Integrable Systems and Invariant Geometric Flows in Affine-related Geometries

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Outline

◇ Introduction

◇ Integrable geometric flows

- Euclidean geometry
- Centro-equiaffine geometry
- Affine geometry
- Centro-affine geometry
- Centro-equiaffine symplectic geometry
- Affine symplectic geometry

◇ Heat flows

1. Introductions

◇ Questions

- Geometric realizations of integrable systems?
- Geometric formulation of various properties to integrable systems?
- Invariant differential equations under certain Lie groups?
- Behaviour, singularities, exac solutions of geometric flows?

There are many works on integrable curve flows:

◇ Integrable curve flows

- Di Rios (1906), Hasimoto (1972) Nonlinear Schrödinger equation
- Lamb (1977) The mKdV and sine-Gordon equations
- Lakshmanan (1979) The Heisenberg spin model
- Langer and Perline (1991) The Schrödinger heirarchy
- Doliwa and Santini (1994) The mKdV hierarchy with lower order term
- Nakayama (1998,1999) The Regge-Lund equation, a couple of systems of the KdV equations and their hyperbolic type
- Goldstein and Petrich (1991) The mKdV hierarchy
- Nakayama, Segur and Wadati (1992) The sine-Gordon equation
- Pinkall (1995) Pinkall's flow in centro-equiaffine geometry

- Terng and Uhlenbeck (1998-2008) Grassmanian and Kähler manifolds, adjoint orbits
- Chou and Qu (2002, 2003) Integrable planar geometric flows in Klein geomery
- Myrzakulov (1997-2000) Motion of curves by endowing an extra spatial variable
- Beffa, Sanders and Jingping (2000) N-dimensional Riemannian manifold with constant curvature
- Anco et al (2006-) Riemannian symmetric space, homogeneous space
- Beffa (2004-) Conformal geometry, homogeneous spaces
- Beffa (2009) Hamiltonian evolutions of curves in classical affine geometries
- Kamran, Olver and Tenenblat (2009) Local affine symplectic invariants for curves

- Olver (2010) Moving frames and differential invariants in centroaffine geometry
- Li and Qu, (2010); Musso, (2012) Curve flows in projective goemetry
- Inoguchi, Kajiwara, Ohta, Hirose, Matsuura (2010–), Discrete integrable curve flows in goemetries
- Calini, Ivey, Beffa (2013) Integable flows for starlike curves in centroaffine space
- Musso (2009) Variational problems for plane curves in centro-affine geometry
- Asadi and Sanders (2009) Integrable systems in symplectic geometry
- Terng, Wu (2015-2020) Gelfand-Dickey hierarchy
- Camassa-Holm type equations (Chou, Qu, 2002; Song, Qu, Yao, 2013; Gui, Liu, Olver, Qu, 2013; Qu, Fu, Liu, 2014)

◇ Invariant differential equations

Definition 1.1. Let G be a Lie group and let $H \subset G$ be a closed subgroup such that G/H is connected. The pair (G, H) is called a *Klein geometry*.

Definition 1.2. A differential invariant of G is a n-th smooth function $I : J^n \to R$ which satisfies $I(g^{(n)}(x, u, \cdots, u^{(n)})) = I(x, u, \cdots, u^{(n)})$ for all $g^{(n)} \in G^{(n)}$ and $(x, u^{(n)}) \in J^n$, where $g^{(n)}(x, u^{(n)})$ denotes the nth prolongation of g.

Consider the evolution equation

$$u_t = K(x, u, u_x, \cdots, u^{(n)}). \tag{1}$$

Theorem 1.3. (Olver, 1993) Let G be a transformation group acting on $E \simeq X \times U$. Suppose $L(x, u^{(n)})$ is a G-Lagrangian such that $E(L) \neq 0$. Then every G-invariant evolution equation has the form

$$u_t = \frac{L}{E(L)} I,$$

where I is an arbitrary differential invariant of G.

Theorem 1.4. (Olver, Sapiro and Tannenbaum, 1994) Let G be a subgroup of projective group $SL(3)$. Let $ds = Ldx$ denote the G-invariant one-form of lowest-order and I its fundamental differential invariant. Then every G-invariant evolution equation has the form

$$u_t = \frac{u_{xx}}{L^2} J,$$

where J is an arbitrary differential invariant of G.

◇ Invariant geometric flows

For a submanifold M, Let α_i, $i = 1, 2, \cdots, n$ be its frame. A submanifold flow is governed by

$$\frac{\partial \gamma}{\partial t} = \sum_{i=1}^{n} f_i \alpha_i,$$

where f_i are the velocities of the flow depending on the curvatures of the curve and its derivatives with respect to the arclength. For the plane curve, the flow is governed by

$$\frac{\partial \gamma}{\partial t} = fN + gT, \tag{2}$$

where f and g are normal and tangent velocities, respectively.

2. Integrable flows in Euclidean geometry

- The mKdV curve flow Consider the flow in Euclidean geometry

$$\gamma_t = \tilde{f} n + \tilde{g} t$$

The Serret-Frenet formulas reads

$$t_s = kn, \quad n_s = -kt,$$

where s is the arc-length, defined by $ds = hdp$, p is a free parameter. By the formula

$$h_t = h(\tilde{g}_s - k\tilde{f}),$$

the first variation formula for the Euclidean perimeter $L = \oint hdp$:

$$\frac{\partial L}{\partial t} = \oint_\gamma (\tilde{g}_s - k\tilde{f}) ds.$$

So, we require

$$\tilde{g} = \partial_s^{-1}(k\tilde{f})$$

and

$$\oint_\gamma k\tilde{f}ds = 0.$$

Then the curvature k satisfies the following equation

$$k_t = \tilde{f}_{ss} + k_s\tilde{g} + k^2\tilde{f} \equiv \Omega\tilde{f},$$

where

$$\Omega = D_s^2 + k^2 + k_s\partial_s^{-1}k.$$

Setting $\tilde{f} = \Omega^{n-1}k_s$, we get the mKdV hierarchy

$$k_t = \tilde{f}_{ss} + k_s\tilde{g} + k^2\tilde{f} \equiv \Omega^n\tilde{f}.$$

(Goldstein, Petrich, 1993, PRL)

