EQUIVARIANT COHOMOLOGY OF CONFIGURATIONS IN $\mathbb{R}^n$

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1. Introduction and statement of results.

Let $C_n(d)$ be the set of ordered configurations of n distinct points in $\mathbb{R}^d$, i.e. $C_n(d) = \{(z_1, \cdots, z_n) | z_i \in \mathbb{R}^d, z_i \neq z_j \text{ if } i \neq j\}$. The symmetric group Sym_n acts on $C_n(d)$, and hence on its (rational) cohomology. The main purpose of this note is to give an explicit formula for the polynomials

$$P_{n,d}(g,t) := \sum_{i \geq 0} \text{ trace } (g, H^i(C_n(d)))t^i,$$

where $g \in \mathrm{Sym}_n$, when d is odd. We shall apply (in §3 below) our formula to study the graded representation of Sym_n on the cohomology ring.

In [L1] $P_{n,d}$ was computed for $d = 2$. The results of that work apply (except for a degree change) whenever d is even. The polynomials arising for d odd are a little different and although they are computed using the algebra of differential forms in [L1], it seems desirable for them to be recorded. The result we shall prove is

Theorem A. *Let n, m, d be positive integers with d odd. Define polynomials*

$$r_n(t) = \sum_{e|n} \mu\left(\frac{n}{e}\right) t^{n-e} \quad \textit{and}$$
$$R_n^{(m)}(t) = r_n(t)(r_n(t) + nt^n)(r_n(t) + 2nt^n) \cdots (r_n(t) + (m-1)nt^n)$$

If $g \in Sym_n$ has cycle decomposition $(i^{m_i})_{i \geq 1}$ (i.e. m_i cycles of length $i, i = 1, 2, \cdots$) then

$$P_{n,d}(g,t) = \prod_{i \geq 1} R_i^{(m_i)}(t^{d-1})$$

Both the formula and proof are similar to the case of even d. One may combine the cases of d odd and even to obtain the following statement.

Theorem B. *Let n, m, d be positive integers. Define polynomials*

$$R_{n,d}^{(m)}(t) = r_n(t)(r_n(t) + (-1)^{d-1}nt^n)(r_n(t) + 2(-1)^{d-1}nt^n)\cdots (r_n(t) + (m-1)(-1)^{d-1}nt^n)$$

where $r_n(t)$ is the polynomial defined in the statement of Theorem A and $g \in Sym_n$ has cycle decomposition $(i^{m_i})_{i\geq 1}$. Then

$$P_{n,d}(g,t) = \Pi_{i\geq 1} R_{i,d}^{(m_i)}((-t)^{d-1}) .$$

When $d = 2$, $C_n(d)$ is a hyperplane complement over $\mathbb{C}$. There are consequently three ways of obtaining the corresponding formula in that case: firstly one has the method of [L1]; secondly one can apply the base change method of [L2], and count rational points over $\mathbb{F}_q$ (a finite field) and thirdly, one has the "function of a centraliser" method of [L3]. Of these, only the first applies easily to the case of d odd. The action of the symmetric group on $H^*(C_n(d), \mathbb{Z})$ is also the subject of [CT], where an approach analogous to that of [LS] for the case where d is even is given. However the explicit determination of the graded character $P_{n,d}(g,t)$ is not addressed there.

2. Proofs.

We follow the steps of the proof given in [L1] for the case of d even, pointing out where statements (and proofs) require modification. It is known (cf. e.g. [C]) that $H^*(C_n(d))$ is generated as associative graded $\mathbb{Q}$-algebra by elements $\{A_{ij} | 1 \leq i \neq j \leq n\}$, which satisfy the following relations.

