RATIONAL POINTS AND COHOMOLOGY OF DISCRIMINANT VARIETIES

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Abstract. Let G be a finite unitary reflection group acting in a complex vector space $V = \mathbb{C}^n$. The *discriminant variety* X_G of G is defined as the space of regular orbits of G on V. Classical examples include the varieties of complex polynomials of degree n with distinct (resp. non-zero distinct) roots. The normaliser $\widetilde{G}$ of G in $\mathrm{GL}(V)$ acts on X_G; in this work we determine the action of $\widetilde{G}/G$ on the cohomology of X_G. In the classical cases this amounts to computing the cohomology of X_G with certain local coefficient systems. Our methods are to compute equivariant weight polynomials by means of explicit counting of the rational points of certain varieties over finite fields, and then to exploit the Hodge purity of the relevant varieties. We obtain some power series identities as a byproduct.

1. Introduction

Let G be a (unitary) reflection group acting in the vector space $V = \mathbb{C}^n$. Let M be the complement in V of the reflecting hyperplanes of G and let $X = X_G$ be the complex algebraic variety $X = M/G$. This is the *discriminant variety* of G. When G is the symmetric group Sym_n acting on $\mathbb{C}^n$ by permutation of coordinates, it is well known that X is isomorphic to the variety of complex polynomials $t^n + a_1t^{n-1} + \cdots + a_n$ with distinct roots, whence the term discriminant variety. Let $\widetilde{G}$ be a subgroup of $\mathrm{GL}(V)$ which normalises G. Then $\widetilde{G}$ acts on X and hence on the cohomology of X. We shall explicitly compute this cohomology action in this paper. It is straightforward, using arguments similar to those in [BL, Proposition 6.9], to reduce the geometric part of this analysis to the case where G is irreducible, and we restrict ourselves to that case here.

The most significant and interesting of these cases is the generic, or "imprimitive" case, when in the notation of [ST], G is the group $G(r, s, n)$ (where $s|r$), and $\widetilde{G}$ is the reflection group $G(r, 1, n)$ for integers n, r. It is easy to see (cf. 3.8, 3.9) that it suffices to consider the case $s = r$, i.e. the inclusion $G = G(r, r, n) \leq G(r, 1, n) = \widetilde{G}$. In this case our computation amounts to the computation of the cohomology of classical discriminant varieties, with local

Date: 26th February, 2003.

2000 *Mathematics Subject Classification.* Primary 14F40; Secondary 14G05, 20G40, 14L30.

coefficient systems (cf. 6.3, 6.5). In addition to the generic case there are six exceptional cases (cf. [BMM, 3.12, 3.13]), which we treat individually.

The form in which we shall express our results is to give explicit formulae for the equivariant Poincaré polynomial

$$P^{\widetilde{G}/G}(X_G,t)=\sum_{i\geq 0}H^i(X_G,\mathbb{C})t^i\in R(\widetilde{G}/G)[t] \tag{1.1}$$

where $R(\Gamma)$ is the Grothendieck ring of (finite dimensional complex) representations of the group Γ. In §2 we shall outline a general strategy for computing the polynomials (1.1). This involves equivariant weight polynomials, as well as counting rational points over finite fields of certain varieties. In the imprimitive case, we obtain the rather surprising result that the cyclic group μ_r acts on the cohomology of X_G only though its characters of order 1 and 2, as in the case of the Weyl groups of type D_n and B_n. This is proved through the explicit computation of certain power series. The formulae we obtain for these power series may be interpreted (see (5.5) below) as power series identities which generalise the cyclotomic identity.

The author wishes to express his gratitude to Anthony Henderson, who pointed out ways to considerably shorten the original arguments used to prove these results.

2. Equivariant weight polynomials, minimal purity and rational points

In this section we discuss general methods for the computation of equivariant Poincaré polynomials. Let X be a complex algebraic variety with a Γ-action, where Γ is a finite group. Following [DL], we define the equivariant weight polynomial with compact supports, $W_c^\Gamma(X,t)$, by

$$W_c^\Gamma(X,t)=\sum_{m\geq 0}E^\Gamma_{c,m}(X)t^m, \tag{2.1}$$

where $E^\Gamma_{c,m}=\sum_{j\geq 0}(-1)^j\mathrm{Gr}_mH^j_c(X,\mathbb{C})$ is the "weight m Euler characteristic with compact support" of X, and Gr_m denotes the weight m graded component in the weight filtration of $H^*_c(X,\mathbb{C})$. The coefficient of t^m should be regarded as a virtual Γ-module, i.e. as an element of the Grothendieck group $R(\Gamma)$ of Γ. As such, it defines a (virtual) character of Γ, and for $g\in\Gamma$, we may speak of the "value" $W_c^\Gamma(X,t;g)=\sum_{m\geq 0}\mathrm{trace}(g,E^\Gamma_{i,m})t^m\in\mathbb{C}[t]$, at $g\in\Gamma$.

The variety X is said to be *minimally pure* (mp) if $H^j_c(X,\mathbb{C})=\mathrm{Gr}_{2j-2n}H^j_c(X,\mathbb{C})$, i.e. if H^j_c is a pure Hodge structure of weight $2j-2n$, where $n=\dim X$. If this applies, then

$$W_c^\Gamma(X,t)=t^{-2n}\sum_{j\geq 0}H^j_c(X,\mathbb{C})(-t^2)^j. \tag{2.2}$$

We also have weight polynomials for ordinary cohomology (i.e. not with compact support). If X is smooth, then by Poincaré duality, we have isomorphisms (cf. [DL, p. 164])

$$H^j_c(X,\mathbb{C}) \xrightarrow{\sim} H^{2n-j}(X,\mathbb{C}), \tag{2.3}$$

which take $\mathrm{Gr}_m H^j_c(X,\mathbb{C})$ to $\mathrm{Gr}_{2n-m} H^j(X,\mathbb{C})$. Hence if X is mp, $H^j(X,\mathbb{C})$ is pure, of weight $2j$. Moreover for any smooth variety X, we have ([DL, (1.6)]),

$$W^\Gamma(X,t) = t^{2n} W^\Gamma_c(X,t^{-1}). \tag{2.4}$$

The first main result we shall use for the computation of this polynomial is the following.

Theorem 2.5. *(cf.*[DL, McM]*) Let X be a complex algebraic variety with a Γ-action, Γ being a finite group. Suppose X is a finite disjoint union $X = \coprod_{i\in\mathcal{I}} X_i$ of locally closed subvarieties X_i which are permuted by Γ. Then*

$$W^\Gamma_c(X,t) = \sum_{\iota\in\mathcal{I}/\Gamma} Ind^\Gamma_{\Gamma_i} W^{\Gamma_i}_c(X_i,t), \tag{2.5.1}$$

where the sum is over the Γ-orbits ι in $\mathcal{I}$, i is any element of ι, and Γ_i is the isotropy group of i in Γ.

