On tau-functions for the KdV hierarchy

Di Yang, Hefei

University of Science and Technology of China

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The Korteweg – de Vries (KdV) equation, discovered in the study of the shallow water waves,

$$u_t = uu_x + \frac{1}{12}u_{xxx}$$

is an important integrable nonlinear evolution PDE, which extends to a hierarchy of pairwise commuting evolution PDEs

$$\frac{\partial L}{\partial t_k} = \frac{1}{(2k+1)!!}\Big[\big(L^{\frac{2k+1}{2}}\big)_+, L\Big], \quad k \geq 0$$

called the *KdV hierarchy*, where $L := \partial^2 + 2u$. We will identify x with t_0.

Pairwise commutativity for the KdV hierarchy implies that the whole hierarchy can be solved together. Partial goal: solve

the initial value problem for the KdV hierarchy

To be precise, let V be a ring of functions of x, closed under ∂_x. For a given $f(x) \in V$, the KdV hierarchy has a unique solution $u = u(\mathbf{t})$ in $V[[\mathbf{t}_{>0}]]$, s.t.

$$u(t_0 = x, 0, 0, \dots) = f(x)$$

This gives a one-to-one correspondence:

$$\{\text{solutions in } V[[\mathbf{t}_{>0}]]\} \quad \leftrightarrow \quad \{\text{initial data}\} = V$$

Example 1. $f(x) = x$.

Set $V = \mathbb{C}[[x]]$ (often). According to the celebrated Witten–Kontsevich theorem, the corresponding unique solution $u = u(\mathbf{t})$ in $V[[\mathbf{t}_{>0}]] = \mathbb{C}[[\mathbf{t}]]$ governs the integrals

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{p_1} \cdots \psi_n^{p_n}$$

Here $\overline{\mathcal{M}}_{g,n}$ denotes the Deligne–Mumford moduli space of stable algebraic curves of genus g with n distinct marked points, and ψ_j denotes the ψ-class.

Example 2. $f(x) = \frac{C}{(x-1)^2}$, $C \in \mathbb{C}$.

The corresponding solution $u = u(\mathbf{t})$ in $\mathbb{C}[[\mathbf{t}]]$ is called the generalized BGW solution, introduced by Alexandrov (cf. also Do–Norbury), denoted by $u_{\Theta(C)}$. According to Norbury, $u_{\Theta(1/8)}$ governs the integrals

$$\int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n}\, \psi_1^{p_1} \cdots \psi_n^{p_n}$$

Here $\Theta_{g,n}$ denotes the Norbury Θ-class. We call $u_{\Theta(1/8)}$ the BGW-Norbury solution.

Example 3. $f(x) = C\,\wp(x;\tau)$, with given $C \in \mathbb{C}$.

Set $V = \mathbb{C}[g_2, g_3, \wp, \wp']/(\wp'^2 - 4\wp^3 + g_2\wp + g_3)$ with $g_2 = 60\,G_4$ and $g_3 = 140\,G_6$, where

$$G_{2k} = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(m+n\tau)^{2k}}$$

The corresponding unique solution $u(\mathbf{t})$ in $V[[\mathbf{t}_{>0}]]$ is a modular deformation of $u_{\Theta(C)}$, denoted by u_{elliptic}. We call u_{elliptic} a Lamé solution. Geometric meaning of this solution with a generic C is an open question.

Example 4. $f(x) = J_0^{-1}(x)$.

According to Kaufman-Manin-Zagier and Dubrovin, the unique solution governs the Weil-Petersson volumes of $\overline{\mathcal{M}}_{g,n}$. This was used by Zograf, Dubrovin-Zhang to design efficient algorithms for computing the Weil-Petersson volumes. Other methods: Mirzakhani's recursion, Chekhov-Eynard-Orantin topological recursion.

Tau-Structure for the KdV Hierarchy

Let $\mathcal{A}$ denote the differential polynomial ring of u. A collection $\Omega_{p,q} \in \mathcal{A}$ $(p, q \geq 0)$ is called a tau-structure for the KdV hierarchy, if

$$\Omega_{0,0} = u \,, \quad \Omega_{p,q} = \Omega_{q,p} \,, \quad \partial_{t_r}\left(\Omega_{p,q}\right) = \partial_{t_q}\left(\Omega_{p,r}\right)$$

For $n \geq 3$, define

$$\Omega_{p_1,\ldots,p_n} := \partial_{t_{p_1}} \cdots \partial_{t_{p_{n-2}}} \left(\Omega_{p_{n-1},p_n}\right)$$

