STRONGLY MULTIPLICITY FREE MODULES FOR LIE ALGEBRAS AND QUANTUM GROUPS

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To Gordon James on his 60^{th} birthday

Abstract. Let $\mathcal{U}$ be either the universal enveloping algebra of a complex semisimple Lie algebra $\mathfrak{g}$ or its Drinfel'd-Jimbo quantisation over the field $\mathbb{C}(z)$ of rational functions in the indeterminate z. We define the notion of "strongly multiplicity free" (smf) for a finite dimensional $\mathcal{U}$-module V, and prove that for such modules the endomorphism algebras $\text{End}_{\mathcal{U}}(V^{\otimes r})$ are "generic" in the sense that they are quotients of Kohno's infinitesimal braid algebra T_r (in the classical (unquantised) case) and of the group ring $\mathbb{C}(z)B_r$ of the r-string braid group B_r in the quantum case. In the classical case, the generators are generalisations of the quadratic Casimir operator C of $\mathcal{U}$, while in the quantum case, they arise from R-matrices, which may be thought of as square roots of a quantum analogue of C in a completion of $\mathcal{U}^{\otimes r}$. This unifies many known results and brings some new cases into their context. These include the irreducible 7 dimensional module in type G_2 and arbitrary irreducibles for $\mathfrak{sl}_2$. The work leads naturally to questions concerning non-semisimple deformations of the relevant endomorphism algebras, which arise when the ground rings are varied.

1. Introduction

Suppose $\mathcal{A}$ is an algebraic structure such as a group or a Hopf algebra, which is represented as an algebra of endomorphisms of a finite dimensional vector space V. Then $\mathcal{A}$ has a representation in $V^{\otimes r}$ for $r = 1, 2, \dots$, and the study of the endomorphism algebras $\text{End}_{\mathcal{A}}(V^{\otimes r})$ has been of great interest for more than 100 years. The case when $\mathcal{A} = \text{GL}(V)$ was first considered by Schur, and leads to "Schur-Weyl duality" between representations of $GL(V)$ and representations of the symmetric groups Sym_r. When $\mathcal{A}$ is a classical group, i.e. a symplectic or orthogonal group acting on V, then the Brauer algebras $B_r(\delta)$ arise ([Br]), with the parameter $\delta = \pm \dim V$. These have been of interest in a wide range of questions concerning representation theory and invariants of oriented links.

When $\mathcal{A}$ is a quantised enveloping algebra, there are well-known connections between the study of $\text{End}_{\mathcal{A}}(V^{\otimes r})$, representations of the r-string braid group, and invariants of oriented links.

The purpose of this work is to define the notion of a *"strongly multiplicity free"* (smf) irreducible $\mathcal{A}$ module when $\mathcal{A}$ is either a classical semisimple Lie algebra $\mathfrak{g}$ (or equivalently its universal enveloping algebra), or the quantised universal enveloping algebra of $\mathfrak{g}$. We classify the pairs $(\mathcal{A}, V)$ which are smf and show that for such pairs, $\text{End}_{\mathcal{A}}(V^{\otimes r})$ is generated, in the classical (unquantised) cases, by (generalisations of) the quadratic Casimir elements of $\mathcal{A}$. In the quantum cases, the generating endomorphisms are R-matrices, which are known to provide representations of the braid group B_r, and may be thought of as lying in some completion of a tensor power of $\mathcal{A}$. This theory includes the cases of type A_n, B_n and C_n referred to above, as well as the case of algebras of type G_2 and any irreducible module for a Lie algebra of type A_1.