The varieties with which we shall be dealing are all defined over algebraic number fields, so that the results of [KL, §2] apply. Choose an appropriate prime ℓ, and let X_q be the variety with $\mathbb{F}_q$-structure and associated Frobenius endomorphism $F : X_q \longrightarrow X_q$ (as in [KL]). Assume that an identification $\overline{\mathbb{Q}}_\ell \longrightarrow \mathbb{C}$ has been fixed. Then according to [KL], we have canonical Γ-equivariant isomorphisms $H^i_c(X_q,\overline{\mathbb{Q}}_\ell) = H^i_c(X,\mathbb{C})$, and if X is mp, then for any element $g \in \Gamma$, we have by the Grothendieck fixed-point formula (cf. [DL, Le1, Le3]) that for almost all q,

$$|X^{gF}_q| = W^\Gamma_c(X, q^{\frac{1}{2}}; g) = q^{-n} P^\Gamma_c(X,-q;g), \tag{2.6}$$

from which we deduce immediately that

$$P^\Gamma_c(X,t) = (-t)^n |X^{gF}_q|_{q\mapsto -t}, \tag{2.7}$$

where $P^\Gamma(X,t)$ is the equivariant Poincaré polynomial $\sum H^j(X,\mathbb{C})t^j$ and $P^\Gamma_c(X,t)$ is the corresponding equivariant Poincaré polynomial $\sum H^j_c(X,\mathbb{C})t^j$ with compact supports. If X is smooth,

$$P^\Gamma_c(X,t) = t^{2n} P^\Gamma(X,t^{-1}) \tag{2.8}$$

Moreover if X is mp, the equation (2.2) may be written

$$W^\Gamma_c(X,t) = t^{-2n} P^\Gamma_c(X,-t^2), \tag{2.9}$$

and combining (2.4),(2.8) and (2.9), we see that

$$W^\Gamma(X,t) = P^\Gamma(X,-t^2). \tag{2.10}$$

Combining this with (2.6), we obtain for $g \in \Gamma$,

$$P^\Gamma(X,-q^{-1};g) = q^{-n}|X_q^{gF}|. \tag{2.11}$$

Equivalently,

$$P^\Gamma(X,t;g) = (-t)^n|X_q^{gF}|_{q\mapsto -t^{-1}}. \tag{2.12}$$

The relation between the equivariant Poincaré polynomial and the equivariant weight polynomial with compact support for an mp variety X is of particular relevance, because it is the latter which may be computed by counting rational points. The following relation may be deduced from the above equations. If $\dim(X) = n$,

$$W_c^\Gamma(X,t^{\frac{1}{2}}) = t^n P^\Gamma(X,-t^{-1}) \tag{2.13}$$

To conclude this section, we note that for all the various forms of cohomology considered here we have a transfer result, which may be expressed as follows. If Δ is any normal subgroup of Γ, then

$$H^*(X/\Delta) = H^*(X)^\Delta \tag{2.14}$$

as Γ/Δ-nodules, where, for any Γ-module A, A^Δ denotes its Δ-fixed points.

3. The generic (imprimitive) case

3.1. **Some discriminant varieties and their coverings.** Let r and n be positive integers and consider the inclusion $G = G(r,r,n) \leq G(r,1,n) = \widetilde{G}$. Recall that $\widetilde{G}$ may be thought of as monomial matrices with entries in the group μ_r of r^{th} roots of unity in $\mathbb{C}$; then G is the subgroup of matrices, the product of whose non-zero entries is 1.

In this case the hyperplane complement M is described as follows.

$$M = M_n = \{(z_1,\cdots,z_n) \in V \mid \forall \zeta \in \mu_r \text{ and } i \neq j, z_i \neq \zeta z_j\}. \tag{3.1}$$

Let $\widetilde{M} = \widetilde{M}_n$ be the corresponding space for $\widetilde{G}$, i.e.

$$\widetilde{M} = \{(z_1,\cdots,z_n) \in M \mid z_i \neq 0 \text{ for all } i\}. \tag{3.2}$$

We shall omit the subscripts when the dimension is clear from the context. Now G and $\widetilde{G}$ act on both M and $\widetilde{M}$. Define $X = M/G$, $\widetilde{X} = \widetilde{M}/\widetilde{G}$ and $\widetilde{Y} = \widetilde{M}/G$.

If $\delta : \widetilde{G} \longrightarrow \mathbb{C}^\times$ is the linear character of $\widetilde{G}$ with kernel G then δ induces an isomorphism $\delta : \widetilde{G}/G \xrightarrow{\sim} \mu_r$. Hence μ_r acts on X and on $\widetilde{Y}$, and we have diagrams

$$
\begin{tikzcd}
M \arrow[r] \arrow[dr] & X_n = M/G \arrow[d, "/\mu_r"] \\
& X_n^0 = M/\widetilde{G}
\end{tikzcd}
\qquad
\begin{tikzcd}
\widetilde{M} \arrow[r] \arrow[dr] & \widetilde{Y}_n = \widetilde{M}/G \arrow[d, "/\mu_r"] \\
& \widetilde{X}_n = \widetilde{M}/\widetilde{G}
\end{tikzcd}
\tag{3.3}
$$

The varieties $X^0 = X_n^0$ and $\widetilde{X}_n$ may be identified with classical discriminant varieties or configuration spaces, and the maps in (3.3) have classical descriptions as follows.

Define, for any variety Z,

$$C_n(Z) = \{(z_1, \cdots, z_n) \in Z^n \mid z_i \neq z_j \text{ if } i \neq j\}/\mathrm{Sym}_n, \tag{3.4}$$

where Sym_n is the symmetric group of degree n.

Lemma 3.5. *(i) We have $\widetilde{X}_n \cong C_n(\mathbb{C}^\times) \cong \{f(t) = t^n + a_1 t^{n-1} + \cdots + a_n \in \mathbb{C}[t] \mid f$ has distinct non-zero roots $\} := \mathcal{P}_n$, and $X_n^0 \cong C_n(\mathbb{C}) \cong \{f(t) = t^n + a_1 t^{n-1} + \cdots + a_n \in \mathbb{C}[t] \mid f$ has distinct roots $\} := \mathcal{Q}_n$.*

(ii) Given the identifications in (i), the varieties X_n and $\widetilde{Y}_n$ of (3.3) may be described as follows: $X_n \cong \{(f(t), \xi) \in \mathcal{Q}_n \times \mathbb{C} \mid \xi^r = f(0)\}$, and $\widetilde{Y}_n \cong \{(f(t), \xi) \in \mathcal{P}_n \times \mathbb{C}^\times \mid \xi^r = f(0)\}$.

(iii) In terms of the descriptions in (ii), the vertical maps in the diagrams (3.3) are the projections to the first coordinate.

Proof. The map $(z_1, \ldots, z_n) \longmapsto \prod_{i=1}^r (t + z_i^r) : \widetilde{M} \longrightarrow \mathcal{P}_n$ has fibres which are precisely the $\widetilde{G}$-orbits on $\widetilde{M}$, which proves the first assertion of (i). Similarly, the map $(z_1, \ldots, z_n) \longmapsto (\prod_{i=1}^r (t + z_i^r), \prod_{i=1}^r z_i) : \widetilde{M} \longrightarrow \{(f(t), \xi) \in \mathcal{P}_n \times \mathbb{C}^\times \mid \xi^r = f(0)$ has fibres which are the G-orbits on $\widetilde{M}$, so that $\widetilde{Y}_n \cong \widetilde{M}/G$ has the stated description. The other assertions of the Lemma follow from similar observations. □

When the need arises to vary r, we may occasionally write $\widetilde{Y}_n^{(r)}$ for the variety $\widetilde{Y}_n$ above. Note that $\widetilde{Y}_n \longrightarrow \widetilde{X}_n$ is an unramified μ_r-covering, while the map $X_n \longrightarrow X_n^0$ is ramified over those points of X_n^0 which have a zero coordinate. The group μ_r acts on the covering spaces by multiplication of the second coordinate (ξ in (3.5(ii))).

Our objective is to determine the action of $\mu_r \cong \widetilde{G}/G$ on the cohomology of $X_n = M/G$ and of $\widetilde{Y}_n = \widetilde{M}/G$. As a consequence, we shall determine the (non-equivariant) cohomology of the spaces in the diagrams (3.3); some of these were previously known, since they arise in the context of real reflection groups (cf. [B, Le1]).

We use the two basic methods described in §2, viz. the additivity of equivariant weight polynomials (cf. [DL]) and the minimal purity of the varieties concerned (loc. cit.) which makes it possible to carry our the computations by means of counting rational points over finite fields ([KL, Le3]).

