking the limit $e_k \to 0$, gives

$$\int T_k(z;t) z^{j-k} e^{zt} d\nu(z;0) = 0, \qquad j = 0, 1, \ldots, \lfloor \frac{k}{2} \rfloor - 1,$$

and consequently we have

$$\int P_k(z^{-\frac{1}{2}};t) z^{j-\frac{k}{2}} e^{zt} d\nu(z;0) = 0, \qquad j = 0, 1, \ldots, \lfloor \frac{k}{2} \rfloor - 1,$$

which, in turn, can be written as

$$\int P_k(z;t) z^{k-2j} e^{\frac{t}{z^2}} d\nu(\frac{1}{z^2};0) = 0, \qquad j = 0, 1, \ldots, \lfloor \frac{k}{2} \rfloor - 1,$$

finally resulting in

$$\int S_k(z;t) z^{k-2j} e^{\frac{t}{z^2}} d\nu(\frac{1}{z^2};0) = 0, \qquad j = 0, 1, \ldots, \lfloor \frac{k}{2} \rfloor - 1.$$

Conclusion

- Isospectral flows related to FST polynomials
  - FST polynomials
  - Infinite FST lattice
  - Finite FST lattice, mCH peakon lattice, FST polynomials
  - Extreme degenerate case of the FST lattice: 2mCH interlacing peakon lattice

Thanks a lot for your attention!