In this way, we show that in the classical cases, $\mathrm{End}_{\mathcal{A}}(V^{\otimes r})$ is always a quotient of the well known infinitesimal braid algebra T_r, which arises in the work of Kohno and others on the holonomy of braid spaces (see below), while in the quantum cases $\mathrm{End}_{\mathcal{A}}(V^{\otimes r})$ is a quotient of the group ring of the classical braid group on r strings. This treatment unifies many known results concerning endomorphisms of tensor powers, as well as pointing out analogies with non-standard cases such as the 7-dimensional irreducible module for $\mathfrak{g}$ of type G_2, and irreducible $\mathfrak{sl}_2$-modules of dimension greater than 2. En route, we also define "weakly multiplicity free" modules, and prove some general results concerning subalgebras of the endomorphism algebras of tensor products. The work is partly motivated by ideas explained well in the interesting work [Jo]. Our proof of the sufficiency of R-matrices in the quantum case rests on the observation (see Proposition 8.1) that if V and V_q are corresponding smf modules for $\mathfrak{g}$ and its quantisation, then the endomorphism algebras of $V^{\otimes r}$ and $V_q^{\otimes r}$ have the same dimension. A similar approach may be seen in [LR].

In this work we consider only the case when both $\mathcal{A}$ and V are complex vector spaces, and the quantum group is taken over the field $\mathbb{C}(q)$ of rational functions in the indeterminate q, which precludes the case when q is a root of unity. Thus all our modules are semisimple. We intend in the future to study deformations of the algebras $\mathrm{End}_{\mathcal{A}}(V^{\otimes r})$ in a uniform way along the lines of [GL], and in this way approach the corresponding questions for groups and algebras over fields other than $\mathbb{C}$, and for quantum groups in the case where q may be a root of unity. In all known cases, the finite dimensional quotients which arise as endomorphism algebras of tensor powers, all have a cellular structure, which permits their deformation into non-semisimple algebras by variation of parameters. These include Hecke algebras, the Brauer algebras, and the Birman-Murakami-Wenzl algebras. One obvious question to which we intend to return is whether the endomorphism algebras in the cases of G_2 and the higher dimensional representations of $\mathfrak{sl}_2$ (both of which contain Brauer algebras) have cellular deformations. Note also that generalisations of the action of the braid group on tensor powers to the affine case have been considered in [OR].

Much of this work has its roots in classical invariant theory, and we have made no attempt to provide a comprehensive bibliography of relevant sources. On the other hand, because there is a growing literature on generalisations of Schur-Weyl duality, this paper has a larger than usual expository component, whose purpose is partly to provide a context for our results, and others'. The specific results of this work which we believe to be new are the statements of Theorems 3.13 and 7.5 firstly for the 7-dimensional representation of G_2 and secondly for any irreducible representation of $\mathfrak{sl}_2$, both in the classical and quantum cases. Thirdly, there is the more general Theorem 5.5, which states that for $\mathfrak{sl}_2$ and its quantum analogue (see Remark 8.6), the endomorphism algebra of an arbitrary tensor product of irreducible modules (i.e., not necessarily a power) is generated by (infinitesimal) braids. Fourthly, we believe that the observation that all the cases covered by those theorems fit into the "strongly multiplicity free" context is new, as is the observation, using Lusztig's specialisation method in §8 that in the smf case, the dimensions of the algebras $\mathcal{A}(r)$ and $\mathcal{B}(r)$ are equal.

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2. Infinitesimal braids and endomorphisms

The base field throughout this work will be the field $\mathbb{C}$ of complex numbers.

For every integer $r \geq 2$, let T_r be the associative algebra generated by t_{ij}, $1 \leq i < j \leq r$, subject to the following relations for all pairwise distinct i, j, k, ℓ:

$$[t_{ij}, t_{k\ell}] = 0, \quad [t_{ik} + t_{i\ell}, t_{k\ell}] = 0, \quad [t_{ij}, t_{ik} + t_{jk}] = 0, \tag{2.1}$$

where $[\ ,\ ]$ denotes the usual commutator in an associative algebra: $[X, Y] = XY - YX$. We shall refer to the t_{ij} as *infinitesimal braids*. Note that the algebra T_r occurs in the literature in several contexts, such as the holonomy of connections on spaces of configurations (the KZ connection) (cf. [K1]) and representations of the pure braid group (cf. [K3, Proposition 2.3]). The algebra T_r thus has a close relationship to the pure braid group. We shall see shortly that its elements may be viewed as endomorphisms of tensor powers; these two observations partially justify the terminology.