3.2. **Decomposition of M.** We start with

Lemma 3.6. *Let Z be one of the varieties $M, \widetilde{M}, X_n, X_n^0, \widetilde{X}_n$ or $\widetilde{Y}_n$. Then Z is smooth and minimally pure (mp).*

Proof. The varieties concerned are all quotients of M or $\widetilde{M}$ by a finite group acting freely. The smoothness is therefore clear, because M and $\widetilde{M}$ are hyperplane complements and therefore smooth. For purity, note that hyperplane complements are mp by [Le3, Proposition 2.3]. The statement now follows from (2.14). □

Lemma 3.7. *(i) Let $M^{(i)} \subset M$ be defined by the equation $z_i = 0$. Then each $M^{(i)}$ is closed ($i = 1, \dots, n$), and $M = \coprod_{i=1}^n M^{(i)} \coprod \widetilde{M}$. Further, the $M^{(i)}$ are permuted transitively by $\widetilde{G}$. Let γ be the element of $\widetilde{G}$ given by $z_1 \longmapsto e^{\frac{2\pi i}{r}} z_1$, and $\gamma z_i = z_i (i > 1)$. Then the stabiliser $\widetilde{G}_{M^{(1)}} \cong \langle\gamma\rangle \times \widetilde{G}(r, 1, n-1)$. Denote this group by $\widetilde{G}_1$.*

(ii) We have

$$W_c^{\widetilde{G}}(M, t) = W_c^{\widetilde{G}}(\widetilde{M}, t) + \operatorname{Ind}_{\widetilde{G}_1}^{\widetilde{G}} W_c^{\widetilde{G}_1}(M^{(1)}, t). \tag{3.7.1}$$

Proof. The proof of the statement (i) is straightforward, while (ii) is an application of (2.5) to the situation of (i). □

Corollary 3.8. *We have*

$$W_c^{\mu_r}(M/G, t) = W_c^{\mu_r}(\widetilde{M}/G, t) + W_c(\widetilde{X}_{n-1}), t) 1_{\mu_r}, \tag{3.8.1}$$

where $\widetilde{X}_{n-1}$ is the $(n-1)$ dimensional analogue of $\widetilde{X} = \widetilde{X}_n$.

Proof. Take G-fixed points in the virtual representations of (3.7(ii)) and use (2.14). Clearly $W_c^{\widetilde{G}}(\widetilde{M}, t)^G = W_c^{\widetilde{G}/G}(\widetilde{M}/G, t)$.

On the other hand, note that since $\langle\gamma\rangle$ acts trivially on $M^{(1)}$, it follows that $\langle\gamma\rangle$ acts trivially in $W_c^{\widetilde{G}_1}(M^{(1)}, t)$. Now for any $\widetilde{G}_1$-module σ, we have a linear isomorphism

$$\left(\operatorname{Ind}_{\widetilde{G}_1}^{\widetilde{G}}(\sigma)\right)^G \xrightarrow{\sim} \sigma^{G \cap \widetilde{G}_1} \tag{3.8.2}$$

by Mackey's theorem and Frobenius reciprocity. Moreover if $\langle\gamma\rangle$ acts trivially on σ, then $\langle\gamma\rangle$ acts trivially on both sides of (3.8.2).

Applying this to $\sigma = W_c^{\widetilde{G}_1}(M^{(1)}, t)$, we see that

$$\left(\operatorname{Ind}_{\widetilde{G}_1}^{\widetilde{G}} W_c^{\widetilde{G}_1}(M^{(1)}, t)\right)^G \cong W_c(M^{(1)}/(\widetilde{G}_1 \cap G), t)1_{\mu_r}$$

as (graded) μ_r-space. But $\widetilde{G}_1 \cap G$ induces the group $\widetilde{G}(r, 1, n-1)$ of transformations in $M^{(1)}$, whence the corollary. □

Let us translate (3.8) into the language of rational points over finite fields, using (2.6) and (2.12). In the next statement, we regard $\widetilde{X}_n$ etc. as schemes over a ring of algebraic integers. Recall that $\widetilde{X}_{n,q}$ denotes the $\mathbb{F}_q$-variety corresponding to $\widetilde{X}_n$, etc.

Corollary 3.9. *For any element $\theta \in \mu_r$, we have*

$$W_c^{\mu_r}(M/G, t; \theta) = W_c^{\mu_r}(\widetilde{M}/G, t; \theta) + W_c(\widetilde{X}_{n-1}, t), \tag{3.9.1}$$

and

$$P^{\mu_r}(M_n/G, t; \theta) = P^{\mu_r}(\widetilde{M}_n/G, t; \theta) - tP(\widetilde{X}_{n-1}, t; 1), \tag{3.9.2}$$

where $P(\widetilde{X}_n, t; 1) = (-t)^n |\widetilde{X}_{n,q}^F|_{q \mapsto -t^{-1}}$ and $P^{\mu_r}(\widetilde{M}/G, t; \theta) = (-t)^n |\widetilde{Y}_{n,q}^{\theta F}|_{q \mapsto -t^{-1}}$.

Proof. The statement (3.9.1) is obvious from (3.8). Since, for any mp Γ-scheme Z we have $|Z_q^{gF}| = W_c^\Gamma(Z, q^{\frac{1}{2}}; g)$, we see that

$$|(M/G)_q^{\theta F}| = |(\widetilde{M}/G)_q^{\theta F}| + |(\widetilde{X}_{n-1})_q^F|. \tag{3.9.3}$$

Using (2.12), we obtain

$$P^{\mu_r}(X_n, t; \theta) = P^{\mu_r}(\widetilde{Y}_n, t; \theta) - tP(\widetilde{X}_{n-1}, t; 1), \tag{3.9.4}$$

as stated. Note that (3.9.2) also follows directly by combining (3.9.1) and (2.13). □

4. Rational Points

We now turn to the computation of the cardinalities $|\widetilde{X}_n^F|$ and $|\widetilde{Y}_n^{\theta F}|$ as polynomials in q, where we drop the subscript q when it is clear from the context that we are considering the $\overline{\mathbb{F}}_q$ points of the relevant scheme, which is always defined over a cyclotomic extension of $\mathbb{Q}$. We assume q is such that $\widetilde{G}$ is defined over $\mathbb{F}_q$ so in particular $r|q-1$ and that the results of [KL, §3] hold for the residue field $\mathbb{F}_q$. We know by (loc. cit.) that there are infinitely many q for which this is the case. For convenience we assume also that q is odd. The entries of $\widetilde{G}$ will be thought of as elements of $\mathbb{F}_q$ (recall that the assumptions imply that $\mu_r \subset \mathbb{F}_q{}^\times$), and the isomorphism $\delta : \widetilde{G}/G \xrightarrow{\sim} \mu_r$ of §1 then maps $\widetilde{G}/G$ into $\mathbb{F}_q$. We start with $X = \widetilde{X}_n$. Throughout this section, varieties will be assumed to be over $\overline{\mathbb{F}}_q$. From (3.5) we see that $\widetilde{X}_n^F \cong \mathcal{P}_n^F$, where $\mathcal{P}_n$ is the variety of polynomials

$f(t) = t^n + a_1 t^{n-1} + a_2 t^{n-2} + \cdots + a_n \in \overline{\mathbb{F}}_q[t]$, with distinct non-zero roots (in $\overline{\mathbb{F}}_q$). Here the Frobenius endomorphism acts as the q-power map on the coefficients of polynomials. We use power series generating functions to count the rational points.

