On the nature of Benford's Law

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Abstract

We study multiplicative and affine sequences of real numbers defined by

$$N(j+1) = \zeta(j)N(j) + \eta(j)$$

where $\{\zeta(j)\}$ and $\{\eta(j)\}$ are sequences of positive real numbers (in the multiplicative case $\eta(j) = 0$ for all j). We investigate the conditions under which the leading digits k of $\{N(j)\}$ have the following probability distribution, known as Benford's Law, $P(k) = \log_{10}(\frac{k+1}{k})$. We present two main results. First, we show that contrary to the usual assumption in the literature, $\{\zeta(j)\}$ does not necessarily need to come from a chaotic or independent random process for Benford's Law to hold. The multiplicative driving force may be a deterministic quasi-periodic or even periodic forcing. Second, we give conditions under which the distribution of the first digits of an affine process displays Benford's Law. Our proofs use techniques from ergodic theory.

1 Introduction

Benford's Law has a long history, as reviewed in the excellent survey by R. Raimi [11]. It is based upon the observation that for many types of numerical data, when the data is expressed in decimal notation and grouped according to the first leading digit (without regard to decimal point), then the resulting nine classes often have a distribution such that the integer k, $k = 1, ..., 9$, occurs with frequency $\log_{10}(\frac{k+1}{k})$, as depicted in Fig.1 (+-signs), and not with frequency $\frac{1}{9}$ as perhaps would be expected. Raimi discusses some explanations that have been put forward to account for this peculiar behaviour. They may be classified into density and summability arguments, scale invariance arguments (including mixture of distribution arguments) and arguments based upon multiplicative processes. The mixture of distribution arguments are interesting because tables which occur naturally often are mixed from more than one probability distribution and it has been shown that such a mixing process in the limit leads to a distribution that obeys Benford's Law. It has been suggested that multiplicative processes with a chaotic driving force are responsible for producing Benford's Law. Adikhari and Sarkar [1, 2] have shown that if $\{X_i\}$ are independent random variables uniformly distributed on $[0, 1]$ then the distribution of $\prod_{i=1}^{n} X_i$ tends to obey Benford's Law as n increases.

Pietronero et al in an interesting paper have expanded on Adikhari and Sarkar's result and suggested that Benford's Law will hold for multiplicative processes of the form $N(t + 1) = \zeta(t)N(t)$ where ζ is a random variable.

They have sketched a proof that if $\{\zeta(j)\}$ is a sequence of independent identically distributed (i.i.d.) random variables of mean one and $\{N(j)\}$ is a sequence of numbers defined by

$$N(j + 1) = \zeta(j)N(j)$$

then with probability one the sequence of leading digits of $\{N(j)\}$ has the probability distribution known as Benford's Law $P(k) = \log_{10}(\frac{k+1}{k})$, for $k = 1, ..., 9$. They provide numerical evidence that when ζ, with equal probability, takes one of two values $\{\Delta, \Delta^{-1}\}$ Benford's Law does indeed hold.

In our approach we view the stochastic process as a dynamical system and apply results from ergodic theory. This approach allows us to extend the results of Adhikari-Sarkar and Pietronero et al to a wider class of driving systems. We present mainly two novel results. First of all our method implies that the systems producing Benford's Law do not necessarily have to be random or 'chaotic' but may be deterministic and involve dependent variables. We also show that for some simple affine models Benford's Law holds as well and give sufficient conditions for its existence.

This suggests that Benford's Law is likely to be a quite wide-ranging phenomenon, more related to the multiplicative (or almost multiplicative) nature of a process than to it's chaoticity.

Our proofs are motivated by the classical and elegant example in ergodic theory of the distribution of the leading digits of the sequence $\{2^n\}$ and also by recent work in the ergodic theory of skew-product systems.