All these elements are symmetric in their indices. The first few values are

$$\Omega_{0,1} = \frac{u^2}{2} + \frac{u''}{12}, \quad \Omega_{0,2} = \frac{u^3}{6} + \frac{u'^2}{24} + \frac{uu''}{12} + \frac{u''''}{240}$$

$$\Omega_{1,1} = \frac{u^3}{3} + \frac{u'^2}{24} + \frac{uu''}{6} + \frac{u''''}{144}$$

Tau-Function for the KdV Hierarchy

If $u = u(\mathbf{t})$ is a solution to the KdV hierarchy, then we know from the definition of the tau-structure that there exists a function $\tau = \tau(\mathbf{t})$ such that

$$\Omega_{p,q} = \frac{\partial^2 \log \tau(\mathbf{t})}{\partial t_p \partial t_q}$$

We call τ *the* tau-function of the solution u, although it is determined by u up to adding any linear function of $t_0, t_1, t_2, \dots$. It is easy to see that for all $n \geq 2$,

$$\Omega_{p_1,\dots,p_n} = \frac{\partial^n \log \tau(\mathbf{t})}{\partial t_{p_1} \cdots \partial t_{p_n}}$$

Simple Observation

Since

$$u = \Omega_{0,0} = \partial_x^2 (\log \tau)$$

for $p_1, \dots, p_n \geq 1$,

$$\frac{\partial^n u}{\partial t_{p_1} \cdots \partial t_{p_n}} = \Omega_{0,0,p_1,\dots,p_n}$$

This means that, if we could compute the differential polynomials Ω's, then the initial value problem is solved (polynomiality property needs to be used). The Ω's actually gives all the multi-derivatives of $\log \tau$. Computing $\log \tau$ is the actual goal.

Matrix Lax Operator

Let

$$\mathcal{L} = \partial + \Lambda(\lambda) + q$$

Here

$$\Lambda(\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 0 & 0 \\ -2u & 0 \end{pmatrix}$$

Let $\mathcal{S} = \mathrm{sl}_2(\mathbb{C})((\lambda^{-1}))$. The principal gradation on $\mathcal{A} \otimes \mathcal{S}$ is defined via

$$\deg E = 1, \ \ \deg F = -1, \ \ \deg \lambda = 2, \ \ \deg u_{ix} = 0$$

Here

$$E := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Definition *The matrix resolvent (MR) associated to the matrix Lax operator $\mathcal{L}$ is defined as the unique element $R(\lambda) \in \mathcal{S} \otimes \mathcal{A}$ satisfying*

$$
\begin{aligned}
\left[\mathcal{L}\,, R(\lambda)\right] &= 0 \\
R(\lambda) &= \Lambda(\lambda) + \textit{l.o.t.} \\
\operatorname{Tr} R(\lambda)^2 &= 2\lambda
\end{aligned}
$$

Here "l.o.t" means lower order terms with respect to the principal gradation.

Remark The definition of MR and its application in constructing integrable hierarchy date back to Dickey (1981) and others. We will show that tau-structure, actually all the n-point ($n \geq 2$) generating series of the Ω's, can be obtained by the MR algebraically.

From MR to Tau-Structure

Proposition (Bertola-Dubrovin-Y., Zhou, Dubrovin-Y.-Zagier)
The following identities are true:

$$\sum_{p_1,\dots,p_n\geq 0} \Omega_{p_1,\dots,p_n} \prod_{j=1}^{n} \frac{(2p_j+1)!!}{\lambda_j^{p_j+1}}$$
$$= -\sum_{\sigma\in S_n/C_n} \frac{\operatorname{tr}\left(R(\lambda_{\sigma(1)})\cdots R(\lambda_{\sigma(n)})\right)}{\prod_{i=1}^{n}\left(\lambda_{\sigma(i+1)}-\lambda_{\sigma(i)}\right)}$$
$$- \frac{\lambda+\mu}{(\lambda-\mu)^2}\delta_{n,2}$$

Here S_n denotes the symmetry group, C_n the cyclic group, and it is understood that $\sigma(n+1)=\sigma(1)$.

Proof (Sketch) 0. Let us show that MR is well defined, i.e. existence and uniqueness. Write

$$R = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

The defining equations then read explicitly as follows:

$$a = \frac{1}{2}\,\partial(b), \quad c = (\lambda - 2u_x)\,b \, - \, \frac{1}{2}\,\partial^2(b)$$
$$\partial^3(b) \, - \, 4\,(\lambda - 2u)\,\partial(b) \, + \, 4\,u_x\,b = 0$$
$$b\,\partial^2(b) \, - \, \frac{1}{2}\big(\partial(b)\big)^2 \, - \, 2\,(\lambda - 2u)\,b^2 = -2\lambda$$

Thus by using the condition $R(\lambda) \in \mathcal{S} \otimes \mathcal{A}$ the existence and uniqueness of R follow.