Definition 4.1. *Let $n_d(q)$ (resp. $n'_d(q)$) be the number of irreducible (resp. non-zero irreducible) polynomials of degree d over $\mathbb{F}_q$. Let $\widetilde{n}_d(q) = dn_d(q), \widetilde{n}'_d(q) = dn'_d(q)$; then $\widetilde{n}_d(q)$ is the number of elements of $\overline{\mathbb{F}}_q$ whose Frobenius (F-)orbit has size d. Let $k_n(q)$ (resp $m_n(q)$) be the number of polynomials $t^n + a_1 t^{n-1} + \cdots + a_n \in \mathbb{F}_q[t]$ with distinct (resp. distinct non-zero) roots in $\overline{\mathbb{F}}_q$. Thus $m_n(q) = |\mathcal{P}_n^F|$ and $k_n(q) = |\mathcal{Q}_n^F|$.*

Proposition 4.2. *We have the following identities.*

$$n_d(q) = \tfrac{1}{d} \sum_{e|d} \mu(\tfrac{d}{e}) q^e. \tag{4.2.1}$$

$$\begin{aligned} n'_d(q) &= \begin{cases} n_d(q) & \text{if } d \neq 1 \\ n_1(q) - 1 = & q - 1 \ \text{ if } d = 1 \end{cases} \\ &= \tfrac{1}{d} \sum_{e|d} \mu(\tfrac{d}{e})(q^e - 1). \end{aligned} \tag{4.2.2}$$

$$\prod_{d \geq 1} (1 - t^d)^{-n_d(q)} = \sum_{n \geq 0} q^n t^n = (1 - qt)^{-1}. \tag{4.2.3}$$

$$\prod_{d \geq 1} (1 + t^d)^{n_d(q)} = \sum_{n \geq 0} k_n(q) t^n = \tfrac{1 - t^2 q}{1 - tq}. \tag{4.2.4}$$

$$\prod_{d \geq 1} (1 + t^d)^{n'_d(q)} = \sum_{n \geq 0} m_n(q) t^n = \tfrac{1 - t^2 q}{(1 - tq)(1 + t)}. \tag{4.2.5}$$

Proof. (4.2.1) and (4.2.2) are clear; (4.2.3) is the well-known cyclotomic identity (cf. e.g., [Le2]). To obtain (4.2.4) from (4.2.3), invert (4.2.3), replace t by t^2, and multiply the resulting relation by (4.2.3). To obtain (4.2.5) from (4.2.4), use (4.2.2). $\square$

The next statement is now immediate.

Corollary 4.3. *We have, with notation as in (4.2),*

$$k_n(q) = \begin{cases} q^n - q^{n-1} & \text{if } n > 1 \\ q & \text{if } n = 1 \end{cases}, \tag{4.3.1}$$

and

$$m_n(q) = \tfrac{(q-1)(q^n-(-1)^n)}{q+1} = |\mathcal{P}_n^F|. \tag{4.3.2}$$

We now consider the evaluation of $|\widetilde{Y}_n^{\theta F}|$, where $\theta \in \mu_r$. Recall that in analogy with (3.3) we have a diagram of varieties over $\overline{\mathbb{F}}_q$

$$\begin{tikzcd} \widetilde{M} \arrow[r, "p_Y"] \arrow[dr, "p_X"'] & \widetilde{Y}_n \arrow[d, "\pi"] \\ & \widetilde{X}_n \cong \mathcal{P}_n \end{tikzcd} \tag{4.4}$$

where π is a Galois covering with group μ_r.

Lemma 4.5. *With the above notation, we have*

(i) For any element $y \in \widetilde{Y}_n$, $y \in \widetilde{Y}_n^{\theta F}$ for some $\theta \in \mu_r$ if and only if $\pi(y) \in \widetilde{X}_n^F$.

(ii) Define $\tau_r : \widetilde{X}_n^F \longrightarrow \mu_r$ as follows. If $x \in \widetilde{X}_n^F$, then $x = p_X v$ for some $v \in \widetilde{M}$, and $Fv = gv$ for some $y \in \widetilde{G}$. Note that since $v \in \widetilde{M}$, g is unique. Define $\tau_r(x) = \delta(g)$ (recall that $\delta : \widetilde{G} \longrightarrow \overline{\mathbb{F}}_q^{\times}$ is the character with kernel G, which identifies $\langle\gamma\rangle$ with $\mu_r \subset \mathbb{F}_q^{\times}$). Then τ_r is well-defined and

$$|\widetilde{Y}_n^{\theta F}| = r|{\tau_r}^{-1}(\theta^{-1})| = r|\{x \in \widetilde{X}_n^F | \tau_r(x) = \theta^{-1}\}|. \tag{4.5.1}$$

Proof. (i) Any element $y \in \widetilde{Y}_n$ is (cf. (4.4)) of the form $y = p_Y v$ for some $v \in \widetilde{M}$, and so by the commutativity of the diagram (4.4), $\pi y = p_X v$. Now $y \in \widetilde{Y}_n^{\theta F}$ for some $\theta \in \mu_r$ if and only if $Fy = \theta^{-1} y \iff \pi F y = \pi y$, i.e., since all the maps in the diagram (4.4) commute with the Frobenius endomorphisms F on the relevant varieties, $F\pi y = \pi y$, or $\pi y \in \widetilde{X}_n^F$.

(ii) We show first that τ_r is well-defined. Take $x \in \widetilde{M}^F$ and suppose $x = p_X v = p_x v'$, with $v, v' \in \widetilde{M}$. Then $v' = hv$ for some $h \in \widetilde{G}$ and so, with $g \in \widetilde{G}$ as in the statement above, $F(v') = hF(v) = hg^{-1}v = hg^{-1}h^{-1} \cdot v'$. Since $\delta(hg^{-1}h^{-1}) = \delta(g^{-1})$, τ_r is well-defined.

Further, note that $y \in \widetilde{Y}_n^{\theta F} \iff Fy = \theta^{-1} y$. If $y = p_Y v$ with $v \in \widetilde{M}$, then $Fy = p_Y F v = \theta^{-1} p_Y v \iff fv = \theta g v$ for some $g \in G \iff \tau_r(p_X v) = \theta^{-1}$ (note that here, as explained above, we think of μ_r as $\widetilde{G}/G$). This proves (4.5.1). □

Recall next that by (3.5), $\widetilde{X}_n$ may be identified with $\mathcal{P}_n$. We shall therefore define τ_r directly on $\mathcal{P}_n^F$. Suppose $f = t^n + a_1 t^{n-1} + \cdots + a_n \in \mathcal{P}_n^F$, and that $f = p_X(z_1, \cdots, z_n) = (t + z_1^r)(t + z_2^r) \cdots (t + z_n^r)$.

Then $f \in \mathcal{P}_n^F \iff F(z_1, \cdots, z_n) = (z_1^q, \cdots, z_n^q) = g(z_1, \cdots, z_n)$ for some element $g \in \widetilde{G}$, and $\tau_r(f)$ is then defined as $\delta(g)$. Now g is a product of cycles, each of which has a *type* $\alpha \in \mu_r$ (cf. [BL]) which coincides with δ (for a possibly smaller group) evaluated at that cycle, and $\delta(g)$ is the product of these types over the cycles, which correspond to the irreducible factors of f. The map τ_r may therefore be defined as follows.

We first define $\tau_r : \overline{\mathbb{F}}_q \longrightarrow \mu_r$. The *degree* of $\eta \in \overline{\mathbb{F}}_q$ is the length of its Frobenius orbit $\{\eta, \eta^q, \cdots\}$. Write $\mathbb{F}_d = \mathbb{F}_{q^d}$ and $\mathbb{F}^0_{q^d} = \mathbb{F}_d \backslash \cup_{e|d, e\neq d} \mathbb{F}_{q^e}$ if $d > 1$, and $\mathbb{F}^0_1 = \mathbb{F}_q^\times$. This is the set of non-zero elements of $\overline{\mathbb{F}}_q$ of degree d, and $\overline{\mathbb{F}}_q^{\times} = \coprod_{d\geq 1} \mathbb{F}^0_{q^d}$. Let $\eta(\neq 0) \in \mathbb{F}^0_{q^d}$ and let $\xi \in \overline{\mathbb{F}}_q$ be such that $\xi^r = \eta$. Then $\xi^{q^d} = \alpha\xi$, for some $\alpha \in \mu_r$ (take r^{th} powers of both sides and recall that $\eta = \xi^r \in \mathbb{F}_d$), and we define $\tau_r(\eta) = \alpha$. If ξ' is another r^{th} root of η, then writing $\xi' = \beta\xi$, we have, since $\beta \in \mu_r \subset \mathbb{F}_q$, $\xi'^{q^d} = \xi^{q^d}\beta = \alpha\xi\beta = \alpha\xi'$, so that τ_r is well-defined on $\overline{\mathbb{F}}_q$. Moreover it is clear that for $\eta \in \mathbb{F}^0_{q^d}$, $\tau_r(\eta^q) = \tau_r(\eta)$. Thus τ_r is constant on F-orbits in $\overline{\mathbb{F}}_q^{\times}$, which correspond to irreducible polynomials in $\mathbb{F}_q[t]$. Thus τ_r defines a μ_r-valued function on irreducible polynomials in $\mathbb{F}_q[t]$.