Set $f = -2u_s$, $\kappa = m \equiv u - u_{ss}$. Then $g = -(u^2 - u_s^2) + b$, b is a constant. Hence $u(t,s)$ satisfies the equation

$$m_t + ((u^2 - u_s^2)m)_s + (b+2)u_{sss} - bu_s = 0. \tag{3}$$

After the transformations $t \to t$, $s \to x = s + (b+2)t$, then (3) becomes

$$m_t + ((u^2 - u_x^2)m)_x + 2u_x = 0, \quad m = u - u_{xx},$$

which is equivalent to the geometric flow in R^2

$$\gamma_t = u_s N + \frac{1}{2}(u^2 - u_s^2)T.$$

(Gui, Liu, Olver, Qu, 2013, CMP)

Remark 2.1. The modified μ-Camassa-Holm equation

$$m_t + \frac{1}{2}\Big((2\mu(u)u - u_s^2)m\Big)_s + u_s = 0, \quad m = \mu(u) - u_{ss}.$$

is equivalent to the geometric flow in R^2

$$\gamma_t = u_s N + \frac{1}{2}(2\mu(u)u - u_s^2)T.$$

(Qu, Fu, Liu, 2014, JFA)

3. Integrable flows in $CEA(2, R)$

- Two-dimensional centro-equiaffine geometry $CEA(2, R)$

The centro-equiaffine geometry $CEA(2, R)$ is determined via the centro-equiaffine transformation:

$$\begin{pmatrix} x' \\ u' \end{pmatrix} = A \begin{pmatrix} x \\ u \end{pmatrix},$$

where $A \in SL(2, R)$. Let $\gamma(p) = (\gamma_1(p), \gamma_2(p))$. The centro-affine arclength $d\tilde{s}$ is

$$d\tilde{s} = [\gamma, \gamma_p]dp = hds,$$

where s is the Euclidean arc-length element.

The centro-equiaffine curvature ϕ is

$$\phi = [\gamma_{\tilde{s}}, \gamma_{\tilde{s}\tilde{s}}].$$

The centro-equiaffine tangent and normal vectors: $T = \gamma_{\tilde{s}}$ and $N = -\gamma$. The centro-equiaffine Serret-Frenet formulas

$$T_{\tilde{s}} = \phi N, \quad N_{\tilde{s}} = -T.$$

- The KdV flow

The Pinkall's flow (Pinkall, 1995)

$$\gamma_t = \phi_{\tilde{s}} N + \phi T.$$

gives the KdV equation

$$\phi_t + \phi_{\tilde{s}\tilde{s}\tilde{s}} + 6\phi\phi_{\tilde{s}} = 0$$

- The CH flow

$$m_t + 4u_{\tilde{s}}m + 2um_{\tilde{s}} + u_{\tilde{s}\tilde{s}\tilde{s}} = 0, \quad \phi = m = u - u_{\tilde{s}\tilde{s}}.$$

The corresponding geometric flow:

$$\gamma_t = -u_{\tilde{s}}N - 2uT.$$

(Chou, Qu, 2002)

Now we examine the traveling wave solutions of the KdV equation. Let $q = \phi(z)$, $z = x - ct$, $x = \tilde{s}$, be such a solution. It satisfies

$$q_{zz} - 3q^2 + 4cq - D = 0,$$

and

$$\frac{1}{2}q_z^2 = q^3 - 2cq^2 + Dq + E,$$

for constants D and E. Only when the equation $q^3 - 2cq^2 + Dq + E$ has three real roots, the solution is bounded. Let's consider the generic case that there are three distinct roots c_1, c_2 and c_3, $c_1 < c_2 < c_3$ first. The solution q is given by

$$q = q_0 + A\mathrm{cn}^2 az, \quad z = x - ct,$$

where

$$q_0 = \frac{1}{6}c_2, \quad A = \frac{1}{6}(c_3 - c_2), \quad c = \frac{1}{3}(c_1 + c_2 + c_3),$$
$$a^2 = \frac{1}{24}(c_3 - c_1), \quad k^2 = \frac{c_3 - c_2}{c_3 - c_1},$$

and k is the modulus of the Jacobi elliptic function cn z. The (real) period of cn z is given by $2K$ where

$$K = \int_0^{\pi/2} \frac{d\sigma}{\sqrt{1 - k^2 \sin^2 \sigma}}.$$

Given a solution q of period $\Lambda = \sqrt{2}K/a$, we determine the solution curve whose centro-affine curvature is q.

Let λ_0 be the value such that all solutions of the equation

$$y'' + (\lambda - 2k^2\mathrm{sn}^2 x)y = 0$$

are bounded in x for all $\lambda \geq \lambda_0$.

Theorem 3.1. For any $m \geq 3$, there exist infinitely many $\lambda_j \geq \lambda_0$ such that the corresponding curve $\hat{\gamma}$ is closed and has m-leaves.

- Geometrical Hamiltonian structure

Theorem 3.2. (Calini, Ivey, Beffa, 2009) The following compatible Poisson structures on $\mathcal{L}g^*$, defined by

$$\{\mathcal{H},\mathcal{G}\}(L) = \int_{S^1} \operatorname{trace}\left(\left((\frac{\delta H}{\delta L})_x + \left[L, \frac{\delta H}{\delta L}\right]\right)\frac{\delta \mathcal{G}}{\delta L}\right) dx,$$

$$\{\mathcal{H},\mathcal{G}_0\}(L) = \int_{S^1} \operatorname{trace}\left([A, \frac{\delta H}{\delta L}]\frac{\delta \mathcal{G}}{\delta L}\right) dx, \qquad (4)$$

where $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ in the projective case, can be reduced to the quotient M/LN to produce the second and first Hamiltonian structures, respectively, for KdV equation.