1. In the first step, we give an alternative formula of R. According to Kac and Kostant, we know the following decomposition of $\mathcal{S}$:

$$\mathcal{S} = \mathrm{Ker}\,\mathrm{ad}_\Lambda \oplus \mathrm{Im}\,\mathrm{ad}_\Lambda, \quad \mathrm{Ker}\,\mathrm{ad}_\Lambda \perp \mathrm{Im}\,\mathrm{ad}_\Lambda$$

$$\mathrm{Ker}\,\mathrm{ad}_\Lambda = \left\{ \sum_{\ell \le m} c_\ell \, \Lambda_{1+2\ell} \,\middle|\, m \in \mathbb{Z}\,,\ c_\ell \in \mathbb{C}\,, c_m \neq 0 \right\}$$

Here $\Lambda_{1+2\ell} := \Lambda\,\lambda^\ell$.

Fundamental Lemma (Drinfeld–Sokolov) *There exists a unique pair* (U, H) *such that*

$$U \in \mathcal{A} \otimes (\mathrm{Im}\,\mathrm{ad}_\Lambda)^{<0}, \quad H \in \mathcal{A} \otimes (\mathrm{Ker}\,\mathrm{ad}_\Lambda)^{<0}$$

$$e^{-\mathrm{ad}_U}\mathcal{L} = \partial + \Lambda + H$$

The proof for this lemma is omitted.

Lemma *We have $R = e^{\mathrm{ad}_U}(\Lambda)$.*
The proof for this lemma is achieved by verifying the RHS satisfies all the defining equations of R and by using the uniqueness. This finishes the first step.

2. In the second step, we express the KdV flows in terms of R. Indeed, define

$$V_k := -\frac{1}{(2k+1)!!}\left(\lambda^k R(\lambda)\right)_+ + \frac{1}{(2k+1)!!}\begin{pmatrix} 0 & 0 \\ b_k & 0 \end{pmatrix}$$

Lemma *The KdV flows can be written as*

$$\left[\partial_{t_k} - V_k \, , \, \mathcal{L}\right] = 0$$

We note that this is widely used in the literature (cf. Babelon-Bernard-Talon (2003), Dickey (1981), Dubrovin (1976)).

Lemma *Let* $P_k := e^{\mathrm{ad}_U}(\Lambda_{1+2k})$. *Then*

$$\partial_{t_k}(P_\ell) \;=\; \left[V_k\,,P_\ell\right]$$

We note that $P_0 = R$. To proceed, introduce

$$\nabla(\lambda) = -\sum_{k\geq 0} \frac{(1+2k)!!}{\lambda^{1+k}} \partial_{t_k}$$

Lemma *The following identity holds true:*

$$\nabla(\lambda)\big(R(\mu)\big) \;=\; \frac{[R(\lambda), R(\mu)]}{\lambda-\mu} \;-\; \left[Q(\lambda), R(\mu)\right]$$

Here $Q(\lambda) := \begin{pmatrix} 0 & 0 \\ b(\lambda) & 0 \end{pmatrix}$.

Proof of Lemma We have

$$
\begin{aligned}
&\nabla(\lambda)\big(R(\mu)\big) \\
&= -\sum_{k\geq 0}\frac{1}{\lambda^{1+k}}\left[-\big(\mu^k R(\mu)\big)_+ + \begin{pmatrix}0 & 0\\ b_k & 0\end{pmatrix}, R(\mu)\right] \\
&= \frac{1}{2\pi i}\oint_{|\mu|<|\rho|<|\lambda|} d\rho \frac{[R(\rho), R(\mu)]}{(\lambda-\rho)(\rho-\mu)} - \left[\begin{pmatrix}0 & 0\\ b(\lambda)-1 & 0\end{pmatrix}, R(\mu)\right] \\
&= \frac{[R(\lambda), R(\mu)]}{\lambda-\mu} - \left[\mathrm{Coef}_{\lambda^1} R(\lambda), R(\mu)\right] \\
&\qquad - \left[\begin{pmatrix}0 & 0\\ b(\lambda)-1 & 0\end{pmatrix}, R(\mu)\right].
\end{aligned}
$$

□

This finishes the second step.