(2.1):

(i) degree $(A_{ij}) = d - 1$

(ii) $A_{ij} = (-1)^d A_{ji}$

(iii) $A_{ij}A_{k\ell} = (-1)^{d-1} A_{k\ell}A_{ij} \qquad (1 \leq i \neq j, k \neq \ell \leq n)$

(iv) $A_{ij}A_{ik} = A_{kj}(A_{ik} - A_{ij}) \qquad (\forall i, \text{ and } j \leq k)$

(2.2) Lemma. *([L1, (3.3)]) The $\mathbb{Q}$-algebra $H^*(C_n(d))$ has a $\mathbb{Q}$-basis consisting of monomials in the A_{ij} which have one factor from each column in the array below.*

$$\begin{array}{cccc}
1 & & & \\
A_{12} & 1 & & \\
A_{13} & A_{23} & 1 & \\
\vdots & \vdots & & \ddots \\
A_{1,n-1} & A_{2,n-1} & \cdots & 1 \\
A_{1,n} & A_{2,n} & \cdots & A_{n-1,n}
\end{array}$$

Such monomials will be called *standard*.

The proof of (2.2) in the general case is the same as that given in [L1] for d even. We denote the set of standard monomials by $\mathcal{M}$.

(2.3) Definition. ([L1], (3.4))

(a) Let $1 < m_1 < m_2 < \cdots < m_s \leq n$ be a sequence of natural numbers. The sequence $t_1 \leq \cdots \leq t_s$ is said to be an *admissible* sequence from $m_1 < \cdots < m_s$ if (i) $t_s = m_s$ and (ii) $t_i \in \{m_i, t_{i+1}\}$ for $i = 1, 2, \cdots, s-1$.

(b) For each admissible sequence $t : t_1 \le t_2 \le \cdots \le t_s$ from $m_1 < m_2 < \cdots < m_s$, define $\epsilon(t) = \epsilon(t_1 \le t_2 \le \cdots \le t_s) = (-1)^p$ where $p = \#\{i : t_i = t_{i+1}\}$.

The next lemma and its proof appear in [L1, Lemma (3.5)].

(2.4) Lemma. *For any sequence $1 < m_1 < m_2 < \cdots < m_s \le n$, we have*

$$A_{1m_1}A_{1m_1}\cdots A_{1m_2} = \sum_{t\in\mathcal{S}(m_1<\cdots<m_s)} \epsilon(t)A_1t_1A_{m_1t_2}A_{m_2,t_3}\cdots A_{m_{s-1},t_s}$$

where $\mathcal{S}(m_1 < \cdots < m_s)$ denotes the set of admissible sequences from $m_1 < \cdots < m_s$.

The proof given in [L1] carries over, even when d is odd.

For any element $g \in \mathrm{Sym}_n$ and standard monomial $m \in \mathcal{M}$, denote by $\gamma_g(m)$ the coefficient of m in the expression of gm as a linear combination of standard monomials. Then clearly we have

$$P_{n,d}(g,t) = \sum_{m\in\mathcal{M}} \gamma_g(m)t^{d(m)}, \tag{2.5}$$

where $d(m)$ denotes the degree of m.

We treat the case $g = (1, 2, \cdots, n)$ (an n-cycle) first.

The case when d is even was treated in [L1]. Therefore, we henceforth take d to be odd.

(2.6) Proposition. *Let $g = (1, 2, \cdots, n)$.*

(i) *Then for $m \in \mathcal{M}$, $\gamma_g(m)$ is non-zero if and only if m is of the form*

$$m = A_{1,j_1}A_{2,j_1+1}\cdots A_{\ell_1,n}A_{\ell_1+1,j_2}A_{\ell_1+2,j_2+1}\cdots A_{\ell_2,n}A_{\ell_2+1,j_3}\cdots A_{\ell_{s-1}+1,j_s}\cdots A_{\ell_s,n}$$

where $\ell_1 = 1 + n - j, \ell_i = \ell_{i-1} + 1 + n - j_n (i = 2, \cdots, s)$ and $j_1, \cdots, j_s$ is an admissible sequence from $\ell_1 + 1 < \ell_2 + 1 < \cdots < \ell_s + 1$.

