Assignment $N^o 5$

Pattern Formation in Complex Systems

The Turing instability and pattern formation.

Usually diffusion plays a smoothing rôle in dynamical systems. Diffusion dampens out sharp gradients. For example, the heat equation

$$u_t = Du_{xx},$$

where D is the diffusion coefficient, has solutions which smoothly decay.

In 1952 Alan Turing proposed a mechanism to study morphogenesis. He explained morphogenesis as an instability mediated by diffusion. This is truly remarkable when we think about the smoothing effect of diffusion. He invented the theory of reaction-diffusion equations. These equations look like

$$\mathbf{u}_t = D\mathbf{u}_{xx} + F(\mathbf{u}),$$

where $\mathbf{u}$ is a vector. The term $F(\mathbf{u})$ describes the reaction kinetics of the chemical or biological system under consideration. Turing suggested that, under certain conditions (see **3.**) chemicals can react and diffuse in such a way as to produce *steady heterogeneous spatial patterns* of chemical or morphogen concentration. In particular, a reaction diffusion system exhibits Turing instability when the homogeneous steady state is stable to small perturbations but unstable to small spatial perturbations.

Alan Turing was a mathematical genius. He wrote 3 seminal papers on very different subjects. The Turing machine is named after him.
He was also a homosexual at a time when homosexuality was illegal in Britain (1952!!), labeled a pervert and was charged with gross indecency. He committed suicide in 1954.

1. Build up an argument why diffusion smoothes sharp gradients. Use the one-dimensional heat equation and Fourier transformation in your argument.

2. Show that

$$u(x,t) = \frac{1}{2\sqrt{\pi D t}} \int_{-\infty}^{\infty} u_0(\xi) e^{-\frac{(x-\xi)^2}{4Dt}} d\xi,$$

where $u_0(x)$ is your initial condition, is an explicit solution of the heat equation.

3. Consider the two-component system

$$\begin{aligned} u_t &= \gamma(a - u + u^2 v) + u_{xx} \\ v_t &= \gamma(b - u^2 v) + D v_{xx} \end{aligned}$$

Derive conditions for the Turing instability according to the requirements that the homogeneous steady state is stable (in the absence of diffusion; i.e. for wavenumbers $k = 0$), and that it is unstable with respect to non-zero spatial variations. You may write such perturbations as

$$u = u_0 + U e^{\sigma t + ikx} \quad \text{and} \quad v = v_0 + V e^{\sigma t + ikx},$$

where u_0, v_0 are the homogeneous steady states.

Give the (intrinsic!) critical wave number and a condition on the diffusion coefficient. The criterion $D \neq 1$ is necessary for the formation of a Turing pattern.

4. Come up with a simple physical intuitive explanation how a difference in diffusion coefficients can lead to pattern formation. Imagine two chemical substances with two different diffusion coefficients. One of the substances inhibits the other. Whose diffusion coefficient must be larger to allow for a pattern?

Pattern formation in catalysis

In the catalytic oxidation of CO a platinum wire is heated and facilitates the oxidation of CO in a chamber filled with CO and O_2. A technician would like that the temperature and O_2 concentrations are uniformly distributed along the wire. This is not always the case and disturbing patterns are observed. A 'hot topic' here are so called 'hot spots'. Here the temperature along the wire is not uniform along the wire but exhibits localized sometimes more than one very hot spots.

5. Find a simple mechanism for hot spot generation by assuming that the system can be described by two reaction-diffusion equations for the temperature along the wire T and the concentration of oxygen c respectively. Hint: Don't be fooled by the Turing mechanism, here we do not have an inhibitor!