We shall be interested in a class of finite dimensional representations of T_r constructed in the following way (cf [K1]). Let $\mathfrak{g}$ denote either the reductive complex Lie algebra $\mathfrak{gl}_k$ for some k, or a semisimple Lie algebra. Fix a Borel subalgebra $\mathfrak{b}$ and a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{b}$; such a pair is essentially unique, in that any two are conjugate under the adjoint group. Let R_+ be the set of the positive roots of $\mathfrak{g}$ determined by this choice of Borel subalgebra. Let $\{\alpha_i\}$ be the corresponding set of the simple roots.

Denote by C the quadratic Casimir element in the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of $\mathfrak{g}$. This element can be described explicitly as follows. Let $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be a non-degenerate invariant symmetric bilinear form. We may take κ to be the trace form in the case of $\mathfrak{gl}_k$, and the Killing form for a semisimple Lie algebra. If $\{X_\alpha | \alpha = 1, 2, \cdots, \dim \mathfrak{g}\}$ is a basis of $\mathfrak{g}$, let $\{X^\alpha | \alpha = 1, 2, \cdots, \dim \mathfrak{g}\}$ be the dual basis with respect to κ. Then

$$C = \sum_\alpha X_\alpha X^\alpha = \sum_\alpha X^\alpha X_\alpha.$$

It is well known that C is in the centre of $\mathrm{U}(\mathfrak{g})$.

The enveloping algebra $\mathrm{U}(\mathfrak{g})$ has a well known Hopf algebra structure. If Δ denotes co-multiplication, then for $X \in \mathfrak{g}$, $\Delta(X) = X \otimes 1 + 1 \otimes X$. Define the homomorphism $\Delta^{(r-1)} : \mathrm{U}(\mathfrak{g}) \longrightarrow \mathrm{U}(\mathfrak{g})^{\otimes r}$ recursively by $\Delta^{(r-1)} = (\Delta \otimes \mathrm{id}^{\otimes(r-2)}) \circ \Delta^{(r-2)}$.

Let t be the element of $\mathrm{U}(\mathfrak{g})^{\otimes 2}$ defined by

$$t := \frac{1}{2}\left(\Delta(C) - C \otimes 1 - 1 \otimes C\right).$$

Then clearly $t = \sum_\alpha X_\alpha \otimes X^\alpha = \sum_\alpha X^\alpha \otimes X_\alpha$, and evidently t commutes with $\Delta(\mathrm{U}(\mathfrak{g})) \subset \mathrm{U}(\mathfrak{g})^{\otimes 2}$. In analogy with t, we define elements $C_{ij} \in \mathrm{U}(\mathfrak{g})^{\otimes r}$, $1 \leq i < j \leq r$, by

$$C_{ij} : = \sum_\alpha 1 \otimes \cdots \otimes 1 \otimes \underbrace{X_\alpha}_{i} \otimes 1 \otimes \cdots \otimes 1 \otimes \underbrace{X^\alpha}_{j} \otimes 1 \otimes \cdots \otimes 1. \tag{2.2}$$

Theorem 2.1. *The map $t_{ij} \mapsto C_{ij}$ extends uniquely to an algebra homomorphism $\psi : T_r \longrightarrow \mathrm{U}(\mathfrak{g})^{\otimes r}$. Furthermore, given any $\zeta \in T_r$, we have $[\psi(\zeta), \Delta^{(r-1)}(x)] = 0, \forall x \in \mathrm{U}(\mathfrak{g})$.*

Proof. For the first statement, we need to verify that the C_{ij} satisfy the three relations (2.1). The first relation is evident. Further, it is clear that the second and third equations need only be checked when $r = 3$. Now

$$\begin{aligned} [C_{12} + C_{13},\ C_{23}] &= [(\mathrm{id} \otimes \Delta)t,\ 1 \otimes t],\ \text{and} \\ [C_{12},\ C_{13} + C_{23}] &= [t \otimes 1,\ (\Delta \otimes \mathrm{id})t]. \end{aligned}$$

But $(\mathrm{id} \otimes \Delta)t$ is a sum of terms like $X \otimes \Delta Y$ $(X, Y \in \mathfrak{g} \subset \mathrm{U}(\mathfrak{g}))$, so that the first commutator is 0 because t commutes with $\Delta(\mathrm{U}(\mathfrak{g}))$, and similarly for the second. This proves the first statement.