3. In the third step, we prove for $n = 2$ the RHS produces the tau-structure. First, let us verify the consistency between LHS and RHS. Using

$$R(\mu) \;=\; R(\lambda) + R'(\lambda)(\mu-\lambda) + (\mu-\lambda)^2\,\partial_\lambda\left(\frac{R(\lambda)-R(\mu)}{\lambda-\mu}\right)$$

we find

$$\mathrm{RHS} = \frac{2\lambda}{(\lambda-\mu)^2} - \frac{\mathrm{tr}\big(R(\lambda)R'(\lambda)\big)}{\lambda-\mu} + \mathrm{tr}\left(R(\lambda)\,\partial_\lambda\Big(\frac{R(\lambda)-R(\mu)}{\lambda-\mu}\Big)\right) - \frac{\lambda+\mu}{(\lambda-\mu)^2}\,.$$

It follows from $\mathrm{tr}\,R(\lambda)^2 = 2\lambda$ that $\mathrm{tr}\big(R(\lambda)R'(\lambda)\big) = 1$. So

$$\mathrm{RHS} = \mathrm{tr}\left(R(\lambda)\partial_\lambda\Big(\frac{R(\lambda)-R(\mu)}{\lambda-\mu}\Big)\right) \in \mathcal{A}[[\lambda^{-1},\mu^{-1}]]\,\lambda^{-1}\mu^{-1}$$

This finishes the verification.

Now denote the coefficients of the RHS by $\omega_{p,q}$. The fact that $\omega_{0,0} = u$ can be obtained from a straightforward residue computation. Obviously the RHS is invariant w.r.t. $\lambda \leftrightarrow \mu$, we have $\Omega_{p,q} = \Omega_{q,p}$. Applying $\nabla(\nu)$ to the RHS we obtain

$$\sum_{p,q,r\geq 0} \partial_{t_r}(\omega_{p,q}) \frac{(1+2p)!!(1+2q)!!(1+2r)!!}{\lambda^{p+1}\mu^{q+1}\nu^{r+1}}$$
$$= \frac{\mathrm{Tr}\,[R(\nu),R(\lambda)]R(\mu)}{(\lambda-\mu)^2(\nu-\lambda)} + \frac{\mathrm{Tr}\,R(\lambda)\,[R(\nu),R(\mu)]}{(\lambda-\mu)^2(\nu-\mu)}$$
$$- \frac{\mathrm{Tr}\,[Q(\nu),R(\lambda)]R(\mu)}{(\lambda-\mu)^2} - \frac{\mathrm{Tr}\,R(\lambda)\,[Q(\nu),R(\mu)]}{(\lambda-\mu)^2}$$
$$= -\frac{\mathrm{Tr}\,[R(\nu),R(\lambda)]R(\mu)}{(\lambda-\mu)(\mu-\nu)(\nu-\lambda)}\,.$$

The RHS of this identity is invariant w.r.t. the permutations of λ, μ, ν. This implies that $\partial_{t_r}(\omega_{p,q}) = \partial_{t_p}(\omega_{q,r}) = \partial_{t_q}(\omega_{r,p})$.

4. The proof is completed by doing the mathematical induction using

$$\nabla(\lambda)\big(R(\mu)\big) \;=\; \frac{[R(\lambda), R(\mu)]}{\lambda - \mu} \;-\; \big[Q(\lambda), R(\mu)\big]$$

The details are omitted here. □

Wave Function, the Time-Independent Case

As before, let $f(x)$ be an element of V (the initial data), and let L be the linear Schrödinger operator

$$L \;=\; \partial_x^2 + 2f(x)$$

By a wave function of f we will mean an element $\psi = \psi(z,x)$ in the module $\widetilde{V}((z^{-1}))\, e^{xz}$ satisfying

$$L(\psi) = z^2\psi$$

of the form

$$\psi = \big(1 + \phi_1(x)/z + \phi_2(x)/z^2 + \cdots\big)\, e^{xz}$$

Here $\widetilde{V}$ is any ring with $V \subseteq \partial_x\big(\widetilde{V}\big) \subseteq \widetilde{V}$.

The dual wave function ψ^* of f associated with ψ is defined as the unique element in $\widetilde{V}((z^{-1}))\, e^{-xz}$ s.t.

$$L(\psi^*) = z^2 \psi^*$$

of the form

$$\psi^* = \left(1 + \phi_1^*(x)/z + \phi_2^*(x)/z^2 + \cdots \right) e^{-xz}$$

for which $\partial_x^i(\psi)\, \psi^*$ has residue 0 at $z = \infty$ $\forall\, i \geq 0$. The pair (ψ, ψ^*) is called a pair of wave functions of f. Given f, the wave function ψ of f is unique up to multiplication by any element in $1 + z^{-1}\mathbb{C}((z^{-1}))$.

Wave Function, Time-dependent Case

Similar, using the standard techniques in integrable systems we get a pair $(\psi(z, \mathbf{t}), \psi^*(z, \mathbf{t}))$ of f.

