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A New Proof of Steinberg's Fixed-Point Theorem

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1 Background

Let V be a complex vector space of dimension $\ell > 0$. A linear transformation $A : V \to V$ is a (pseudo)reflection if it has finite order and has fixed-point subspace $\mathrm{Fix}_V(A)$ of codimension one. A finite group generated by (pseudo)reflections in V is called a "reflection group in V." The celebrated theorem of the title is the following.

Theorem 1.1 [15, Theorem 1.5]**.** Let G be a reflection group in $V \cong \mathbb{C}^\ell$ and let $U \subseteq V$ be a linear subspace. Then $G_U := \{g \in G \mid gv = v \text{ for each element } v \in U\}$ is a reflection subgroup of G, that is, G_U is generated by the reflections in G which fix U pointwise. □

This theorem has numerous applications to representation theory and other areas of mathematics (cf. [2, 3, 4, 5, 13]). The proof originally given by Steinberg in [15] involved the algebra of holomorphic functions on V and a subtle characterisation of reflection groups. Another proof has been given by Serre [1, Chapter V, Example 8, page 139], which is no more elementary, and which is algebraic geometric in nature. It is our purpose here to give a short elementary proof of the theorem. We start with three elementary results upon which our proof is based.

Let V^* be the dual space of V and S^* its coordinate ring identified with the symmetric algebra on V^*. Then G acts on S^* via $gF(v) := F(g^{-1}v)$ for $g \in G$, $F \in S^*$, and $v \in V$, and it is well known that the ring ${S^*}^G$ of polynomial invariants of G is free. If $F_1, F_2, \dots, F_\ell$ is a set of homogeneous free generators of ${S^*}^G$, then the degrees $d_i = \deg F_i$ $(i = 1, \dots, \ell)$ are determined by G, and are called the *invariant degrees* of G. The symmetric algebra

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$S = S(V)$ acts as an algebra of differential operators on S^* with $v \in V$ operating as differentiation D_v in the direction v. We will require the following well-known fact.

Assertion 1.2. For $f \in S$, $P \in S^*$, and $g \in G$, we have

$$g\big(D_f(P)\big) = D_{gf}(gP). \tag{1.1}$$

The second basic fact we require is the characterisation of reflection groups due to Shephard and Todd [10], although we use the formulation below, which is to be found in [13].

Assertion 1.3. Let K be any finite group acting on the vector space V of dimension ℓ over $\mathbb{C}$. Suppose that $F_1, F_2, \dots, F_\ell$ are homogeneous algebraically independent elements of $S(V^*)^K$, and set $d_i := \deg F_i$ for $i := 1, 2, \dots, \ell$. Then

(i) $|K| \leq d_1 d_2 \cdots d_\ell$;

(ii) if $|K| = d_1 d_2 \cdots d_\ell$, then K is a reflection group on V and $S(V^*)^K$ is generated by $F_1, F_2, \dots, F_\ell$ as an algebra;

(iii) if $S(V^*)^K$ is generated by $F_1, F_2, \dots, F_\ell$ as an algebra, then equality holds in (i) and K is a reflection group on V.

Suppose G is a reflection group in V. Let $\mathcal{A}$ be the set of reflecting hyperplanes of G, and for each $H \in \mathcal{A}$, let $L_H \in V^*$ be a linear form corresponding to H, let $r_H \in G$ be a generator of the cyclic group of reflections in H, and let e_H be the order of r_H. Define

$$\Pi_G := \prod_{H \in \mathcal{A}} L_H^{e_H - 1}. \tag{1.2}$$

The third basic result which we will require is as follows.

Assertion 1.4 (Steinberg, cf. [14]). If $F_1, \dots, F_\ell$ are basic invariants for the reflection group G in V and $v_1, \dots, v_\ell$ is a basis of V with dual basis $X_1, \dots, X_\ell$ of V^*, we have

$$\det\left(D_{v_i} F_j\right) = \frac{\partial\left(F_1, \dots, F_\ell\right)}{\partial\left(X_1, \dots, X_\ell\right)} = \Pi_G. \tag{1.3}$$

2 Proof of the theorem

We will now give a proof of Theorem 1.1. Since G acts linearly on V, it clearly suffices to consider the isotropy subgroup of an arbitrary point $v \in V$. We therefore fix $v \in V$, and set $K = G_v = \{g \in G \mid gv = v\}$. Let K_0 be the subgroup of K which is generated by the reflections which fix v; clearly, K_0 is a normal subgroup of K. Our task is to prove that $K = K_0$.

Take any element $g \in K$, and let $P_1, \ldots, P_\ell \in S^*$ be a set of basic invariants of the reflection group K_0. Since g normalises K_0, we may suppose (cf. [13, 6]) that the P_i are eigenfunctions of g, that is

$$gP_i = \epsilon_i P_i \quad \text{for } i = 1, 2, \ldots, \ell, \tag{2.1}$$

for some roots of unity ϵ_i. Suppose we knew that, for each $g \in K$,

$$\epsilon_i = 1 \quad \text{for } i = 1, \ldots, \ell. \tag{2.2}$$

It would then follow that ${S^*}^{K_0} \subseteq {S^*}^K$, and hence by Assertion 1.3(i), that $|K| \leq \prod_i \deg P_i = |K_0|$, whence $K = K_0$. Hence we have shown the following.

Assertion 2.1. The statement (2.2) implies Theorem 1.1.