Background and notation

The ergodic theorem relates the time-average of an observable on a dynamical system to the spatial average of the observable on the system with respect to a reference measure. It is natural to try to determine the frequency of leading digits (which may be considered a time-average of an observable) by identifying it with a spatial average on the system via the ergodic theorem.

A probability measure space is a set X equipped with a measure μ such that $\mu(X) = 1$. A map $f : X \to X$ preserves μ if $\mu(f^{-1}(A)) = \mu(A)$ for every measurable set A. A measure-preserving map f is called ergodic if for every measurable set A, for μ a.e. $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n}\{\#k : f^k(x) \in A : 1 \leq k \leq n\} = \mu(A)$$

i.e. f is ergodic if with probability one the orbit of a point spends a fraction $\mu(A)$ of its time in the set A. This is equivalent to the property that for all integrable observables $\phi : X \to \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k(x)) = \int \phi d\mu$$

This is often described as *time-average is equal to spatial average.*

A map $f : X \to X$ is called *uniquely ergodic* if

$$\lim_{n\to\infty} \frac{1}{n} \{\#k : f^k(x) \in A : 1 \leq k \leq n\} = \mu(A)$$

holds for every $x \in X$.

In this paper the main measure space that is mentioned is the unit interval $[0, 1)$ modulo 1 equipped with Lebesgue measure m. We will sometimes think of $[0, 1)$ (mod 1) as the group of the unit circle S^1.

If $[0, 1)$ is the unit interval (mod 1) then the map $f_\alpha(x) = x + \alpha$ (mod 1) is ergodic if and only if α is irrational. In the case of an irrational rotation of the circle ergodicity implies unique ergodicity.

The notation $\log x$ denote the logarithm of the number x to the base 10. Sometimes we will write $\log_{10}(x)$ to emphasize we are taking logs to the base 10.

The distribution of leading digits of α^n in base k

We first consider the elementary example from ergodic theory of the distribution of the leading digits of the sequence 2^n [3, Appendix 12]. Note that 2^n has leading digit k if and only if there exists r such that $k10^r \leq 2^n < (k + 1)10^r$. Taking logs to the base 10 this is equivalent to $\log k + r \leq n \log 2 < \log(k + 1) + r$ and when we take modulo 1 this is equivalent to $n \log 2 \in [\log k, \log(k + 1))$. The mapping $f(x) = x + \log 2$ is ergodic as $\log 2$ is irrational. In fact it is uniquely ergodic. Hence $f^n(0) = n \log 2$ spends a fraction of time $m[\log k, \log(k+1)) = \log(\frac{k+1}{k})$ in the interval $[\log k, \log(k+1))$ since this is precisely the Lebesgue measure of this interval. But this implies Benford's Law, since the relative frequency of $1's$ is thus $\log 2$, the relative frequency of $2's$ is $\log \frac{3}{2}$ and so on.

Note that the same proof will hold for any number α such that $\log_{10} \alpha$ is irrational. In fact the same result holds for any integer base k, not just base 10, as long as $\log_k \alpha$ is irrational. For example if we consider the distribution of the leading digits of $\sqrt{2}^n$ in base 3 we will find that $1'$s will occur a fraction $\log_3(2)$ of the time and $2'$s will occur a fraction $\log_3(\frac{3}{2})$ of the time.

Chaotically forced multiplicative processes

Now we consider the case of a multiplicative process,

$$N(j+1) = \zeta(j)N(j) \tag{1.1}$$

where for the moment we assume $\{\zeta(i)\}$ is a sequence of independent identically distributed (i.i.d) random variables (not necessarily of mean one).

For notational convenience we further assume that the i.i.d. random variables are discrete and take values in the set $\{\beta_1, ..., \beta_m\}$ $(\beta_i > 0,\ i = 1, ..., m)$.