Let us introduce:

$$D(z, w) \;:=\; \frac{\psi(z)\,\psi^*_x(w) \;-\; \psi^*(w)\,\psi_x(z)}{w^2 \;-\; z^2}$$

Theorem 1 (Dubrovin-Y.-Zagier) $\forall\, n \geq 2$, *we have*

$$\sum_{p_1,\dots,p_n} \Omega_{p_1,\dots,p_n} \prod_{j=1}^{n} \frac{(2p_j+1)!!}{z_j^{2p_j+2}}$$
$$= - \sum_{\sigma \in S_n/C_n} \prod_{i=1}^{n} D\big(z_{\sigma(i)}, z_{\sigma(i+1)}\big) - \frac{\delta_{n,2}}{(z_1 - z_2)^2}$$

Remark Under $\psi \mapsto g(z)\psi(z) = \widetilde{\psi}(z)$, the ψ^* associated to ψ is mapped to $\psi^*(z)/g(z) = \widetilde{\psi}^*(z)$. Then

$$\widetilde{D}(z,w) = \frac{g(z)}{g(w)} D(z,w)$$

However, the RHS remain unchanged because products of the factors of the form $g(z)/g(w)$ cancel.

Remark For a given $f(x) \in V$, one can find an explicit recursion for solving the time-independent wave function $\psi(z, x)$; however, we do not know an efficient way of solving $\psi(z, \mathbf{t})$. However, using Theorem 1 the knowledge of $\psi(z, x)$ already gives rise to a construction of $\log \tau$. More precisely, if we specialize the formula in Theorem 1 to $\mathbf{t} = (x, \mathbf{0})$, then it gives $\Omega_{p_1,\ldots,p_n}$ evaluated at $\mathbf{t} = (x, \mathbf{0})$ for all $n \geq 2$ and hence gives the entire Taylor series of $\log \tau$.

Proof of Theorem 1 Let $u = u(\mathbf{t})$ be an arbitrary solution to the KdV hierarchy and (ψ, ψ^*) a pair of wave functions of u (time dependent).

Lemma *Define*

$$\Psi(z,\mathbf{t}) \;=\; \begin{pmatrix} \psi(z,\mathbf{t}) & \psi^*(z,\mathbf{t}) \\ -\psi_x(z,\mathbf{t}) & -\psi^*_x(z,\mathbf{t}) \end{pmatrix}$$

Then we have

$$\det \Psi(z,\mathbf{t}) \;\equiv\; 2z$$

and

$$b \;=\; \psi(z,\mathbf{t})\,\psi^*(z,\mathbf{t})$$

Here $\lambda = z^2$. *Moreover,*

$$R = -\,\Psi(z,\mathbf{t}) \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix} \Psi^{-1}(z,\mathbf{t})$$

Proof of Lemma Denote $W = \psi_x \psi^* - \psi \psi_x^*$. The proof for $W = 2z$ is known, but we give a review. Using $L\psi = z^2 \psi$ and $\psi_{t_k} = \left(\frac{1}{(2k+1)!!} L^{\frac{2k+1}{2}}\right)_+ \psi$, we have $W_{t_k} = 0$. So W has the form $W = 2z + \sum_{k\geq 0} s_k z^{-k}$, where s_k are constants. $\forall\, i \geq -1$,

$$
\begin{aligned}
&\mathrm{res}_{z=\infty}\, z^i\, W\, dz \\
&= \mathrm{res}_{z=\infty} \Big(\partial_x \circ \Phi \circ \partial_x^i \big(e^{\sum \frac{t_k z^{2k+1}}{2k+1)!!}} \big) \left(\Phi^{-1}\right)^* \big(e^{-\sum \frac{t_k z^{2k+1}}{2k+1)!!}} \big) \\
&\qquad - (-1)^i\, \Phi \big(e^{\sum \frac{t_k z^{2k+1}}{2k+1)!!}} \big)\, \partial_x \circ \left(\Phi^{-1}\right)^* \circ \partial_x^i \big(e^{-\sum \frac{t_k z^{2k+1}}{2k+1)!!}} \big) \Big)\, dz \\
&= \mathrm{res}_{\partial_x} \left(\partial_x \circ \Phi \circ \partial_x^i \circ \Phi^{-1} + \Phi \circ \partial_x^i \circ \Phi^{-1} \circ \partial_x \right) \\
&= \mathrm{res}_{\partial_x} \left(\partial_x \circ L^{\frac{i}{2}} + L^{\frac{i}{2}} \circ \partial_x \right) \;=\; 0
\end{aligned}
$$