(ii) *The standard monomials m such that $\gamma_g(m) \ne 0$ are in bijective correspondence with sequences $1 \le \ell_1 < \ell_2 < \cdots < \ell_s < n$ which satisfy the condition that if $j_1 = 1 + n - \ell_1, j_i = 1 + n + \ell_{i-1} - \ell_i (i = 2, \cdots, s)$, then $j_1, \cdots, j_s$ is an admissible sequence from $\ell_1 + 1 < \cdots < \ell_s + 1$.*

(iii) *Let m be the monomial (see (ii)) corresponding to the sequence $1 \le \ell_1 < \ell_2 < \cdots < \ell_s < n$. Then $\gamma_g(m) = (-1)^s\epsilon(j_1, \cdots, j_s)$.*

The statements and proofs of Proposition (2.6) (i) and (ii) are the same as those of (4.3) and (4.3)$'$ in [L1]. Part (iii) is the variation of (4.3) [loc.cit] which applies when d is odd.

(2.7) Proposition.

(i) *For every divisor e of n and each ordered factorisation $\frac{n}{e} = (i_h + 1)(i_{h-1} + 1)\cdots(i_1 + 1)(i_j \ge 1)$ there is a unique sequence of the type in Proposition (2.6) (ii) above.*

(ii) *The monomial m corresponding to this sequence has degree $(h - 1)(d - 1)$.*

(iii) *With m as in (ii), we have $\gamma_g(m) = (-1)^h$.*

Proposition (2.7) is proved in the same way as (4.5), (4.6), (4.8) and (4.8)$'$ of [L1]. Parts (i) and (ii) are identical to the statements there, while (iii) involves a computation which is modified to take account of the odd nature of d.

(2.8) Lemma. *For any positive integer k, we have*

$$\sum_{k=(i_h+1)\cdots(i_1+1)} (-1)^h = \mu(k)$$

where the sum is over the ordered factorisations of k and μ is the arithmetic Möbius function.

(2.9) Corollary. *Let $g = (1, 2, \cdots, n)$. Then (for d odd)*

$$P_{n,d}(g,t) = r_n(t^{d-1}) \ .$$

This follows immediately from (2.5), (2.7) and (2.8). The deduction of the general case proceeds as in [L1].

(2.10) Lemma. *Let $g = g_1 g_2$ with $g_1 = (1, \cdots, k)$ (a k-cycle) and $g_2 \in$ Sym $\{k+1, \cdots, n\}$. Let $\mathcal{M}_1$ be the set of standard monomials $X_1 X_2 \cdots X_{n-1}$ (see (2.2); X_i from column i) where $X_i = 1$ for $i > k$. Let H_1 be the (linear) span of $\mathcal{M}_1$.*

Then

$$P_{n,d}(g,t) = \sum_{m_1 \in \mathcal{M}_1} \gamma_g(m_1) t^{d(m_1)} P_{n-k,d}(g_2, t) \ .$$

The proof is exactly the same as that of [L1, Lemma (5.2)].

(2.11) Lemma. *Let g be as in (2.10) and suppose that $m_1 = X_1 \cdots X_k$ is a standard monomial with X_i in column i of (2.2). Then $\gamma_g(m_1) \neq 0$ if and only if $m_1 = m_{10} m_{11}$, where $m_{10} = X_1 X_2 \cdots X_{k_s} \in H(k)$, $H(k)$ being the subalgebra of $H^*(C_n(d))$ generated by $\{A_{ij}, 1 \leq i,j \leq k\}$ and $\gamma_{g_1}(m_{10}) \neq 0$ and either $m_{11} = 1$ on $m_{11} = A_{k_s+1,gj} A_{k_s+2,g^2 j} \cdots A_{k,j}$, where $g^{k-k_s} j = j$ (some $j > k$).*

The proof is similar to that of [L1, (5.3) and (5.3)$'$].

(2.12) Lemma.

(i) *Let g and m_1 be as in (2.11). Then $\gamma_g(m_1) = \gamma_{g_1}(m_{10})$.*

(ii) *We have*

$$\sum_{m_1 \in \mathcal{M}_1} \gamma_g(m_1) t^{d(m_1)} = r_k(t^{(d-1)}) + t^{k(d-1)} \sum_{m \in H(k) \cap \mathcal{M}} \gamma_{g_1}(m) N(k - d(m)),$$

where $H(k)$ is the subalgebra of $H^*(C_n(d))$ defined in (2.11) and for any integer i, $N(i) = \#\{j| > k, g^{d_j} = j\}$.