Thus $N(j) = \prod_{i=0}^{j-1} \zeta(i)N(0)$ and $N(j)$ has leading digit k if and only if for some positive integer r,

$$k10^r \leq N(j) < (k+1)10^{r+1} \tag{1.2}$$

Taking logs of Equation 1.2 above we see that $N(j)$ has leading digit k if and only if $\sum_{i=0}^{j-1} \log(\zeta(i)) \in [\log(k) - \log(N(0)), \log(k+1) - \log(N(0)))$ (mod 1).

Thus we may consider the additive process,

$$\log N(j+1) = \log \zeta(j) + \log N(j)$$

and consider the sequence $\sum_{i=0}^{j-1} \log(\zeta(i))$ taking values in $[0, 1)$ (mod 1) or equivalently the group S^1.

We will interpret this process now as a dynamical system in order to use results from ergodic theory. We first define an ergodic map with respect to a relevant measure

μ on the space of all possible successive choices of the numbers $\{\beta_i\}$, which we denote Ω. We write an element of Ω as $w = (w_0 w_1 w_2 w_3 ... w_n ...)$ where $w_i \in \{\beta_1, ..., \beta_m\}$. The probability of choosing the numbers $\{\beta_i\}$, (say $p(\beta_i) = p_i$ such that $\sum_{i=1}^m p_i = 1$) induces a probability measure μ on Ω in a standard way. For example $\mu\{w \in \Omega : w_1 = \beta_3, w_4 = \beta_1\} = p_3 p_1$. The shift map σ on the space Ω is defined by

$$\sigma(w_0 w_1 w_2 w_3 ... w_n ...) = (w_1 w_2 w_3 ... w_n ...) \ . \tag{1.3}$$

σ preserves the measure μ and in fact is ergodic with respect to μ.

A point $w \in \Omega$ may be identified with a fixed sequence of multiplications $\{\beta_{w_i}\}$. In fact, since $\log \beta_{w_i} = \log \beta_{(\sigma^n w)_0}$ it is useful to define an observable of the dynamical system by the map $\phi : \Omega \to S^1$

$$\phi(w) = \log \beta_{w_0} \ . \tag{1.4}$$

The observable ϕ takes values in S^1 since we are interested in the distribution of the sums of the form $\phi(w) + \phi(\sigma w) + ... \phi(\sigma^n w)$ modulo 1.

Now comes the central part of our proof. We define a map T which relates the driving ergodic dynamical system on $\sigma : \Omega \to \Omega$ to the observable ϕ taking values in S^1

$$T(w, x) = (\sigma w, \phi(w) + x),$$

where $x = N(0)$ is an initial condition (mod 1). The map T maps $\Omega \times S^1$ to itself and preserves the measure $\mu \times m$ where m is Lebesgue measure on S^1. We define $\phi^n(w, x)$ by $T^n(w, x) = (\sigma^n w, \phi^n(w, x))$.

We note that $N(j)$ has leading integer k if and only if $\log k \leq \phi^j(w, x) < \log(k+1)$.

Benford's Law will hold for initial conditions $N(0)$ with probability one if $T : \Omega \times S^1 \to \Omega \times S^1$ is ergodic with respect to $\mu \times m$, since in this case the fraction of times j that $\phi^j(w, x) \in [\log k, \log(k+1))$ will be precisely $\log(\frac{k+1}{k})$ for almost every pair (w, x).

Recent work [6, 9, 4] on the ergodic theory of compact group extensions has shown that T is ergodic if and only if for every nontrivial irreducible representation R of S^1 there are no Hölder solutions $h : \Omega \to S^1$ to the equation