Definition 4.6. *(i) For a non-zero irreducible polynomial $f \in \mathbb{F}_q[t]$ of degree d, define $\tau_r(f) \in \mu_r$ by $\tau_r(f) = \tau_r(\eta)$, where $-\eta$ is any root of f in $\overline{\mathbb{F}}_q$. For an arbitrary polynomial $f \in \mathcal{P}^F_n$, define $\tau_r(f) = \prod_i \tau_r(f_i)$.*

(ii) With the above notation, define $m^{(r)}_n(\alpha, q) \in \mathbb{C}[q]$ by

$$m^{(r)}_n(\alpha, q) = \#\{f \in \mathcal{P}^F_n \mid \tau_r(f) = \alpha\}. \tag{4.6.1}$$

Remark 4.7. If $f(t) \in \mathbb{F}_q[t]$ is an irreducible monic polynomial of degree d over $\mathbb{F}_q$, and $-\eta$ is a root of $f(t)$ in $\overline{\mathbb{F}}_q$, it is easy to see that

$$\tau_r(f) = \tau_r(\eta) = \eta^{\frac{q^d-1}{r}} = f(0)^{\frac{q-1}{r}} \tag{4.8}$$

It follows from the above definition that the definition

$$\tau_r(f) = f(0)^{\frac{q-1}{r}} \tag{4.9}$$

applies in general for any polynomial $f(t) \in \mathcal{P}^F_n$.

With τ_r interpreted thus, the relation

$$|\widetilde{Y}^{\alpha F}_n| = rm_n(\alpha^{-1}, q), \tag{4.10}$$

where $m_n(\alpha, q)$ is as in (4.6.1), is precisely what is obtained from the description in (3.5(ii)) of $\widetilde{Y}_n$.

Note that from (3.9) and (4.5.1,4.10) it follows that our task is essentially to compute the polynomials $m^{(r)}_n(\alpha, q)$.

We record the main property of the map $\tau_r : \mathcal{P}^F_n \longrightarrow \mu_r$ in

Lemma 4.11. *The function $\tau_r : \overline{\mathbb{F}}_q^{\times} \longrightarrow \mu_r$ defined above is given by $\tau(\eta) = N(\eta)^{\frac{q-1}{r}}$, where the norm map $N : \overline{\mathbb{F}}_q^{\times} \longrightarrow \mathbb{F}_q^{\times}$ is defined by $N(\eta) = \eta^{\frac{q^d-1}{q-1}}$ for $\eta \in \mathbb{F}^0_{q^d}$.*

To see how knowledge of the function τ_r is used to determine the equivariant Poincaré polynomials (cf. (3.9.2,4.5.1), consider, for $\lambda \in \hat{\mu}_r$, the power series

$$F_\lambda^{(r)}(t) = F_\lambda(t) = 1 + \sum_{n\geq 1}\sum_{\alpha\in\mu_r} m_n^{(r)}(\alpha, q)\lambda(\alpha)t^n \in \mathbb{C}[q][[t]], \tag{4.12}$$

Since $m_d^{(r)}(\alpha, q)$ is the number of polynomials f in $\mathcal{P}_d^F$ such that $\tau_r(f) = \alpha$, we have, taking into account that any polynomial in $\mathcal{P}^F$ is uniquely the product of distinct irreducible polynomials in $\mathbb{F}_q[t]$, that

$$\begin{aligned} F_\lambda^{(r)}(t) =& 1 + \sum_{n\geq 1}\sum_{\alpha\in\mu_r} m_n^{(r)}(\alpha, q)\lambda(\alpha)t^n \\ =& 1 + \sum_{n\geq 1}\sum_{f\in\mathcal{P}_n^F} \lambda(\tau_r(f))t^n \\ =& \prod_{\substack{f\in\mathcal{P}^F \\ f \text{ is irreducible}}} \left(1 + \lambda(\tau_r(f))t^{\deg f}\right). \end{aligned} \tag{4.13}$$

Notice that (cf. 4.1,4.3.2)

$$\sum_{\alpha\in\mu_r} m_n^{(r)}(\alpha, q) = m_n(q) = |\mathcal{P}_n^F|. \tag{4.14}$$

It follows from (4.13) that for fixed $\beta \in \mu_r$, we have

$$\sum_{\lambda\in\hat{\mu}_r} F_\lambda^{(r)}(t)\overline{\lambda}(\beta) = \sum_{n\geq 0} k_n(\beta, q)t^n, \tag{4.15}$$

where

$$\begin{aligned} k_n(\beta, q) &= \sum_\lambda\sum_\alpha m_n(\alpha, q)\lambda(\alpha)\overline{\lambda}(\beta) \\ &= \sum_\alpha m_n(\alpha, q)\cdot r\delta_{\alpha,\beta} \\ &= rm_n(\beta, q). \end{aligned} \tag{4.15.1}$$

Hence we have shown (cf. 4.5.1, 3.9).

Proposition 4.16. *Let $F_\lambda^{(r)}(t) = \prod_{d\geq 1}(1 + \lambda(\alpha)t^d)^{n_d^{(r)}(\alpha,q)}$. Then the coefficient of t^n in $\sum_\lambda \lambda(\alpha)F_\lambda^{(r)}(t)$ is $rm_n^{(r)}(\alpha^{-1}, q) = |\widetilde{Y}_n^{\alpha F}|$ (see 4.6.1). That is, for $\alpha \in \mu_r$,*

$$\sum_{\lambda\in\hat{\mu}_r} \lambda(\alpha)F_\lambda^{(r)}(t) = \sum_{n\geq 0} |\widetilde{Y}_n^{\alpha F}|t^n \tag{4.16.1}$$

.

Thus to compute the μ_r-action on the cohomology of $\widetilde{Y}_n$, we need only compute $F_\lambda^{(r)}(t)$ for each $\lambda \in \hat{\mu}_r$. In more explicit terms, we have

Corollary 4.17. *We have, for $\sigma \in \hat{\mu}_r$,*

$$F_\sigma^{(r)}(t) = \sum_{n\geq 0} t^n q^{-n} \langle P_c^{\mu_r}(\widetilde{Y}_n, -q), \sigma\rangle_{\mu_r}. \tag{4.17.1}$$

and

$$F_\sigma^{(r)}(t) = \sum_{n\geq 0} t^n q^{n} \langle P^{\mu_r}(\widetilde{Y}_n, -q^{-1}), \sigma\rangle_{\mu_r}. \tag{4.17.2}$$

where $P_c^{\mu_r}(\widetilde{Y}_n, x)$ is the virtual μ_r-module $\sum_{i\geq 0} H_c^i(\widetilde{Y}_n)x^i$, and $P^{\mu_r}(\widetilde{Y}_n, x)$ is defined similarly, without compact supports.

Proof. From (4.16), we have

$$\sum_{\lambda\in\hat{\mu}_r} \lambda(\alpha) F_\lambda(t) = \sum_{n\geq 0} |\widetilde{Y}_n^{\alpha F}| t^n \tag{4.17.3}$$

.