Here Φ is the dressing operator, and we used the fact that for two ΨDOs P, Q, $\mathrm{res}_{z=\infty} P\left(e^{xz}\right) Q\left(e^{-xz}\right) dz = \mathrm{res}_{\partial_x} P \circ Q^*$. and the fact $(L^{1/2})^* = -L^{1/2}$. To show $b = \psi\psi^*$ it suffices to show that $\psi\psi^*$ satisfies the determining equation for b as it has a unique solution starting with 1. The rest is a direct check. □

Using the lemma, we can write R in terms of ψ, ψ^* as

$$R = z + \begin{pmatrix} \psi \\ -\psi_x \end{pmatrix} \begin{pmatrix} \psi_x^* & \psi^* \end{pmatrix}$$

Then we have

$$\begin{aligned} &\operatorname{tr} R(z_1^2)\, R(z_2^2) \\ &= \big(\psi_x^*(z_1)\,\psi(z_2) - \psi^*(z_1)\psi_x(z_2)\big)\big(\psi_x^*(z_2)\,\psi(z_1) - \psi^*(z_2)\psi_x(z_1)\big) \\ &\qquad - 2z_1 z_2 \end{aligned}$$

Hence

$$\begin{aligned} &\sum_{p_1,p_2} \Omega_{p_1,p_2} \frac{(2p_1+1)!!\,(2p_2+1)!!}{z_1^{2p_1+2}\, z_2^{2p_2+2}} \\ &= -D(z_1,z_2)\,D(z_2,z_1) - \frac{1}{(z_1-z_2)^2} \end{aligned}$$

This shows the $n=2$ case. The $n \geq 3$ cases can be proved in a similar way. □

Introduce

$$K(z,w) \;:=\; \frac{b(z^2)\, b_x(w^2) - b(w^2)\, b_x(z^2)}{2\,(w^2-z^2)} \;-\; \frac{w\, b(z^2) + z\, b(w^2)}{w^2-z^2}$$

Theorem 2 (Dubrovin-Y.-Zagier) $\forall\, n \geq 2$, *we have*

$$\sum_{p_1,\dots,p_n} \Omega_{p_1,\dots,p_n} \prod_{j=1}^{n} \frac{(2p_j+1)!!}{z_j^{2p_j+2}}$$
$$= \; - \; \frac{\sum_{\sigma\in S_n/C_n} \prod_{i=1}^{n} K\big(z_{\sigma(i)}, z_{\sigma(i+1)}\big)}{\prod_{i=1}^{n} b\big(z_i^2\big)} \; - \; \frac{\delta_{n,2}}{(z_1-z_2)^2}$$

This theorem is quite unexpected. Geometric meaning of K is an open question.

Proof of Theorem 2 The functions $\psi, \psi^*, \psi_x, \psi_x^*$ and b satisfy the following three relations:

$$\begin{aligned} \psi\,\psi^* &= b\,, \\ \psi_x\,\psi^* - \psi\,\psi_x^* &= 2z\,, \\ \psi_x\,\psi^* + \psi\,\psi_x^* &= b_x\,. \end{aligned}$$

Solving this system we obtain

$$\psi^* = \frac{b}{\psi}\,, \quad \psi_x = \psi\,\frac{b_x + 2z}{2\,b}\,, \quad \psi_x^* = \frac{b_x - 2z}{2\,\psi}\,.$$

Theorem 2 follows easily from Theorem 1, where it is amazing that the factors of the form $\psi(z)/\psi(w)$ cancel in each product of the sum. □

Example 1. Intersection numbers of ψ-classes

In this case, $f(x) = x$.

Corollary 1 (Bertola-Dubrovin-Y.) For $n \geq 2$, we have

$$\sum_{g,p_1,\dots,p_n} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{p_1} \cdots \psi_n^{p_n} \prod_{j=1}^{n} \frac{(2p_j+1)!!}{\lambda_j^{p_j+1}}$$

$$= -\sum_{\sigma \in S_n/C_n} \frac{\operatorname{tr} M(\lambda_{\sigma(1)}) \dots M(\lambda_{\sigma(n)})}{\prod_{i=1}^{n}\left(\lambda_{\sigma(i+1)} - \lambda_{\sigma(i)}\right)} - \delta_{n,2} \frac{\lambda_1 + \lambda_2}{(\lambda_1 - \lambda_2)^2}$$

Here

$$M(\lambda) = \begin{pmatrix} \frac{1}{2}\sum_{g=1}^{\infty} \frac{(6g-5)!!}{24^{g-1}\,(g-1)!}\lambda^{-3g+2} & \sum_{g=0}^{\infty} \frac{(6g-1)!!}{24^g\,g!}\lambda^{-3g} \\ \sum_{g=0}^{\infty} \frac{1+6g}{1-6g}\frac{(6g-1)!!}{24^g\,g!}\lambda^{-3g+1} & -\frac{1}{2}\sum_{g=1}^{\infty} \frac{(6g-5)!!}{24^{g-1}\,(g-1)!}\lambda^{-3g+2} \end{pmatrix}$$