$$R(f(w)) = h(\sigma w)h(w)^{-1}$$

Suppose for some j, $\log \beta_j$ is irrational and consider a periodic point $w = (jjj\cdots...)$, say. Then as $\sigma w = w$ we must have $R(\phi(w)) = Id$ but the group element $f(w)$ generates S^1 and hence R is the trivial representation. Thus the group extension is ergodic and Benford's Law follows in the case that a single number $\log \beta_j$ is irrational, which clearly is a condition which is satisfied with probability one. We chose to present this proof in a skew-product setting since it generalizes to more general base transformations. We have not used the fact that $\{\zeta(i)\}$ is i.i.d. In fact if $\sigma : \Omega \to \Omega$ is an ergodic subshift of finite type (SSFT) we would have the same result. Thus if instead of driving by the shift-map $\sigma : \Omega \to \Omega$ we have a driving map which is a deterministic chaotic system which has been shown to be ergodic with respect to a physically relevant measure (a Sinai-Ruelle-Bowen measure) such as an Anosov system or more generally an Axiom-A attractor then Benford's Law will also hold with probability one. Benford's Law will also hold with probability one for multiplicative processes forced by, for example, n-step Markov processes.

If $\{\zeta(i)\}$ are i.i.d then an alternative is to appeal to the fact that a random walk on a compact Lie group coming from a distribution which has a set of generators in it's support is ergodic and converges in distribution to Haar measure.

Periodic and quasiperiodically forced processes

It is often claimed that the presence of Benford's Law is due to an underlying chaotic or at least random process. In [10], for example, it is suggested that the chaotic nature of the stock prices (in logarithmic space) is the reason that stock prices obey Benford's Law. In this section we show that deterministic multiplicative forcing also generally results in Benford's Law. In particular we show that quasiperiodic and even periodically forced multiplicative processes lead to Benford's Law.

It follows easily from the results of Section 1 that if the multiplicative forcing term is periodic with an irrational frequency then we expect Benford's Law to hold with probability one. This is confirmed by our numerical analysis. We note that for too many iterations the results are not reliable because one runs quickly into numerical round-off errors when either zero or infinity gets approached.

The case of quasiperiodic forcing requires more explanation. Suppose now that $\{\zeta(i)\}$ is a series of positive numbers arising from a quasiperiodic system. More precisely suppose $R_\beta : [0,1) \to [0,1) \pmod 1$ is an irrational rotation, $R_\beta(x) = x + \beta$ and the multiplicative forcing is defined by a function $\zeta(x)$

$$\zeta(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}$$

where the coefficients $\{a_n\}$ satisfy the reality condition $a_n = \bar{a}_n$. (For simplicity we will assume that $\zeta(x)$ is analytic though similar results will hold under weaker regularity assumptions but with a more complicated statement of the precise assumptions).

The sequence $\{\zeta(j)\}$ of variables is then given by $\zeta(R_\beta^j(.))$ so that

$$\zeta(j)(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n (x + j\beta)}$$

where we have explicitly indicated the dependence on the initial condition x. Since we have assumed $\zeta(j) > 0$ for all j we may define $\log(\zeta(j)(x))$. Note that $\log(\zeta(j)(x))$ is also quasiperiodic and so

$$\log(\zeta(j)(x)) = \sum_{n=-\infty}^{\infty} \gamma_n e^{2\pi i n (x + j\beta)}$$

for some coefficients $\{\gamma_n\}$ which satisfy the reality condition $\gamma_n = \bar{\gamma}_n$. We define

$$\log(\zeta(x)) = \sum_{n=-\infty}^{\infty} \gamma_n e^{2\pi i n x}$$

The quasiperiodic multiplicatively forced system will satisfy Benford's Law if the skew-product $T(x,y) = (x+\beta, y+\log\zeta(x))$, $T : S^1 \times S^1 \to S^1 \times S^1$ is ergodic. It is well-known that two mild Diophantine conditions are sufficient for the skew-product to be ergodic. We let $[x]$ denote the real number x mod 1. The two conditions (which follow easily from the analysis of [7, Theorem 7.4]) are there exists $C, \epsilon > 0$ such that for all nonzero integers j, k:

(1) $[\beta j] > C j^{1+\epsilon}$

(2) $j\beta + k\gamma_0 \neq 0$

Condition (2) states that γ_0 and β are not rationally commensurate, ie the map T is in resonance with the inherent quasiperiodicity of the dynamical system. These conditions will be satisfied with probability one [7, Proposition 7.1,7.2] so that quasiperiodic multiplicatively forced systems will also satisfy Benford's Law except in degenerate resonant cases. A similar analysis shows Benford's Law will also hold with probability one for higher dimensional quasiperiodic forcing, on which the base map acts on a higher dimensional torus.