But by (2.6), $|\widetilde{Y}_n^{\alpha F}| = q^{-n} P_c^{\mu_r}(\widetilde{Y}_n, -q; \alpha)$, so that

$$\sum_{\lambda\in\hat{\mu}_r} \lambda(\alpha) F_\lambda^{(r)}(t) = \sum_{n\geq 0} t^n q^{-n} P_c^{\mu_r}(\widetilde{Y}_n^{\alpha F}). \tag{4.17.4}$$

Now multiply both sides of (4.17.4) by $\overline{\sigma}(\alpha)$ and sum over α to obtain the statement (4.17.1). The relation (4.17.2) is obtained from (4.17.3) similarly using the equation (2.11). □

The following assertion follows immediately (2.12) and (4.17.2).

Corollary 4.18. *Let $F_\sigma^{(r)}(t) = \sum_{n\geq 0} a_n(\sigma, q) t^n$. Then*

$$\langle P^{\mu_r}(\widetilde{Y}_n, x), \sigma\rangle_{\mu_r} = (-x)^n a_n(\sigma, -x^{-1}). \tag{4.18.1}$$

5. The power series $F_\lambda^{(r)}(t)$

In this section we carry out the computation of the power series $F_\lambda^{(r)}(t)$, where $\lambda \in \hat{\mu}_r$ (see (4.12)). We shall prove

Theorem 5.1. *Let r be any positive integer, let q be a prime power such that $r|q-1$, and let λ be a character of the cyclic group μ_r. Let $F_\lambda^{(r)}(t)$ be the power series defined in (4.12). Then $F_\lambda^{(r)}(t)$ is given explicitly by*

$$F_\lambda^{(r)}(t) = \begin{cases} 1 \text{ if } |\lambda| \neq 1 \text{ or } 2 \\ \frac{1-qt^2}{1-t^2} \text{ if } |\lambda| = 2 \\ \frac{1-t^2q}{(1-tq)(1+t)} \text{ if } \lambda = 1. \end{cases} \tag{5.1.1}$$

Proof. We have

$$\begin{aligned} F_\lambda^{(r)}(t) =& 1+\sum_{n\geq 1}\sum_{f\in\mathcal{P}_n^F}\lambda(\tau_r(f))t^n \\ =&1+\sum_{n\geq 1}\sum_{f\in\mathcal{P}_n^F}\chi(f(0))t^n \\ =&1+\sum_{f\in\mathcal{P}^F,\deg f>0}\chi(f(0))t^{\deg f}, \end{aligned} \tag{5.1.2}$$

where $\chi:\mathbb{F}_q{}^\times\longrightarrow\mathbb{C}^\times$ is the character defined by

$$\chi(\eta)=\lambda(\eta^{\frac{q-1}{r}})\quad(\eta\in\mathbb{F}_q{}^\times). \tag{5.1.3}$$

For any character $\chi:\mathbb{F}_q{}^\times\longrightarrow\mathbb{C}^\times$, write $G_\chi(t)$ for the power series in the last line of (5.1.2). Since any polynomial $f\in\mathcal{P}^F$ is uniquely the product of distinct irreducible polynomials in $\mathcal{P}^F$, we have

$$G_\chi(t)=\prod_{\substack{f\in\mathcal{P}^F\\ f\text{ is irreducible}}}\left(1+\chi(f(0))t^{\deg f}\right). \tag{5.1.4}$$

Now let

$$\begin{aligned} H_\chi(t) =&1+\sum_{\substack{f\in\mathbb{F}_q[t],f\text{ is monic}\\ \deg f>0,f(0)\neq 0}}\chi(f(0))t^{\deg f} \\ =&\prod_{\substack{f\in\mathcal{P}^F\\ f\text{ irreducible}}}(1-\chi(f(0))t^{\deg f})^{-1}, \end{aligned} \tag{5.1.5}$$

the equality following from the fact that every polynomial $f\in\mathbb{F}_q[t]$ with $f(0)\neq 0$ has a unique expression as a product of irreducible polynomials in $\mathcal{P}^F$. Since $1+a=\frac{1-a^2}{1-a}$, it follows that

$$G_\chi(t)=\frac{H_\chi(t)}{H_{\chi^2}(t^2)}. \tag{5.1.6}$$

Further, it is evident that

$$H_\chi(t)=\begin{cases}1\text{ if }\chi\neq 1\\ \frac{1-t}{1-qt}\text{ if }\chi=1.\end{cases} \tag{5.1.7}$$

Combining (5.1.6) and (5.1.7), we see that for any character χ,

$$G_\chi(t)=\begin{cases}1\text{ if }\chi^2\neq 1\\ \frac{1-qt^2}{1-t^2}\text{ if }\chi^2=1,\chi\neq 1\\ \frac{1-t^2q}{(1-tq)(1+t)}\text{ if }\chi=1.\end{cases} \tag{5.1.8}$$

Finally, applying (5.1.8) with χ equal to the character defined in (5.1.3), we obtain the assertion of the Theorem. $\square$

Theorem (5.1) may be used (cf. (4.16)) to give explicit expressions for the polynomials

$$m_n^{(r)}(\alpha, q) = |\{f \in \mathcal{P}_n^F \mid \tau_r(f) = \alpha \in \mu_r\}| = \frac{1}{r}|\widetilde{Y}_n^{\alpha F}|.$$

These are by (4.16.1) the coefficients of t^n in $\frac{1}{r}\sum_{\lambda\in\hat{\mu}_r}\lambda(\alpha)F_\lambda(t)$, which may be evaluated directly using (5.1) and the elementary

Lemma 5.2. *Let r be a positive integer and suppose $t|r$. Then*

$$\sum_{u|r}\frac{\phi(u)}{\phi(\frac{u}{\gcd(t,u)})}\mu(\tfrac{u}{\gcd(t,u)}) = \begin{cases} 0 \text{ if } t \neq r \\ r \text{ if } t = r \end{cases} \tag{5.2.1}$$

The result is

Corollary 5.3. *Suppose $\alpha \in \mu_r$ has order $s|r$. Then*
(i) If r is odd, then

$$\sum_{\lambda\in\hat{\mu}_r}\lambda(\alpha)F_\lambda(t) = \frac{t(q-1)}{(1-tq)(1+t)} + r\delta_{s,1}. \tag{5.3.1}$$

(ii) If r is even, then

$$\sum_{\lambda\in\hat{\mu}_r}\lambda(\alpha)F_\lambda(t) = \frac{t(q-1)}{(1-tq)(1-t^2)}\Big((1-t)-(-1)^{\frac{r}{s}}(t-t^2q)\Big) + r\delta_{s,1} \tag{5.3.2}$$

where $\delta_{i,j}$ is the Kronecker delta.

Note that $(-1)^{\frac{r}{s}}$ is just the value of the character ϵ_{μ_r} of order 2 of μ_r at any element α of order s in μ_r, so the relation (5.3.2) may be written with the right side transparently a function of α.

Remark 5.4. (i) It may be easily checked that in all cases,

$$\frac{1}{r}\sum_{\alpha\in\mu_r}\left(\sum_{\lambda\in\hat{\mu}_r}\lambda(\alpha)F_\lambda(t)\right) = \frac{1-t^2q}{(1-tq)(1+t)},$$

which is the relation (4.14).

(ii) Our results (e.g. (5.3)) show that the polynomials $m_n^{(r)}(\alpha, q) = \#\{f \in \mathcal{P}_n^F \mid f(0)^{\frac{q-1}{r}} = \alpha\}$ (for $\alpha \in \mu_r \subset \mathbb{F}_q$), depend only on the parity of $\frac{r}{|\alpha|}$. These polynomials can of course be written down explicitly from the formulae in (5.3).