A particular pair of wave functions of $f(x) = x$ are

$$\psi(z,x) = \frac{\sqrt{z}}{(z^2-2x)^{\frac{1}{4}}}\, e^{\frac{1}{3}z^3 - \frac{1}{3}(z^2-2x)^{\frac{3}{2}}} \sum_{k\geq 0} \frac{(-1)^k}{288^k} \frac{(6k)!}{(3k)!\,(2k)!} (z^2-2x)^{-\frac{3k}{2}}$$

$$\psi^*(z,x) = \frac{\sqrt{z}}{(z^2-2x)^{\frac{1}{4}}}\, e^{-\frac{z^3}{3} + \frac{1}{3}(z^2-2x)^{\frac{3}{2}}} \sum_{k\geq 0} \frac{1}{288^k} \frac{(6k)!}{(3k)!\,(2k)!} (z^2-2x)^{-\frac{3k}{2}}$$

Explicit expression for $D(z,w)$ was obtained by Jian Zhou; later a second proof of Zhou's formula was given by Balogh-Y. using the Sato Grassmannian.

Example 2. Intersection numbers of ψ-classes coupling the Norbury class

In this case, $f(x) = \frac{C}{(x-1)^2}$. (The Norbury class corresponds to the particular $C = 1/8$.)

Corollary 2 (Dubrovin-Y.-Zagier, Bertola-Ruzza) *For* $n \geq 2$,

$$\sum_{g,p_1,\dots,p_n} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{p_1} \cdots \psi_n^{p_n} \cdot \Theta_{g,n} \prod_{j=1}^{n} \frac{(2p_j+1)!!}{\lambda_j^{p_j+1}}$$

$$= - \sum_{\sigma \in S_n / C_n} \frac{\operatorname{tr} M(\lambda_{\sigma(1)}) \dots M(\lambda_{\sigma(n)})}{\prod_{i=1}^{n} \left(\lambda_{\sigma(i+1)} - \lambda_{\sigma(i)}\right)} - \delta_{n,2} \frac{\lambda_1 + \lambda_2}{(\lambda_1 - \lambda_2)^2}$$

Here

$$M(\lambda) = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} + \sum_{k \geq 0} \left[\frac{(2k-1)!!}{2^k} \right]^3 \begin{pmatrix} k & 1 \\ -\frac{8k^3+12k^2+4k+1}{8(k+1)} & -k \end{pmatrix} \frac{\lambda^{-k}}{k!}$$

Corollary 3 (Dubrovin-Y.-Zagier, Bertola-Ruzza) *For a general* $C \in \mathbb{C}$*, we have*

$$R = \begin{pmatrix} -\frac{C\,\lambda^{\frac{1}{2}}}{\zeta^{\frac{3}{2}}} G_{\frac{3}{2}}(\zeta) & G_{\frac{1}{2}}(\zeta) \\ \frac{\lambda}{\zeta}\left((\zeta - 2C)\,G_{\frac{1}{2}}(\zeta) - \frac{3C}{\zeta}\,G_{\frac{3}{2}}(\zeta) - \frac{6C(C+1)}{\zeta^2} G_{\frac{5}{2}}(\zeta)\right) & \frac{C\,\lambda^{\frac{1}{2}}}{\zeta^{\frac{3}{2}}} G_{\frac{3}{2}}(\zeta) \end{pmatrix}$$

Here $\zeta^{3/2} := \lambda^{3/2}\,(x-1)^3$ *and*

$$\Delta \;:=\; 1 - 8\,C\,, \quad G_\alpha(\zeta) \;:=\; {}_3F_0\left(\alpha,\, \alpha + \frac{\sqrt{\Delta}}{2},\, \alpha - \frac{\sqrt{\Delta}}{2};\,;\, \frac{1}{\zeta}\right).$$

Corollary 4 (Dubrovin-Y.-Zagier) *Let* $\alpha = (\frac{1}{4} - 2C)^{1/2}$*. Then*

$$\psi(z,x) := \sum_{k\ge 0} \frac{a_k(\alpha)}{z^k(1-x)^k}\, e^{zx}\,, \quad \psi^*(z,x) := \psi(-z,x)$$

form a pair of wave functions of $f = C/(x-1)^2$*. Here*

$$a_k(\alpha) := \frac{\left(\alpha - k + \frac{1}{2}\right)_{2k}}{2^k\, k!} = \frac{(-1)^k}{k!} \prod_{j=1}^{k}\left(C + \binom{j}{2}\right) \in \mathbb{Q}[C]$$