Since each generator F_i of ${S^*}^G$ is K_0-invariant, there exist unique polynomials $Q_1, Q_2, \ldots, Q_\ell \in \mathbb{C}[y_1, \ldots, y_\ell]$ such that, for each i,

$$F_i(X_1, \ldots, X_\ell) = Q_i(P_1, \ldots, P_\ell). \tag{2.3}$$

The polynomials Q_i may be thought of as the coordinate functions of the polynomial map $\omega_{G,K_0} : \mathbb{C}^\ell \cong V/K_0 \to V/G \cong \mathbb{C}^\ell$ below:

$$\begin{array}{ccc} V & \xrightarrow{\omega_{K_0}} & V/K_0 \\ & \underset{\omega_G}{\searrow} & \downarrow{\scriptstyle \omega_{G,K_0}} \\ & & V/G. \end{array} \tag{2.4}$$

Since $\omega_G = \omega_{G,K_0} \circ \omega_{K_0}$, it follows from the chain rule that

$$\frac{\partial(F_1, \ldots, F_\ell)}{\partial(X_1, \ldots, X_\ell)} = \frac{\partial(Q_1, \ldots, Q_\ell)}{\partial(P_1, \ldots, P_\ell)} \times \frac{\partial(P_1, \ldots, P_\ell)}{\partial(X_1, \ldots, X_\ell)} \in S(V^*). \tag{2.5}$$

Consequently, by applying Assertion 1.4 in turn to G and to K_0, we see that

$$\frac{\partial(Q_1, \ldots, Q_\ell)}{\partial(P_1, \ldots, P_\ell)}(X_1, \ldots, X_\ell) = \prod_{H \in \mathcal{A},\, v \notin H} L_H^{e_H - 1}, \tag{2.6}$$

from which it is evident that $(\partial(Q_1, \ldots, Q_\ell)/\partial(P_1, \ldots, P_\ell))(v) \neq 0$, and hence, that there is some permutation π of $\{1, \ldots, \ell\}$ such that

$$\prod_{i=1}^{\ell} \frac{\partial Q_i}{\partial P_{\pi i}}(v) \neq 0. \tag{2.7}$$

On the other hand, consider $(\partial Q_i/\partial P_j)(X_1,\dots,X_\ell) \in S^*$. By Assertion 1.2 applied to V/K_0, we have, writing p_j for the coordinate in V/K_0, which corresponds to P_j, $g(\partial Q_i/\partial P_j) = g(D_{p_j}Q_i) = D_{gp_j}(gQ_i) = \epsilon_j^{-1}D_{p_j}Q_i$ since $gp_j = \epsilon_j^{-1}p_j$, and Q_i is K-invariant since F_i is G-invariant. It follows that

$$\left(g\frac{\partial Q_i}{\partial P_j}\right)(v) = \frac{\partial Q_i}{\partial P_j}(g^{-1}v) = \frac{\partial Q_i}{\partial P_j}(v) = \epsilon_j^{-1}\frac{\partial Q_i}{\partial P_j}(v), \tag{2.8}$$

from which it is clear that, for any i,

$$\frac{\partial Q_i}{\partial P_j}(v) = 0 \quad \text{if } \epsilon_j \neq 1. \tag{2.9}$$

But for any $j \in \{1,\dots,\ell\}$, there is an index i with $\pi i = j$ in (2.7). Hence, by (2.9), $\epsilon_j = 1$ for each j, and by Assertion 2.1, the proof of Theorem 1.1 is complete.

Note that arguments similar to those above have been used in [4, 6, 7].

3 Remarks concerning the case of positive characteristic

In this section, we consider the applicability of the above argument to the case where the field $\mathbb{C}$ of complex numbers is replaced by an arbitrary field $\mathbb{F}$, the interesting case being when $\mathbb{F}$ has characteristic $p > 0$. Apart from this change, notation is as above, so that $V = \mathbb{F}^\ell$, and S^* is the symmetric algebra on the dual space of V, and so forth. The following result is easily deduced from [8, Theorem 1.4]. Another proof may be found in [11].

Assertion 3.1. Let G be any finite group acting on V and assume that ${S^*}^G$ is a polynomial algebra (i.e., is free). Then for any subspace $U \subseteq V$, ${S^*}^{G_U}$ is again a polynomial algebra.

Note that when $\mathbb{F}$ has characteristic zero, in view of Assertion 1.3, this statement is equivalent to Theorem 1.1. One might ask whether the method above might be adapted to prove Assertion 3.1.

The principal obstacles are, firstly, that when $\mathbb{F}$ has characteristic $p > 0$, one does not have the characterisation of Assertion 1.3, so that in the notation of Section 2, there is no natural candidate for $K_0 \leq K$. However, a Galois-theoretic argument could be substituted, so that K_0 could be defined as a maximal subgroup of K with free ring of invariants using Noether's normalisation lemma. Secondly, one does not have the result of Assertion 1.4, and an alternative method would be necessary to show that the Jacobian of ω_{G,K_0} does not vanish at v. This type of issue is addressed in [12]. Thirdly, since one does not necessarily have complete reducibility, one would obtain only that K/K_0 acts unipotently on V/K_0, so additional arguments such as those in [8] would be required to show that the invariant rings of K and K_0 coincide.

We finally remark that in [9] there is a discussion of which properties of the ring S^{*G} of invariants (such as being polynomial, Cohen-Macaulay, or complete intersection) are inherited by the rings S^{*G_u}.

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