Affine processes

It is a natural step to consider the distribution of leading digits arising from affine processes, as they are perhaps the simplest generalization of multiplicatively forced systems. This study is relevant because affine processes model the effect of additive noise on a multiplicative process and so we are considering how robust the distribution of Benford's Law is to an additive background noise which on its own would lead to a uniform distributions of first digits. In this section we consider affine processes of form

$$N(j+1) = \zeta(j)N(j) + \eta(j)$$

where $\zeta(j)$ and $\eta(j)$ are bounded i.i.d. processes with values in $(0,\infty)$, independent of each other.

There is a difference in the behaviour of such systems determined by whether the expectation value $E(\log \zeta)$ is positive or negative. The number $E(\log \zeta)$ may be thought of as the Lyapunov exponent of the system. We will show that when $E(\log \zeta) > 0$ Benford's Law holds. In this case the i.i.d assumption is not necessary and we may consider $\zeta(j)$ arising from a periodic, quasiperiodic or ergodic deterministic dynamical system as in Section 1.

However we consider two trivial examples to show that if $E(\log \zeta) < 0$ then Benford's Law may not hold. This is confirmed by our numerics. Consider the two maps $f_1(x) = \lambda_1 x + 1$, $f_2(x) = \lambda_2 x + 2$ with $\lambda_i \leq .001$. If both maps are chosen with equal probability then the leading digit will be either 1 or 2 and no other digits will occur. An interesting intermediate case occurs when $\lambda_1 < 1$ and $\lambda_2 > 1$. If $\eta(j) > 0$ then the resulting dynamics will be unbounded but it is not obvious for what classes of such systems Benford's Law will hold. In Fig.1 we show the numerical simulation for $\lambda_1 = 1.5$, $\lambda_2 = 1/1001$ and $\eta(j)$ a uniformly distributed random variable.

However if the system has a positive Lyapunov exponent, that is if $E(\log \zeta) > 0$, then Benford's Law is the stationary distribution.

For simplicity of exposition we will prove this only in the i.i.d case. The proofs of Sections 1 are easily modified to show that with probability one Benford's Law will hold if $\zeta(j)$ come from a periodic or quasiperiodic process. To simplify notation we assume that the random variables ζ and η take only finitely many values $\{\zeta_j\}, \{\eta_k\}$ so that there are, say, m distinct values for the output pairs (ζ_i, η_i), $i = 1, ..., m$.

The stochastic process (1.5) may be modelled as a skew-product over a one-sided shift space of m symbols $\Omega = \{1, ..., m\}^{\mathbb{Z}^+}$.

In this case Ω consists of all one-sided sequences of symbols from the set $\{1, ..., m\}$ so that an element of Ω is of form $w = (w_0 ... w_n ...)$ where $w_i \in \{1, ..., m\}$. The probability p_i of choosing the pair (ζ_i, η_i) is the product of the probability of η assuming the value η_i with the probability of the random variable ζ assuming the value ζ_i, so that $\sum_{i=1}^{m} p_i = 1$. As before the probabilities p_i induce a probability measure μ on

Ω in a standard way. The shift-map $\sigma : \Omega \to \Omega$ given by $(\sigma w)_i = w_{i+1}$ preserves the measure μ and is ergodic with respect to μ.

Let $f_i : \mathbb{R} \to \mathbb{R}$ be the map $f_i(x) = \zeta_i x + \eta_i$, $i = 1, ...m$, and define the skew-product $T(w, x) = (\sigma w, f_{w_0}(x))$. We define $f^n(w, x)$ by $T^n(w, x) = (\sigma^n w, f^n(x))$.