They are equivalent respectively to

$$\{h, f\}(k) = \int_{S^1} \frac{\delta f}{\delta k}\left(-\frac{1}{2}D^3 - kD - Dk\right)\frac{\delta h}{\delta k}dx,$$

and

$$\{h, f\}_0(k) = 2\int_{S^1} \frac{\delta f}{\delta k} D \frac{\delta h}{\delta k}dx.$$

4. Curve flows in RP^1

As mentioned in (P. Olver, Equivalence, Invariants and Symmetry, Cambridge Univ. Press, 1995), there are three inequivalent geometries in the real line, which are described by its Lie group determined by the following corresponding algebras:

$$E(1): V_1 = \partial_x; \;\; Curvature: \kappa = x';$$

$$S(1): V_1 = \partial_x, \;\; V_2 = x\partial_x; \;\; Curvature: \kappa = \frac{x''}{x'};$$

$$P(1): V_1 = \partial_x, \;\; V_2 = x\partial_x, \;\; V_3 = x^2\partial_x;$$

$$Curvature: \kappa = (\frac{x''}{x'})' - \frac{1}{2}(\frac{x''}{x'})^2.$$

• The Burgers flow in $S(1)$

Consider the flow in the similarity geometry $S(1)$:

$$\frac{\partial x}{\partial t}(p,t) = F(\chi)\chi_p, \tag{5}$$

where the curvature $\chi = x_{pp}/x_p$. We obtain the equation for the curvature χ:

$$\chi_t = (\partial_p + \chi + \chi_p \partial_p^{-1}) F_p.$$

Let $F = \chi$, then we get the Burgers equation

$$\chi_t = \chi'' + 2\chi\chi',$$

where $'$ denotes the derivative with respect to p.

• The KdV flow in $P(1)$

Consider the flow in the geometry $P(1)$:

$$\frac{\partial x}{\partial t}(p,t) = F(\mathcal{S})\chi_p, \tag{6}$$

where the curvature $\mathcal{S} = (x_{pp}/x_p)_p - \frac{1}{2}(x_{pp}/x_p)^2$. In this case, we obtain the equation for the curvature $\mathcal{S}$:

$$\mathcal{S}_t = (\partial_p^2 + 2\mathcal{S} + \mathcal{S}_p\partial_p^{-1})F_p.$$

Let $F = \mathcal{S}$, then we get the KdV equation.

$$\mathcal{S}_t = \mathcal{S}''' + 3\mathcal{S}\mathcal{S}'.$$

(Chou, Qu, JNS, 20003)

• The relationship between the flows in RP^1 and $CEA(2,R)$

Let's review the geometry of regular parameterized curves in RP^1. Let γ: $I \to R^2$ be a curve in R^2, $\pi : R^2 \to RP^1$, $\phi = \pi \circ \gamma : I \to RP^1$. We now establish the relationship between the invariant curve flows in imprimitive cetro-equiaffine geometry and one-dimensional projective space.

Proposition 4.1. The Maurer-Cartan matrix is given by

$$K = \begin{pmatrix} \lambda & 1 \\ k-\lambda^2 & -\lambda \end{pmatrix} \tag{7}$$

where $k = \frac{1}{2}\mathcal{S}(u)$, where $k = -\frac{1}{2}\mathcal{S}(u)$ is the projection curvature of ϕ, $\mathcal{S}(u) = (\frac{f''}{f'})' - \frac{1}{2}(\frac{f''}{f'})^2$ is the Schwartzian derivative.

Let $\gamma = (\gamma_1, \gamma_2)^T$ and assume that $\gamma_1 \neq 0$ on an open interval. The projection ϕ of γ is a given by $u = \frac{\gamma_2}{\gamma_1}$. The general flow for the parametrized planar curves $\gamma(x)$ is invariant under the centro-equiaffine action is of the form

$$\gamma_t = r_1\gamma + r_2\gamma_s, \tag{8}$$

where r_1 and r_2 are differential invariants, s is the centro-equiaffine arc-length, which is invariant under the flow, one requires

$$2r_1 + r_{2,s} = 0.$$

So that the flow reduces to the Pinkall's flow

$$\gamma_t = \frac{1}{2}r_s\gamma + r\gamma_s \tag{9}$$

for a differential invariant r. Indeed, we have

Proposition 4.2. Let $\gamma(x)$ be a star-shaped curve parametrized by centro-affine arc-length, and let $u(x)$ be its projective ration in an affine chart. Then the centro-affine curvature satisfies $p(x) = \frac{1}{2}\mathcal{S}(u)$. if γ evolves by (9), then $u = \frac{\gamma_2}{\gamma_1}$ satisfies the flow (6)

$$u_t = ru'. \tag{10}$$

In particular, if γ evolves by (9), then u fulfills the Schwartzian KdV equation. (Chou, Qu, 2003; Calini, Ivey, Beffa, 2009)

5. Affine geometry

The geometry is invariant under the unimodular affine transformations

$$\begin{pmatrix} x' \\ u' \end{pmatrix} = A \begin{pmatrix} x \\ u \end{pmatrix} + B,$$

where $A \in SL(2, R)$, $B \in R^2$.

The affine Serret-Frenet formulas:

$$\begin{pmatrix} T \\ N \end{pmatrix}_\rho = \begin{pmatrix} 0 & 1 \\ -\mu & 0 \end{pmatrix} \begin{pmatrix} T \\ N \end{pmatrix}.$$

The affine perimeter evolution:

$$\frac{dL}{dt} = \oint_\gamma \left(\frac{k_t}{3k} + \frac{s_t}{s}\right) d\rho, \\ = \oint \left(\frac{1}{3} f_{\rho\rho} - \frac{2}{3}\mu f + g_\rho\right) d\rho.$$

Impose

$$\oint \mu f d\rho = 0,$$

and

$$g = -\frac{1}{3} f_\rho + \frac{2}{3}\partial_\rho^{-1}(\mu f).$$

Also

$$\begin{pmatrix} T \\ N \end{pmatrix}_t = \begin{pmatrix} g_\rho - \mu f & f_\rho + g \\ H_1 & H_2 \end{pmatrix} \begin{pmatrix} T \\ N \end{pmatrix},$$

where $H_1 = g_{\rho\rho} - 2\mu f_\rho - \mu_\rho f - \mu g$ and $H_2 = f_{\rho\rho} + 2g_\rho - \mu f$.