Remark 5.5. For $\alpha \in \mu_r$, define $n_d^{(r)}(\alpha, q)$ to be the number of irreducible polynomials f of degree d in $\mathcal{P}^F$ such that $\tau_r(f) = \alpha$. It is not difficult to show that $n_d^{(r)}(\alpha, q)$ depends only on the order $s = |\alpha|$, and hence may be written $n_d^{(r)}(s, q)$ (for $s|r$). The polynomial $r\phi(s)n_d^{(r)}(s, q)$ (where ϕ is the Euler ϕ-function) is then the number of elements of $\overline{\mathbb{F}}_q$ which have degree ds over $\mathbb{F}_q$, and whose r^{th} power has degree d. Using elementary Möbius inversion arguments, one may prove that

$$n_d^{(r)}(s, q) = \frac{1}{rd(r)} \sum_{e|\frac{r}{s}, e|d(r)} \mu(e) e n'_{d'(r)}\left(q^{\frac{d(r)}{e}}\right), \tag{5.5.1}$$

where $d = d(r)d'(r)$, with $d(r)$ involving only primes which divide d, and $\gcd(d'(r), r) = 1$ and where $n'_d(q)$ is defined in (4.2.2).

It is clear from (4.13) that

$$F_\lambda^{(r)}(t) = \prod_{\substack{d \geq 1 \\ \alpha \in \mu_r}} \left(1 + \lambda(\alpha)t^d\right)^{n_d^{(r)}(\alpha, q)}. \tag{5.6}$$

For any integer $s > 0$ define polynomials $\psi_s(X)$ by

$$\psi_s(X) = \prod_{\substack{\theta \in \mathbb{C}^\times \\ |\theta| = s}} (1 + \theta X). \tag{5.7}$$

If $\lambda \in \hat{\mu}_r \subset \mathbb{C}^\times$ has order r (i.e. is faithful), then clearly

$$F_\lambda^{(r)}(t) = F_r(t) = \prod_{d \geq 1} \prod_{s|r} \psi_s(t^d)^{n_d^{(r)}(s, q)}. \tag{5.8}$$

It follows from Theorem 5.1 that the right side of (5.8) is equal to 1 unless $r = 1$ or 2, in which case the values are given by (5.1.1). The last statement may be thought of as a set of power series identities which generalise the identities (4.2.3),(4.2.4) and (4.2.5).

6. Equivariant Poincaré polynomials

We interpret the results of the last section in terms of equivariant Poincaré polynomials.

Theorem 6.1. *Let $\widetilde{Y}_n$ ($n \geq 1$) be the complex algebraic variety $\widetilde{M_n}/G(r, r, n)$ defined in (3.2). Then $\widetilde{Y}_n$ is an unramified cyclic covering of degree r of the discriminant variety $\mathcal{P}_n$ of polynomials $f(t) = t^n + a_1 t^{n-1} + \cdots + a_n$ with distinct non-zero roots. The μ_r-equivariant Poincaré polynomial $P^{\mu_r}(\widetilde{Y}_n, t) :=$*

$\sum_i H^i(\widetilde{Y}_n, \mathbb{C})t^i \in R(\mu_r)[t]$ *is given by*
(6.1.1)

$$P^{\mu_r}(\widetilde{Y}_n, t) = \begin{cases} (1+t)(1+t+t^2+\cdots+t^{n-1})1_{\mu_r} \text{ if either } r \text{ or } n \text{ is odd} \\ (1+t)(1+t+t^2+\cdots+t^{n-1})1_{\mu_r} + (t^{n-1}+t^n)\epsilon_{\mu_r} \text{ otherwise,} \end{cases}$$

where ϵ_{μ_r} is the unique character of μ_r of order two.

Proof. It follows from (4.18) and (5.1) that the graded multiplicity of a character $\sigma \in \hat{\mu}_r$ in $P^{\mu_r}(\widetilde{Y}_n, t)$ is $(-t)^n a_n(\sigma, -t^{-1})$, where $a_n(\sigma, q)$ is the coefficient of t^n in $F_{|\sigma|}(t)$. It follows immediately from (5.1), that this graded multiplicity is zero unless $|\sigma| = 1$ or 2, i.e. unless $\sigma = 1_{\mu_r}$ or ϵ_{μ_r}. The graded multiplicities in these two cases are easily computed from the formulae given in Theorem (5.1). □

Note that $\widetilde{Y}_0$ is just μ_r, so that $P^{\mu_r}(\widetilde{Y}_0, t)$ is the regular representation of μ_r, concentrated in degree zero.

Theorem 6.1 may be interpreted in terms of the cohomology of the discriminant variety $\mathcal{P}_n$ with local coefficients. For each character ω of μ_r, there is a local system ζ_ω on $\mathcal{P}_n = \widetilde{X}_n$ which arises from the covering $\widetilde{Y}_n^{(r)}$ of $\widetilde{X}_n$. Note that if the order of ω is $s(|r)$, this local system is the same as that arising from the covering $\widetilde{Y}_n^{(s)}$ of $\widetilde{X}_n$ and corresponding to a faithful character ω_0 of μ_s. This is because of the commutativity of the diagram

(6.2)

$$\begin{tikzcd} \widetilde{Y}_n^{(r)} \arrow[r, "\sigma_{rs}"] \arrow[dr, "\pi_r"'] & \widetilde{Y}_n^{(s)} \arrow[d, "\pi_s"] \\ & \widetilde{X}_n = \mathcal{P}_n, \end{tikzcd}$$

where $\sigma_{rs}(f(t), \xi) := (f(t), \xi^{\frac{r}{s}})$. It follows that it suffices to consider faithful characters ω.

Note that the local systems ζ_ω are different from those considered in [SS] and [DPS].

Corollary 6.3. *The Poincaré polynomial of the cohomology of $\mathcal{P}_n$ with local coefficient system ζ_ω of order $r > 0$ as above is given by*
(6.3.1)

$$P(\mathcal{P}_n, \zeta_\omega, t) = \sum_{i \geq 0} \dim H^i(\mathcal{P}_n, \zeta_\omega)t^i = \begin{cases} (1+t)(1+t+t^2+\cdots+t^{n-1}) \text{ if } r = 1 \\ t^{n-1} + t^n \text{ if } r = 2 \\ 0 \text{ otherwise.} \end{cases}$$

The next result is well known.

Corollary 6.4. *Let $\widetilde{X}_n = \widetilde{Y}_n/\mu_r \cong \widetilde{M}_n/G(r,1,n) \cong \mathcal{P}_n$. Then $P(\widetilde{X}_n, t) = (1+t)(1+t+t^2+\cdots+t^{n-1})$.*

This follows immediately from (6.1), taking into account that by (2.14), $P(\widetilde{X}_n,t)$ is the 1_{μ_r}-isotypic component of $P^{\mu_r}(\widetilde{Y}_n,t)$.

Corollary 6.5. *Let X_n be the discriminant variety of the unitary reflection group $G(r,r,n)$. Then $\mu_r \cong G(r,1,n)/G(r,r,n)$ acts on X_n (with quotient the discriminant variety $\mathcal{Q}_n$) and the equivariant Poincaré polynomial is given by*

$$(6.5.1) \quad P^{\mu_r}(X_n,t) = \begin{cases} (1+t)1_{\mu_r} \text{ if either } r \text{ or } n \text{ is odd} \\ (1+t)1_{\mu_r} + (t^{n-1}+t^n)\epsilon_{\mu_r} \text{ if } r \text{ and } n \text{ are both even,} \end{cases}$$

where ϵ_{μ_r} is the unique character of μ_r of order two.

Proof. This is a straightforward application of (3.9.2), taking into account (6.1,6.4). □

Our final remark in this section is that Theorem 6.1 may be used (together with (3.9.2) and (4.16.1)) as an alternative way to perform the computation in (5.3).