If we define $\lambda := \sum_i p_i \log \zeta_i$ then for μ a.e. $w \in \Omega$ by the multiplicative ergodic theorem [12, Theorem 10.2]

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{n} \log \zeta(\sigma^i w) = \lambda$$

In other words for μ a.e. $w \in \Omega$ given $\epsilon > 0$ there exists $N(w, \epsilon)$ such that

$$(\lambda - \epsilon)^n \leq \prod_{i=0}^{n-1} \zeta(\sigma^i w) \leq (\lambda + \epsilon)^n$$

for all $n \geq N(w, \epsilon)$.

Now we consider a realization of the stochastic process w starting at an initial condition $x = N(0)$.

We calculate $f^n(w, x) = \prod_{i=0}^{n-1} \zeta(\sigma^i w)(x + \frac{\eta_{w_0}}{\zeta_{w_1}} + ... + \frac{\eta_{w_n}}{\prod_{i=0}^{n-1} \zeta(\sigma^i w)}) = \prod_{i=0}^{n-1} \zeta(\sigma^i w)(C(n, w))$.

By our observations for μ a.e. $w \in \Omega$, $C(n, w)$ converges to a constant $C(w)$ as $n \to \infty$.

We note that $N(j)$ has leading integer k if and only if

$$\log k \leq \log(\prod_{i=0}^{j-1} \zeta(\sigma^i w)(C(j, w))) < \log(k + 1)$$

Since $C(j, w)$ converges to a constant the result follows from Section 1 since the fraction of times j that $\sum_{i=0}^{j-1} \log \zeta(\sigma^i w) \in [\log k - \log C(w), \log(k + 1) - \log C(w))$ will be precisely $\log(\frac{k+1}{k})$ for almost every realization w.

Conclusion

We have shown that Benford's Law holds for the numerical output of a wide variety of deterministic and randomly forced multiplicative and even affine processes.

Most of our observations on the occurrence of Benford's Law have been based upon the uniform distribution of an additive process in S^1 which arises by mapping a multiplicative process via the log function to S^1. As we have seen this uniform distribution is commonly produced by ergodic deterministic as well as chaotic driving systems.

Benford's Law is one scaling law amongst many others showing scale invariance in the sense that if a numerical output $\{N(j)\}$ displays Benford's Law then so will $\{\beta N(j)\}$ for any positive constant β. Scale invariance corresponds in our approach to changing the initial condition $N(0)$. The fact that Benford's Law holds for the first digit of stock market prizes for any currency conversion stems from the underlying scale invariance of the process. Another empirical probability distribution often linked to Benford's Law is Zipf's Law which has been found in such diverse areas as city-development [13], earth-quake distributions [5] or linguistics [13]. Zipf's Law states that when ranked by their size, such phenomena as earth-quakes or populations of cities scale with a power-law behaviour. It has been argued that Benford's Law and Zipf's Law are related. We stress here though that Zipf's Law will not be as easily amenable to our dynamical systems point of view. The scaling in Zipf's Law is a global property and ranking is not a simple observable on a system but instead involves the knowledge of all outcomes of the stochastic process. A first promising approach to understand Zipf's Law has been presented in [8]. In a further paper we will consider various models for phenomena in which Zipf's Law has been found from a dynamical systems viewpoint.

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Figure Captions

Fig. 1 Numerical simulation (crosses) for the maps $f_1(x) = \lambda_1 x + \eta(j), f_2(x) = \lambda_2 x + \eta(j)$ with $\lambda_1 = 1.5$ and $\lambda_2 = 1/1001$. The +-signs represent Benford's Law. For the numerical simulation 10^5 initial seeds in the interval $(0, 1)$ were used and each seed was iterated 10^5 times.