The compatibility condition $\Rightarrow$

$$\begin{aligned}\mu_t &= \frac{1}{3}(D_\rho^4 + 5\mu D_\rho^2 + 4\mu_\rho D_\rho \\ &\quad + \mu_{\rho\rho} + 4\mu^2 + 2\mu_\rho \partial_\rho^{-1}\mu) f.\end{aligned}$$

$f = -3\mu_\rho \Rightarrow$ SK equation

$$\mu_t + \mu_5 + 5\mu\mu_3 + 5\mu_1\mu_2 + 5\mu^2\mu_1 = 0.$$

In general, $f = -3(D_\rho^2 + \mu + \mu_\rho\partial_\rho^{-1})u$, $u = \Omega^{n-1}(\mu)\mu_\rho$, $\Rightarrow$ Sawada-Kotera hierarchy.

$$\mu_t = -\Omega^n(\mu)\mu_\rho.$$

(Chou, Qu, 2002; Olver 2009)

This is based on the operator identity:

$$\begin{aligned}\Omega = &(\partial_y^3 + 2Q\partial_y + 2\partial_y Q) \cdot (2\partial_y^3 + 2\partial_y^2 Q \partial_y^{-1} \\ &+2\partial_y^{-1} Q\partial_y^2 + Q^2 \partial_y^{-1} + \partial_y^{-1} Q^2) \\ &= 2(\partial_y^4 + 5Q\partial_y^2 + 4Q_y\partial_y + Q_{yy} + 4Q^2 \\ &\quad +2Q_y\partial_y^{-1}Q)\left(\partial_y^2 + Q + Q_y\partial_y^{-1}\right).\end{aligned}$$

6. Centro-affine geometry $CA(2,R)$

The geometry is invariant under the centro-affine transformation

$$\begin{pmatrix} x' \\ u' \end{pmatrix} = A \begin{pmatrix} x \\ u \end{pmatrix},$$

where $A \in GL(2,R)$. The arc-length s and curvature κ in centro-affine geometry are given respectively by

$$ds = u^{-1} Q_2^{\frac{1}{2}} dx,$$

and

$$\kappa = Q_2^{-\frac{3}{2}} (u^2 u_3 - 9 u u_1 u_2 + 12 u_1^3),$$

where

$$Q_{k+2} = (k+1)u_k u_{k+2} - (k+2)u_{k+1}^2.$$

Consider the flow

$$\gamma_t = FN + GT,$$

where the tangent and normal are respectively

$$T = uQ_3^{-\frac{1}{2}}(1, u_x) \quad \text{and} \quad N = u(0, 1).$$

The inextensibility condition means

$$\oint (G_s + \frac{1}{2}F_{ss} + \frac{1}{4}\kappa F_s)ds = 0.$$

So we require

$$\oint \kappa_s F ds = 0$$

and

$$G = -\frac{1}{2}F_s - \frac{1}{4}\partial_s^{-1}(\kappa F_s).$$

The equation for the curvature κ is

$$\kappa_t = (\partial_s^2 - \frac{1}{4}\kappa^2 - \frac{1}{4}\kappa_s\partial^{-1}\kappa - 4)F. \tag{11}$$

If $F = k_s$, we get the defocusing mKdV equation

$$\kappa_t = \kappa_{sss} - \frac{3}{8}\kappa^2 k_s - 4\kappa_s. \tag{12}$$

(Chou,Qu, JNS, 2003; Park, Kajiwara, Kurose, Matsuura, 2018)

By a straightforward computation, we find that (12) is related to the KdV equation

$$\phi_t = \phi_{sss} + 3\phi\phi_s$$

by the Miura transformation

$$\phi = -\frac{1}{2}\kappa_s - \frac{1}{8}\kappa^2 - 2,$$

which relates the flows in $P(1)$ and centro-affine geometry $CA(2, R)$. (Chou,Qu, JNS, 2003)

7. Centro-equiaffine symplectic geometry

- Centro-equiaffine symplectic geometry for curves

The symplectic space $(M = \mathbf{R}^{2n}, \Omega)$ is the vector space $\mathbf{R}^{2n}$ endowed with the standard symplectic form Ω, given by $\Omega = \sum_{i=1}^{n} dx^i \wedge dy^i$. Each tangent space is endowed with the symplectic inner product $< u; v >$. The symplectic group $\mathrm{Sp}(2n, \mathbf{R})$ is the subgroup of $\mathrm{GL}(2n, R)$:

$$\mathrm{Sp}(2n, \mathbf{R}) = \{M \in GL(2n, \mathbf{R}) : JMJ^t = M\},$$

where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

◇ Moving coframe method

Definition 7.1. (Fels and Olver, 98,99) Given a transformation group G acting on a manifold M, a moving frame ρ is a smooth G-equivariant map

$$\rho : M \to G.$$

Lemma 7.2. (Fels and Olver, 98,99) If G acts on M, then a moving frame exists in a neighborhood of a point $x \in M$ if and only if G acts freely and regularly near x.

Lemma 7.3. If $\rho(x)$ is a moving frame, then the components of the map I: $M \to M$ defined by $I(x) = \rho(x) \cdot x$ provide a complete set of invariants for the group.

A symplectic frame is an ordered basis of vectors $\{a_1, \cdots, a_n, b_1, \cdots, b_n\}$ such that

$$\begin{gathered} < a_i; a_j > = < b_i; b_j > = 0, \quad 1 \leq i, j \leq n, \\ < a_i; b_j > = 0, \quad 1 \leq i \neq j \leq n, \\ < a_i; b_i > = 1, \quad 1 \leq i \leq n. \end{gathered}$$

Here the rigid motion $T \in \mathrm{Sp}(2n, \mathbf{R})$ is the linear transformation acting on the points $u \in \mathbf{R}^{2n}$ via

$$u \to Tu.$$

Definition 7.4. A curve $u(p) : I \to \mathbf{R}^{2n}$ is said to be a star-shaped centro-equiaffine symplectic curve if

$$\hat{K}_0 = < u; u' > \neq 0.$$

Definition 7.5. A curve $u(p) : I \to R^{2n}$ is said to be a regular star-shaped centro-equiaffine symplectic curve if

$$\hat{K}_i =< u_i; u_{i+1} >\neq 0, \quad i = 0, 1, \cdots, 2n-1.$$

Definition 7.6. (Song, Qu, Physica D, 2012) The arc-length s for a star-shaped curve in centro-equiaffine symplectic geometry is defined by