This is an immediate consequence of (2.11) and (2.9).

We are now able to give the

Proof of Theorem A. Use induction on the number of cycles of $g \in \mathrm{Sym}_n$. If g has one cycle, the result is (2.9). Write the cycle type of g as $\ell_1^{m_1}, \ell_2^{m_2}, \cdots, \ell_p^{n_p}$, where $\ell_1 < \ell_2 < \cdots < \ell_p$. We may assume $g = g_1 g_2$, where $g_1 = (1, 2, \cdots, \ell_1)$ and g_2 has cycle type $(\ell_1^{m_1-1}, \ell_2^{m_2}, \cdots, \ell_p^{m_p})$. Then from (2.10) and (2.12) we have (in the notation of (2.12),

$$P_{n,d}(g,t) = \left(r_{\ell_1}(t^{d-1}) + (t^{\ell_1(d-1)} \sum_{m \in H(\ell_1) \cap \mathcal{M}} \gamma_{g_1}(m) N(\ell_1 - d(m)) \right) P_{n-\ell_1,d}(g_2,t)$$

Now $N(\ell_1 - d(m))$ is the number of $j > l_1$ such that j lies in a g-orbit of size dividing $\ell_1 - d(m)$. Since ℓ_1 is the smallest orbit size, this is non-zero only when $d(m) = 0$ (i.e. $m = 1$), and $N(\ell_1) = (n_1 - 1)\ell_1$. Hence

$$P_{n,d}(g,t) = (r_{\ell_1}(t^{d-1}) + (n_1 - 1)\ell_1 t^{\ell_1(d-1)}) P_{n-\ell_1,d}(g_2,t)$$

and the result follows by induction. □

3. Some applications.

(3.1) Theorem. *Let d, n be positive integers with d odd. Then*

(i) *The representation of Sym$_n$ on $H^*(C_n(d)) = \oplus_{i \geq 0} H^i(C_n(d))$ is the regular representation of Sym$_n$.*

(ii) *The representation of Sym$_n$ on $H^{(n-1)(d-1)}(C_n(d))$ (the top cohomology) is $Ind_{<c_n>}^{\mathrm{Sym}_n}\xi$, where $< c_n >$ is the cyclic group generated by the n-cycle c_n and ξ is a faithful character of $< c_n >$ (if $d \geq 3$).*

Proof. For (i) put $t = 1$ in the formula for $P_{n,d}(g,t)$ to see that the character is the character of the regular representation. For (ii) look at the coefficient of $t^{(n-1)(d-1)}$ in the formula for $P_{n,d}(g,t)$. □

Note that for d even, the statement analogous to corollary A$'$ was given in [L1]; it asserts that for d even, the representation of Sym$_n$ on $H^*(C_n(d))$ is $2Ind_{<r>}^{\mathrm{Sym}_n}(1)$, where r is any reflection (2-cycle) in Sym$_n$ and that the representation of Sym$_n$ on $\mathrm{H}^{(n-1)(d-1)}(C_n(d))$ is the sign representation times $\mathrm{Ind}_{<c_n>}^{\mathrm{Sym}_n}\xi$ in the notation of (3.1). The latter of these assertions goes back to Stanley [S].

Theorem (3.1) shows that the sign representation ϵ of Sym$_n$ occurs with multiplicity one in $H^*(C_n(d))$ when d is odd. Our next result shows that the cohomology degree where ϵ occurs is the "middle degree".