Moreover, for any $n \ge 2$*,*

$$\sum_{p_1,\dots,p_n} \Omega^{\Theta(C)}_{p_1,\dots,p_n} \prod_{j=1}^{n} \frac{(2p_j+1)!!}{z_j^{2p_j+2}}$$

$$= -\sum_{\sigma\in S_n/C_n} \prod_{i=1}^{n} D_{\Theta(C)}\left(z_{\sigma(i)}, z_{\sigma(i+1)}\right) - \frac{\delta_{n,2}}{(z_1-z_2)^2}$$

…Corollary 4 continues *Here* $D_{\Theta(C)}(z,w)$ *has the following explicit expression:*

$$D_{\Theta(C)}(z,w) = \frac{1}{z-w} + \sum_{m,n} \frac{A_{mn}(\alpha)}{z^m\,(-w)^n}$$

with A_{mn} *given by*

$$A_{mn} = \sum_{\substack{r\geq m,\ s\geq 0\\ r+s=m+n-1}} \frac{r-s}{r+s}\, a_r(\alpha)\, a_s(\alpha) = \sum_{\substack{r\geq n,\ s\geq 0\\ r+s=m+n-1}} \frac{r-s}{r+s}\, a_r(\alpha)\, a_s(\alpha)$$

$$= 2\,\frac{m!\,n!\,a_m(\alpha)\,a_n(\alpha)}{(m+n-1)\,(m+n-1)!} \sum_k (-1)^k \frac{\binom{m+n-1}{m+k}\binom{m+n-1}{n+k}}{\alpha-k-\frac{1}{2}}$$

Proof We omit the proof of the formula for $a_k(\alpha)$. To prove the formulas for D, let

$$A(z) := \psi(z,0) = \sum_{k=0}^{\infty} \frac{a_k}{z^k}, \quad B(z) := \psi_x(z,0) = z + \sum_{k=0}^{\infty} \frac{b_k}{z^k}$$

with $b_k(\alpha) = a_{k+1} + k a_k$. One checks easily that $(A(z), B(z))$ satisfies the differential system

$$\theta_z \begin{pmatrix} A(z) \\ B(z) \end{pmatrix} = \begin{pmatrix} z & -1 \\ 2C - z^2 & 1+z \end{pmatrix} \begin{pmatrix} A(z) \\ B(z) \end{pmatrix}$$

Here $\theta_z := z\frac{d}{dz}$. Setting $y = -w$, we find two identities

$$\sum_{m,n\geq 0} \frac{(m+n)(A_{m+1,n} - A_{m,n+1})}{z^m\, y^n} = (\theta_z + \theta_y)\left(\frac{A(z)B(y) - B(z)A(y)}{z+y}\right)$$

$$\sum_{m,n\geq 0} \frac{(n-m)a_m a_n}{z^m\, y^n} = (\theta_z - \theta_y)\big(A(z)\, A(y)\big)$$

Using the differential system we find that the right hand sides of both of these formulas are equal to

$$(z-y)A(z)A(y) + A(z)B(y) - B(z)A(y)$$

This gives the following recursion

$$A_{m,n+1}(\alpha) - A_{m+1,n}(\alpha) \; = \; \frac{m-n}{m+n}\, a_m(\alpha) a_n(\alpha)$$

The first two closed formulas for $A_{m,n}$ are then obtained. The third closed formula can be obtained by analyzing the residues and by doing the partial fraction decomposition. □

Example 3. Lamé tau-function

Here we recall that $f(x) = C\wp(x;\tau)$, $C \in \mathbb{C}$. In this case, we do not have a closed formula for a general C. For $C = -p(p+1)/2$ with $p \geq 0$, however, we know that u is a special finite gap solution to the KdV hierarchy. We have for $p = 1$,

$$b = 2\sqrt{\lambda}\frac{\lambda - \wp}{\sqrt{4\lambda^3 - g_2\lambda - g_3}}$$

For $p = 2$,

$$b = \frac{1}{2}\sqrt{\lambda}\frac{4\lambda^2 - 12\wp\lambda + 36\wp^2 - 9g_2}{\sqrt{(\lambda^2 - 3g_2)(4\lambda^3 - 9g_2\lambda + 27g_3)}}$$

For $p = 3$,

$$b = \frac{4\lambda^3 - 24\wp\lambda^2 + 60(3\wp^2 - g_2)\lambda - 900\wp^3 + 225g_2\wp + 225g_3}{\sqrt{16\lambda^6 - 504g_2\lambda^4 + 2376g_3\lambda^3 + 4185g_2^2\lambda^2 - 36450g_2g_3\lambda - 3375g_2^3 + 91125g_3^2}}$$

Thank You!