$$s = \int_{p_0}^{p} < u; u_p > dp.$$

A curve is parametrized by the arc-length parameter s iff it satisfies

$$< u; u_s >= 1.$$

- Centro-equiaffine symplectic frames and Frenet formulae

For any such curve, an adapted symplectic frame is $(a_1, \cdots, a_n, a_{n+1}, \cdots, a_{2n})$.
The Frenet formulae is

$$\begin{pmatrix} a_1 \\ \vdots \\ a_{2n} \end{pmatrix}_s = B \begin{pmatrix} a_1 \\ \vdots \\ a_{2n} \end{pmatrix}, \tag{13}$$

with

$$B = \begin{pmatrix} & \begin{matrix} 1 & & & & \\ & H_2 & & & \\ & & H_3 & & \\ & & & \ddots & \\ & & & & H_n \end{matrix} \\ \begin{matrix} K_1 & 1 & & & \\ 1 & K_2 & 1 & & \\ & 1 & K_3 & 1 & \\ & & \ddots & \ddots & 1 \\ & & & 1 & K_n \end{matrix} & \end{pmatrix}.$$

The existence and uniqueness theorem for centro-equiaffine symplectic regular curves.

Theorem 7.7. Let $H_2, \cdots, H_n$ and $K_1, \cdots, K_n$ be $2n-1$ smooth real valued functions defined on an interval with $H_j \neq 0$ for all $2 \leq j \leq n$. There exists, up to a centro-equiaffine symplectic motion, a unique regular symplectic curve $u : I \to \mathbf{R}^{2n}$, parametrized by the centro-equiaffine symplectic arclength, whose local centro-equiaffine symplectic invariants are the given functions $H_2, \cdots, H_n$ and $K_1, \cdots, K_n$.

- Gauge transformation and natural frame

Theorem 7.8. Let $\{a_1, \cdots, a_n, a_{n+1}, \cdots, a_{2n}\}$ be Frenet frame along a regular star-shaped symplectic curve. Then there exists another symplectic frame $\{\tilde{a}_1, \cdots, \tilde{a}_n, \tilde{a}_{n+1}, \cdots, \tilde{a}_{2n}\}$, which are called the natural frame, obtained by a symplectic transformation T:

$$\begin{pmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_n \\ \tilde{a}_{n+1} \\ \vdots \\ \tilde{a}_{2n} \end{pmatrix} = \begin{pmatrix} 1 & & & & & & \\ & \frac{1}{\sqrt{\epsilon_2 H_2}} & & & & & \\ & & \ddots & & & & \\ & & & \frac{1}{\sqrt{\epsilon_n H_n}} & & & \\ 0 & & & & 1 & & \\ & (\frac{1}{\sqrt{\epsilon_2 H_2}})' & & & & \sqrt{\epsilon_2 H_2} & & \\ & & \ddots & & & & \ddots & \\ & & & (\frac{1}{\sqrt{\epsilon_n H_n}})' & & & & \sqrt{\epsilon_n H_n} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \\ a_{n+1} \\ \vdots \\ a_{2n} \end{pmatrix},$$

where $\epsilon_i = \mathrm{sgn}(\mathrm{H_i})$ ($i = 1, 2, \cdots, n$), and the associated structure equation of the natural frame $\{\tilde{a}_1, \cdots, \tilde{a}_n, \tilde{a}_{n+1}, \cdots, \tilde{a}_{2n}\}$ is given by

$$\begin{pmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_n \\ \tilde{a}_{n+1} \\ \vdots \\ \tilde{a}_{2n} \end{pmatrix}_s = \begin{pmatrix} & & & & 1 & & & \\ & & & & & \epsilon_2 & & \\ & & & & & & \ddots & \\ & & & & & & & \epsilon_n \\ \tilde{K}_1 & \tilde{P}_1 & & & & & & \\ \tilde{P}_1 & \tilde{K}_2 & \ddots & & & & & \\ & \ddots & \ddots & \tilde{P}_n & & & & \\ & & \tilde{P}_n & \tilde{K}_n & & & & \end{pmatrix} \begin{pmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_n \\ \tilde{a}_{n+1} \\ \vdots \\ \tilde{a}_{2n} \end{pmatrix},$$

where $\tilde{K}_i$ and $\tilde{P}_i$ are centro-equiaffine symplectic invariants satisfying

$$\begin{aligned} \tilde{P}_i &= \sqrt{|H_i H_{i+1}|} = \sqrt{\epsilon_i \epsilon_{i+1} H_i H_{i+1}}, \quad 1 \leq i \leq n-1, \\ \tilde{K}_i &= \epsilon_i [H_i K_i - \frac{1}{2}(\frac{H_{i,ss}}{H_i} - \frac{3}{2}\frac{H_{i,s}^2}{H_i^2})], \quad 1 \leq i \leq n. \end{aligned} \tag{14}$$

• Integrable matrix-KdV flows

Consider the non-stretching geometric flows for a regular star-shaped curve in four-dimensional symplectic space with non-degenerate curvature $H_2 \neq 0$.

Let $u(s,t)$ be a regular star-shaped curve parametrized by the centro-equiaffine symplectic arclength s. The natural structure equation is

$$\begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{b}_1 \\ \tilde{b}_2 \end{pmatrix}_s = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \epsilon_2 \\ \tilde{K}_1 & \tilde{P} & 0 & 0 \\ \tilde{P} & \tilde{K}_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{b}_1 \\ \tilde{b}_2 \end{pmatrix},$$

where $(\tilde{a}_1 = u, \tilde{a}_2, \tilde{b}_1 = u_s, \tilde{b}_2)$ is the natural symplectic frame.