7. The six exceptional cases

It follows from [BMM, Proposition 3.13] that in addition to the generic (imprimitive) cases treated above, there are just six other cases to consider, which we refer to as exceptional. Of the six, three are in dimension two.

7.1. **The two-dimensional cases.** We first make some general observations about the two-dimensional situation, and then treat the three specific cases.

Let G be a unitary reflection group in $V = \mathbb{C}^2$ and let $\widetilde{G} \geq G$ be a subgroup of $GL(V)$ which normalises G. Let M be the complement of the reflecting lines of G in V. It is well-known that as G-module (resp. $\widetilde{G}$-module), $H^1(M)$ is the permutation module Π_G of G (resp. $\Pi_{\widetilde{G}}$ of $\widetilde{G}$) on the reflecting lines. Since the Lefschetz module $\Lambda(M) = H^0(M) - H^1(M) + H^2(M) = 0$ (see, e.g. [Le3]), it follows that $H^2(M) = \Pi_G - 1_G$.

One deduces easily, using (2.14), that the action of $\widetilde{G}/G$ on $H^*(X_G = M/G)$ is given by the Poincaré polynomial (cf. 1.1)

$$(7.1) \qquad P^{\widetilde{G}/G}(X_G,t) = 1_{\widetilde{G}/G} + t\widetilde{\Pi}_{\widetilde{G}/G} + t^2(\widetilde{\Pi}_{\widetilde{G}/G} - 1_{\widetilde{G}/G}),$$

where $\widetilde{\Pi}_{\widetilde{G}/G}$ is the permutation action of $\widetilde{G}/G$ on the G-orbits of reflecting lines of G.

7.1.1. *Case 1: $G = G(4,2,2)$, $\widetilde{G} = G_6$.* We use the Shephard-Todd notation in referring to the irreducible reflection groups. Here $\widetilde{G}$ induces an automorphism α of order 3 on G. Now G has 6 reflecting lines, which divide into 3 G-orbits, each with 2 lines. If α fixes these G-orbits, it must act trivially on the lines, since it has order 3. Hence for similar reasons, it must centralise each reflection, and hence acts trivially on G, which is false. It follows that α permutes the 3 orbits cyclically and hence by (7.1), since $\widetilde{G}/G \cong \mu_3$ here,

$$P^{\widetilde{G}/G}(X_G, t) = (1+t)1_{\mu_3} + (t+t^2)(\omega+\omega^2), \tag{7.2}$$

where ω and ω^2 are the characters of order 3 of μ_3.

7.1.2. *Case 2:* $G = G_5$, $\widetilde{G} = G_14$. Here $\widetilde{G}/G \cong \mu_2$. The following facts may be found in [Co] or [LT, Ch. 5, §5, p. 56]: G has 8 reflecting lines, which fall into two G-conjugacy classes. These classes are interchanged by $\widetilde{G}$. Hence in this case, we have from (7.1),

$$P^{\widetilde{G}/G}(X_G, t) = (1+t)1_{\mu_2} + (t+t^2)\epsilon, \tag{7.3}$$

where ϵ is the character of order 2 of μ_2.

7.1.3. *Case 3:* $G = G_7$, $\widetilde{G} = G_10$. Again $\widetilde{G}/G \cong \mu_2$, and the following facts may be found in [Co] or [LT, *loc. cit.*]: G has 14 reflecting lines, which divide into 3 G-orbits of size $4, 4$ and 6; $\widetilde{G}$ interchanges the two orbits of size 4. Hence in this case, we have from (7.1),

$$P^{\widetilde{G}/G}(X_G, t) = (1+t)^2 1_{\mu_2} + (t+t^2)\epsilon, \tag{7.4}$$

where ϵ is the character of order 2 of μ_2.

This completes our treatment of the two dimensional exceptional cases.

7.2. **The higher dimensional exceptional cases.** Finally, we discuss the three exceptional cases where G acts in dimension greater than 2.

7.2.1. *Case 1:* $G = G(3,3,3)$, $\widetilde{G} = G_{26}$. It follows from (6.5) above that the discriminant variety X_G of G has cohomology of dimension 1 in degrees 0 and 1, and vanishes in other degrees. But by the argument used to prove (7.1), $\widetilde{G}$ acts trivially on $H^1(X_G)$, whence $\widetilde{G}$ acts trivially on $H^*(X_G)$. From (6.5) it follows that

$$P^{\widetilde{G}/G}(X_G, t) = (1+t)1_{\widetilde{G}/G}, \tag{7.5}$$

7.2.2. *Case 2:* $G = W(D_4) = G(2,2,4)$, $\widetilde{G} = W(F_4) = G_{28}$. In this case $\widetilde{G}/G \cong \mathrm{Sym}_3$. If the triality automorphism of G is denoted τ, then $\mu_2 = \mathrm{Sym}_3/\langle\tau\rangle$ acts on $H^*(X_G)$ as $G(2,1,4)/G(2,2,4) = \mu_2$. Hence it remains only to determine the action of τ. But by (6.5), every non-trivial cohomology space of X_G has dimension 1; moreover the action of τ on each $H^i(X_G)$ is defined over $\mathbb{Q}$ (even $\mathbb{Z}$), by [B] or [D]. It follows, since τ has order 3, that τ acts trivially on $H^*(X_G)$. Hence

$$P^{\mathrm{Sym}_3}(X_G, t) = (1+t)^2 1_{\mu_2} + (t^3+t^4)\epsilon, \tag{7.6}$$

where ϵ is the non-tivial one dimensional character of Sym_3.

The final case is

7.2.3. *Case 3: $G = W(F_4)$ and $\widetilde{G}$ is the extension of G by its graph automorphism.* It is convenient to recall (cf. [B] and [LS, Lemma 2.5]) that in the above notation, if G is any reflection group, we have for any integer $p \geq 0$,

$$H^p(M_G) = \oplus_{\substack{\bar{X}\in L(\mathcal{A}_G)/G \\ \operatorname{codim} X = p}} \operatorname{Ind}^G_{N_G(G_X)} H^p(M_{G_X}), \tag{7.7}$$

where $\mathcal{A}_G$ is the arrangement of reflecting hyperplanes of G, $L(\mathcal{A}_G)$ is the lattice of intersections of these hyperplanes, and for $X \in L(\mathcal{A}_G)$, $\bar{X}$ is its G-orbit in $L(\mathcal{A}_G)$, and G_X is its pointwise stabiliser in G. It follows, taking G-fixed points on the right side that

$$\dim H^p(M_G/G) = \sum_{\substack{\bar{X}\in L(\mathcal{A}_G)/G \\ \operatorname{codim} X = p}} \langle H^p(M_{G_X}), 1\rangle_{N_G(G_X)}, \tag{7.8}$$

For $G = W(F_4)$, these dimensions are given (see [B], [Le1]) in the following table

$$\begin{array}{rcccccc} p & : & 0 & 1 & 2 & 3 & 4 \\ \dim H^p(M_G/G) & : & 1 & 2 & 2 & 2 & 1 \end{array} \tag{7.9}$$

Now denote by σ the graph automorphism of F_4, regarded as an orthogonal transformation of the space on which $W(F_4)$ acts. Then σ permutes the summands on the right side of (7.7), and acts correspondingly on the spaces of G-fixed points. Using the Orlik-Solomon presentation of $H^*(M_G)$ as described in this context in [LS, §2], the action of σ on $H^p(M_G/G)$ may readily be determined. The result is

$$P^{\langle\sigma\rangle}(M_G/G, t) = (1 + 2t^3 + t^4)1_{\langle\sigma\rangle} + (t + t^2)\operatorname{Reg}_{\langle\sigma\rangle}, \tag{7.10}$$

where $1_{\langle\sigma\rangle}$ and $\operatorname{Reg}_{\langle\sigma\rangle}$ respectively denote the trivial and regular representations of $\langle\sigma\rangle$.

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