(3.2) Theorem. *Let d, n be positive integers with d odd. Then the sign representation ϵ of Sym$_n$ occurs with multiplicity one in $H^{[\frac{n}{2}](d-1)}(C_n(d))$ and nowhere else in*

$H^*(C_n(d))$. That is, the unique (up to multiplication by a scalar) non-zero cohomology class which is alternating with respect to Sym_n has degree $\left[\frac{n}{2}\right]$ in the A_{ij} of (2.1)

Proof. If we regard $P_{n,d}(g,t)$ as the value of the polynomial-valued class function $P_{n,d}(t)$ on Sym_n at $g \in \mathrm{Sym}_n$, then for any character χ of Sym_n, we may express the generating function $\sum_i < H^i(C_n(d)), \chi > t^i = P^{\chi}_{n,d}(t)$ for the multiplicities of χ in the cohomology modules as a sum over Sym_n:

$$P^{\chi}_{n,d}(t) = \sum_i < H^i(C_n(d)), \chi > t^i = \frac{1}{n!} \sum_{g \in \mathrm{Sym}_n} P_{n,d}(g,t)\chi(g) \tag{3.2.1}$$

Our task is to evaluate $P^{\epsilon}_{n,d}(t)$. We first rewrite the formula of Theorem A in a form which suits our computation. First, following [L4], write, for any positive integer i,

$$q_i(t) = \frac{1}{i} \sum_{e|i} \mu\left(\frac{i}{e}\right)(-t)^{-e} \in \mathbb{C}[t^{-1}]. \tag{3.2.2}$$

For any polynomial f and positive integer m, let the binomial $\binom{f}{m}$ denote the polynomial $\binom{f}{m} = \frac{1}{m!} f(f-1)(f-2)...(f-m+1)$. Then

$$R^{(m)}_n(t) = (-nt^n)^m m! \binom{-q_n(-t)}{m}. \tag{3.2.3}$$

It follows that if $g \in \mathrm{Sym}_n$ has cycle type $(i^{m_i})_{i\geq 1}$ as in the statement of Theorem A,

$$P_{n,d}(g,t) = \prod_{i \geq 1} (-iT^i)^{m_i} m_i! \binom{-q_i(-T)}{m_i}, \quad \text{where } T = t^{d-1}. \tag{3.2.4}$$

Taking into account that the number of elements g of Sym_n of cycle type $(i^{m_i})_{i\geq 1}$ is $\frac{n!}{\prod m_i! i^{m_i}}$ and that for such g we have $\epsilon(g) = (-1)^{\sum_i (i-1)m_i}$, it follows from (3.2.1) that

$$\begin{aligned} P^{\epsilon}_{n,d}(t) &= \sum_{(i^{m_i})} \prod_{i \geq 1} (-T)^{im_i} \binom{-q_i(-T)}{m_i} \\ &=: \theta_n(T) \end{aligned} \tag{3.2.5}$$

where the sum is over the partitions $(i^{m_i})_{i \geq 1}$ of n. We shall compute $\theta_n(T)$ by determining the element

$$\Theta(T,z) := \sum_{n \geq 0} \theta_n(T) z^n, \tag{3.2.6}$$

of the ring $\mathbb{C}[[z]][T,T^{-1}]$, where $\theta_0(T)=1$ and for $n\geq 1$, $\theta_n(T)$ is defined as in (3.2.5). Formal expansion, together with (3.2.5) and (3.2.6), yields immediately that

$$\Theta(T,z)=\prod_{i\geq 1}(1+(-Tz)^i)^{-q_i(-T)}. \tag{3.2.7}$$

Now to obtain an explicit form for (3.2.7), we use the cyclotomic identity ([L4, (6.2)]), which asserts that in the ring $\mathbb{C}[[x,t]][t^{-1}]$, where x and t are indeterminates over $\mathbb{C}$, we have

$$\prod_{i\geq 1}(1-x^i)^{q_i(t)}=1+\frac{x}{t}. \tag{3.2.8}$$

Replacing x by x^2 in (3.2.8) and dividing the respective sides of the resulting equation by the corresponding side of (3.2.8), we obtain, after replacing t by $-t$ and q_i by $-q_i$,

$$\prod_{i\geq 1}(1+x^i)^{-q_i(-t)}=\frac{x-t}{x^2-t}. \tag{3.2.9}$$

Finally, the substitutions $x\mapsto -Tz, t\mapsto T$ give the formula

$$\Theta(T,z)=\frac{1+z}{1-Tz^2}=(1+z)(1+Tz^2+T^2z^4+...),$$

from which we deduce that for $n\geq 1$, $\theta_n(T)=T^{[\frac{n}{2}]}$, as required. $\square$

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