The general curve flow is

$$u_t = f_1\tilde{a}_1 + f_2\tilde{a}_2 + g_1\tilde{b}_1 + g_2\tilde{b}_2, \tag{15}$$

where f_1, f_2, g_1 and g_2 are functions depending on $\tilde{K}_1$, $\tilde{K}_2$ and $\tilde{P}$ and their derivatives with respect to s. Furthermore, assume that the induced evolution of the frame is governed by

$$\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix}_t = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix}, \tag{16}$$

where A, $B = B^t$, $C = C^t$ are the matrix with entries given by centro-equiaffine symplectic invariants. The flow is intrinsic gives rise to the system

$$\begin{aligned} &A_s = EC - B\tilde{K}, \quad A_s^t = CE - \tilde{K}B, \\ &B_s = -(EA^t + AE), \quad \tilde{K}_t - C_s + \tilde{K}A + A^t\tilde{K} = 0. \end{aligned}$$

Thus we have

$$\tilde{K}_t = -\frac{1}{2}EB_{sss}E + EB_s\tilde{K} + \tilde{K}B_sE + \frac{1}{2}EB\tilde{K}_s + \frac{1}{2}\tilde{K}_sBE$$
$$+\frac{1}{2}\tilde{K}\partial_s^{-1}(B\tilde{K} - E\tilde{K}BE) + \frac{1}{2}\partial_s^{-1}(\tilde{K}B - EB\tilde{K}E)\tilde{K}. \qquad (17)$$

Now, we consider two cases:

Case 1. $H_2 > 0$. In this case, we get

$$\tilde{K}_t = -\frac{1}{2}B_{sss} + \{\tilde{K}, B_s\} + \frac{1}{2}\{\tilde{K}_s, B\} - \frac{1}{2}[\tilde{K}, \partial_s^{-1}[\tilde{K}, B]].$$

Use the notations

$$\Gamma_A B = \{A, B\} \triangleq AB + BA, \quad \Lambda_A B = [A, B] \triangleq AB - BA.$$

It implies that

$$\tilde{K}_t = -\frac{1}{2}(\partial_s^3 - \Gamma_{\tilde{K}}\partial_s - \partial_s\Gamma_{\tilde{K}} + \Lambda_{\tilde{K}}\partial_s^{-1}\Lambda_{\tilde{K}})B.$$

Setting $B = -2\tilde{K}$, we get the matrix KdV equation (Olver and Sokolov, 1998)

$$\tilde{K}_t = \tilde{K}_{sss} - 3\{\tilde{K}, \tilde{K}_s\}. \tag{18}$$

So the corresponding flow is

$$u_t = \tilde{K}_{1s}a_1 - 2\tilde{K}_1 b_1 + \tilde{P}_s a_2 - 2\tilde{P} b_2. \tag{19}$$

This is a generalization of the Pinkall's flow in the centro-equiaffine geometry. The recursion operator of the matrix KdV is

$$R = \epsilon_2 \circ \epsilon_1^{-1} = \partial_s^2 - 2\Gamma_{\tilde{K}} - \Gamma_{\tilde{K}_s}\partial_s^{-1} + \Lambda_{\tilde{K}}\partial_s^{-1}\Lambda_{\tilde{K}}\partial_s^{-1}.$$

(Song, Qu, 2012, Physica D)

Case 2. $H_2 < 0$, Namely, $\epsilon_2 = -1$. Taking $B = -2E\tilde{K}E$ in (17), we obtain a Jordan KdV system

$$\begin{aligned} \tilde{K}_t &= \tilde{K}_{sss} - 3\tilde{K}_s E\tilde{K} - 3\tilde{K}E\tilde{K}_s \\ &= \tilde{K}_{sss} - 3\{\tilde{K}, E, \tilde{K}_s\}. \end{aligned} \tag{20}$$

The corresponding geometric flow is

$$u_t = \tilde{K}_{1s}\tilde{a}_1 - 2\tilde{K}_1\tilde{b}_1 + \tilde{P}_s\tilde{a}_2 + 2\tilde{P}\tilde{b}_2.$$

Define $A * B = AEB$, then the system (20) becomes

$$\tilde{K}_t = \tilde{K}_{sss} - 3\tilde{K} * \tilde{K}_s - 3\tilde{K}_s * \tilde{K}.$$

(Olver, Sokolov, 1998)

In general, setting $B=-2EB_0E$, (17) becomes

$$\begin{aligned}\tilde{K}_t = {} & B_{0sss}-2B_{0s}E\tilde{K}-2\tilde{K}EB_{0s}-B_0E\tilde{K}_s-\tilde{K}_sEB_0 \\ & +\tilde{K}\partial_s^{-1}(E\tilde{K}EB_0-EB_0E\tilde{K})-\partial_s^{-1}(\tilde{K}EB_0E-B_0E\tilde{K}E)\tilde{K},\end{aligned}$$

that is

$$\tilde{K}_t=(\partial_s^3-2\Gamma^*_{\tilde{K}}\partial_s-\Gamma^*_{\tilde{K}_s}+\Lambda^*_{\tilde{K}}\partial_s^{-1}\Lambda^*_{\tilde{K}})B_0\equiv D_2(B_0), \qquad (21)$$

where

$$\Gamma^*_A B\triangleq AEB+BEA,\quad \Lambda^*_A B\triangleq AEB-BEA,$$
$$D_2=\partial_s^3-2\Gamma^*_{\tilde{K}}\partial_s-\Gamma^*_{\tilde{K}_s}+\Lambda^*_{\tilde{K}}\partial_s^{-1}\Lambda^*_{\tilde{K}}.$$

The recursion operator is

$$R_E=D_2\cdot D_1^{-1}=\partial_s^2-2\Gamma_{\tilde{K}}-\Gamma_{\tilde{K},s}\partial_s^{-1}+\Lambda_{\tilde{K}}*\partial_s^{-1}*\Lambda_{\tilde{K}}\partial_s^{-1}.$$

(Song, Qu, 2012, Physica D)

8. Integrable affine symplectic curve flows

- Curves in affine symplectic geometry

A symplectic frame is an ordered basis of vectors $\{a_1, \cdots, a_n, b_1, \cdots, b_n\}$ such that

$$\begin{aligned} < a_i; a_j >=< b_i; b_j >= 0, \quad 1 \leq i, j \leq n, \\ < a_i; b_j >= 0, \quad 1 \leq i \neq j \leq n, \\ < a_i; b_i >= 1, \quad 1 \leq i \leq n. \end{aligned}$$

This frame can be determined by the Gram-Schmidt algorithm or the moving frame method.

The structure equations:

$$da_i = \sum_k \omega_{ik} a_k + \sum_k \theta_{ik} b_k,$$
$$db_i = \sum_k \phi_{ik} a_k - \sum_k \omega_{ki} b_k, \quad 1 \le i \le n,$$

where ω_{ik}, θ_{ik} and ϕ_{ik} satisfy

$$\theta_{ij} = \theta_{ji}, \quad \phi_{ij} = \phi_{ji}, \quad 1 \le i, j \le n.$$

Thus the matrix valued 1-from

$$\Theta = \begin{pmatrix} \omega & \theta \\ \phi & -\omega^t \end{pmatrix}$$

takes the value in the Lie algebra $sp(2n, R)$. Here the rigid motion $T \in \mathrm{Sp}(2n, R) \ltimes R^n$ is the ordinary linear transformation acting on the points $u \in R^{2n}$ via

$$u \to Tu + B.$$

Definition 8.1. (Kamran, Olver, Tenenblat, 2009) A curve $u(x) : I \to R^{2n}$ is said to be a star-shaped centro-equiaffine symplectic curve if

$$\hat{K}_0 =< u_x; u_{xx} >\neq 0.$$

Definition 8.2. A curve $u(x) : I \to R^{2n}$ is said to be a regular star-shaped centro-equiaffine symplectic curve if

$$\hat{K}_i =< u_i; u_{i+1} >\neq 0, \quad i = 0, 1, \cdots, 2n - 1.$$

A curve is parametrized by the arc-length parameter if and only if it satisfies

$$< u_s; u_{ss} >= 1.$$

Definition 8.3. (Kamran, Olver, Teneblat, 2009) The affine symplectic arc-length parameter s is defined by

$$s = \int_{x_0}^{x} < u_x; u_{xx} >^{\frac{1}{3}} dx.$$

- Affine symplectic frames and Frenet formulae

For any such curve, we can associate an adapted symplectic frame $(a_1, \cdots, a_n, a_{n+1}, \cdots, a_{2n})$. They satisfy the Frenet formulae

$$\begin{pmatrix} a_1 \\ \vdots \\ a_{2n} \end{pmatrix}_s = B \begin{pmatrix} a_1 \\ \vdots \\ a_{2n} \end{pmatrix}, \tag{22}$$

with

$$B = \begin{pmatrix} & & & & & 1 & & & & \\ & & & & & & H_2 & & & \\ & & & & & & & H_3 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & H_n \\ K_1 & 1 & & & & & & & & \\ 1 & K_2 & 1 & & & & & & & \\ & 1 & K_3 & 1 & & & & & & \\ & & \ddots & \ddots & 1 & & & & & \\ & & & 1 & K_n & & & & & \end{pmatrix}.$$

• Integrable geometric flows

Consider the geometric flow in $AP(4, R)$ (Valiquette, 2012)

$$\frac{d\gamma}{dt} = V = \sum_{i=1}^{4} F_i a_i, \tag{23}$$

Valiquetue proved that:

Theorem 8.4. (Valiquette, 2012) An invariant geometric flow (23) preserves the sympletic arc-length provided if it is of the form

$$V = F_1 a_1 + F_2 a_2 + F_3 a_3 - \frac{1}{2}(3F_1' + F_3'' + 2F_3 I)a_4,$$

where I is an arbitrary differential invariant, where the curvatures H, I and J satisfy the system

$$\begin{aligned}
H_t &= (6HD + DH)\,V_1 + \left(2HC_{4,2} + DC_{7,2} + DH\right)V_2 \\
&\quad + \left(6HD^2 + 2JH + 2DHD + D^2H\right)V_3 \\
&\quad -\frac{1}{2}\left(2HC_{4,4} + DC_{7,4}\right)\left(3V_1' + V_3'' + 2V_3 I\right), \\
I_t &= \left(D^3 + DI - 2ID\right)V_1 + (2HD + 3DH)\,V_2 \\
&\quad + \left(2H - 2I^2 + D^2 I + DID\right)V_3 \\
&\quad - \left(2D^2 + HJ - I\right)\left(3V_1' + V_3'' + 2V_3 I\right), \\
J_t &= \left(DC_{10,1} - 6JD\right)V_1 + \left(DC_{10,2} - 2D - 2JC_{4,2}\right)V_2 \\
&\quad + \left(C_{10,3} - 2 - 6JD^2 - 2HJ^2\right)V_3 \\
&\quad -\frac{1}{2}\left(C_{10,4} + 2HJ + 2D^2 - 2J^2\right)\left(3V_1' + V_3'' + 2V_3 I\right),
\end{aligned} \tag{24}$$

where

$$
\begin{aligned}
C_{4,2} &= D^3 + D(JH) + JHD + JDH - ID, \\
C_{4,4} &= D^2H + D(JD) + HJ^2 + JD^2 - IJ + 1, \\
C_{7,2} &= HD^2 + JH^2 + D(HD) + D^2H - IH, \\
C_{7,4} &= HDJ + HJD + D(HJ) + D^3ID, \\
C_{10,1} &= \frac{1}{H}(6D^2 + JH), \\
C_{10,2} &= \frac{1}{H}(D^4 - D(ID) - IJH + D^2(JH) + D(JHD) \\
&\quad + D(JDH) + H + JHD^2 + J^2H^2 + JD(HD) + JD^H),
\end{aligned}
$$

$$\begin{aligned} C_{10,3} &= \frac{1}{H}(4D^3 + D(JH) + 2DI + 2JHD + JDH), \\ C_{10,4} &= \frac{1}{H}(D^3J - JDI - IDJ + D^2(JD) + 4D + D(HJ^2) \\ &\quad + D(JD^2) + JHDJ + HJ^2D + JD(HJ) + JD^3). \end{aligned}$$

Theorem 8.5. (Valiquette, 2012) The invariant curve flow in $AP(4,R)$

$$\gamma_t = a_1$$

gives the following bi-Hamiltonian system

$$K_t = \mathcal{P}_r \delta h_1 = \mathcal{P}_{\tilde{r}} \delta h_2, \tag{25}$$

with

$$h_1 = \int (\frac{1}{4} I^2 - H) dx,$$
$$h_2 = \int (\frac{1}{8} H^3 + \frac{1}{16} I'^2 + JH^2 - \frac{3}{2} HI) dx.$$

and

$$P_r = \begin{pmatrix} HP_1 - DP_2 & DH + 3HD & HP_3 - DP_4 \\ HD + 3DH & DI + ID - \frac{1}{2}D^3 & JD + \frac{3}{2}D^2(\frac{1}{H}D) - 3D \\ P_t & D(J + \frac{3}{2H}D^2) - 3JD & P_s \end{pmatrix},$$

$$P_{\tilde{r}} = \begin{pmatrix} -HD - DH & 0 & DJ - JD - D^2(\frac{1}{2H}D) \\ 0 & 2D & 0 \\ JD - DJ - D(\frac{1}{2H}D^2) & 0 & P_z \end{pmatrix},$$

$$
\begin{aligned}
P_t &= \left(D(\frac{1}{2H}D) - J\right)P_1 - D(\frac{J}{H})P_2, \\
P_s &= \left(D(\frac{1}{2H}D) - J\right)P_3 - D(\frac{J}{H})P_4, \\
p_z &= D\frac{J^2}{H} + \frac{J^2}{H}D - D\left(\frac{J}{2H}D\frac{1}{H}D\right) - D\left(\frac{1}{H}D(\frac{J}{2H}D)\right), \\
P_1 &= D^3 + JHD + 2JDH - ID + 2D(JH), \\
P_2 &= JH - D^2H - JH^2 - D(\frac{1}{2}HD) - \frac{1}{2}HD^2, \\
P_3 &= J^2D - 2JDJ - 2DJ^2 + (1 - IJ + JD^2 + D^2J + DJD)(\frac{1}{H}D), \\
P_4 &= (D^2 - I + JH)J + 1 - (D^3 + D(JH) + HDJ + JHD \\
&\qquad -ID)(\frac{1}{2H}D).
\end{aligned}
$$

Indeed, (25) can be written explicitly as

$$
\begin{aligned}
H_t = & -H''' + 3HI' + \frac{3}{2}IH' - 3(JH^2)', \\
I_t = & -\frac{1}{4}I''' + \frac{3}{2}II' - 3J', \\
J_t = & -J''' + 3J^2H' + \frac{3}{2}IJ' + \frac{3}{4}(\frac{I''}{J})' - 3(\frac{(H'J)'}{H})'.
\end{aligned}
$$

Question: How to classify the integrable systems (24)?

9. Heat flow in the affine geometry $A(2,R)$

Consider the heat flow in $A(2,R)$

$$\gamma_t = \gamma_{ss}.$$

In terms of the affine curvature ϕ and local graph $(x, u(x,t))$, the heat flow can be written respectively as the equations

$$\phi_t = \phi_{ss} + \phi^2, \tag{26}$$

$$u_t = u_{xx}^{\frac{1}{3}}, \tag{27}$$

where s is the arc-length parameter, θ is the angle between the tangent vector and a fixed direction.

Theorem 9.1. (Angenent, Sapiro, Tanenbaum, 1997) The curve governed by the affine heat flow shortening flow (26) preserves convexity and shrinks any closed convex curve to an elliptic point. Furthermore, the family of dilated normalized curves converges in the Hausdorff metric to an ellipse.

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◇ Exact solutions

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10. Heat flow in centro-equiaffine geometry

Consider the heat flow in $CEA(2, R)$

$$\gamma_t = \gamma_{ss}. \tag{28}$$

In terms of the centro-equiaffine curvature ϕ and the support function $h = -(\gamma, n)$, the heat flow can be written as

$$\phi_t = \phi_{ss} + 4\phi^2, \tag{29}$$

or

$$h_t = -\frac{1}{h^2(h_{\theta\theta} + h)}, \tag{30}$$

where s is the centro-equiaffine arc-length parameter, θ is the angle between the tangent vector and a fixed direction.

Theorem 10.1. (Wo, Wang, Qu, MZ, 2018) (The centro-equiaffine isoperimetric inequality) For any closed convex curve γ, there exists a point (x_0, y_0), such that the centro-affine isoperimetric inequality

$$A \int_\gamma \phi d\sigma \leq 2\pi^2, \tag{31}$$

holds, where A is area enclosed by the curve γ, ϕ is the centro-equiaffine curvature of γ with respect to (x_0, y_0) and equality holds if and only if γ is an ellipse. Furthermore, if the curve γ is origin symmetric, then inequality (31) also holds for $(x_0, y_0) = (0, 0)$.

Theorem 10.2. (Wo, Wang, Qu, 2018) The solution to the centro-equiaffine heat flow (28) with the initial value γ_0 continues until the area A enclosed by the curve $\gamma(t)$ goes to zero.

Let

$$\tilde{\gamma} = (\frac{\pi}{A})^{\frac{1}{2}}\gamma,$$

where A is the area of γ. In terms of the support function and Euclidean curvature,

$$\tilde{h} = (\frac{A}{\pi})^{-\frac{1}{2}}h,$$

and

$$\tilde{k} = (\frac{A}{\pi})^{\frac{1}{2}}k.$$

Theorem 11.3. (Wo, Wang, Qu, 2018) The normalized curve $\tilde{\gamma}$ converges smoothly to an ellipse centered at the origin.

11. Heat flow in the centro-affine geometry

Consider the heat flow in $CA(2, R)$ (Olver, Qu, Yang, 2020)

$$\gamma_t = \gamma_{ss}. \tag{32}$$

In terms of the centro-affine curvature κ, the heat flow (32) can be written as

$$\kappa_t = \kappa\kappa_s, \tag{33}$$

where s is the arc-length parameter.

Theorem 11.1. (Olver, Qu, Yang, SIGMA, 2020) The centro-affine curve evolution process

$$\frac{\partial\gamma}{\partial t} = \gamma_{ss}, \qquad \gamma(s,0) = \gamma_0,$$

is equivalent to the initial value problem of the inviscid Burgers' equation:

$$\frac{\partial\kappa}{\partial t} - \kappa\frac{\partial\kappa}{\partial s} = 0, \qquad \kappa(s,0) = \varphi(s),$$

where $\varphi(s)$ is the signed centro-affine curvature κ of the initial curve γ.

Theorem 11.2. (Olver, Qu, Yang, 2020) If the initial centro-affine curvature $\kappa_0(s) \equiv 0$, then at any time $t \geq 0$, the curve $\gamma(t)$ is centro-affine equivalent to the initial curve γ_0.

Theorem 11.3. (Olver, Qu, Yang, 2020) The curve family $C(\bar{p}, t)$ converges smoothly to the origins as $t \to \infty$.

Thank you!