The second statement follows from the fact, implicit in the argument above, that for any $i < j$, $[C_{ij}, \Delta^{(r-1)}(x)] = 0$ for all $x \in \mathrm{U}(\mathfrak{g})$. □

Next let (π, V) be any finite dimensional $\mathrm{U}(\mathfrak{g})$-module, that is, a finite dimensional vector space V with an algebra homomorphism $\pi : \mathrm{U}(\mathfrak{g}) \longrightarrow \mathrm{End}_{\mathbb{C}}(V)$. (In general, we shall use the notation (π_μ, L_μ) for a finite dimensional irreducible $\mathfrak{g}$-module with highest weight μ.) Since every $\mathfrak{g}$-module can be regarded as a module over its universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ in a natural way, we shall use the terms "$\mathfrak{g}$-module" and "$\mathrm{U}(\mathfrak{g})$-module" interchangeably.

Now $\pi^{\otimes r}$ clearly defines an action of $\mathrm{U}(\mathfrak{g})^{\otimes r}$ on $V^{\otimes r}$, which makes $V^{\otimes r}$ into an $\mathrm{U}(\mathfrak{g})^{\otimes r}$-module. The homomorphism $\Delta^{(r-1)}$ from $\mathrm{U}(\mathfrak{g})$ to $\mathrm{U}(\mathfrak{g})^{\otimes r}$ therefore makes $V^{\otimes r}$ into a $\mathrm{U}(\mathfrak{g})$-module by restriction (i.e. via the action $\pi^{(r)} \circ \Delta^{(r-1)}$). In this work our main concern is the identification of the centraliser algebra of $V^{\otimes r}$ as $\mathrm{U}(\mathfrak{g})$-module.

Given (π, V) as above, define

$$\psi_r := \pi^{\otimes r} \circ \psi : T_r \longrightarrow \mathrm{End}_{\mathbb{C}}(V^{\otimes r}). \tag{2.3}$$

Evidently the image of ψ_r is a finite dimensional algebra $\mathcal{A}(r)$ of endomorphisms of $V^{\otimes r}$.

Theorem 2.2. *The T_r-module $(\psi_r, V^{\otimes r})$ is semisimple. Thus the image $\mathcal{A}(r)$ of ψ_r is a semisimple algebra.*

Proof. We postpone the proof to the Appendix. □

Theorem 2.3. *The commutant $C(\mathcal{A}(r))$ of $\mathcal{A}(r)$ in $\mathrm{End}_{\mathbb{C}}(V^{\otimes r})$ is semisimple, and the commutant of $C(\mathcal{A}(r))$ is $\mathcal{A}(r)$ itself. Furthermore, $V^{\otimes r}$ decomposes into a direct sum of inequivalent irreducible $\mathcal{A}(r) \otimes C(\mathcal{A}(r))$-modules each appearing with multiplicity 1.*

Proof. Because of the semisimplicity of $\mathcal{A}(r)$, the double centraliser theorem (see, e.g., [CR, P. 175]) applies to the present situation. □

As a first step in the identification of $C(\mathcal{A}(r))$, we have the following statement.

Proposition 2.4. *Suppose that (π, V) is a representation of $\mathfrak{g}$. Then for any $r \geq 2$ we have*

(1) $C(\mathcal{A}(r)) \supseteq \pi^{\otimes r} \circ \Delta^{(r-1)}(\mathrm{U}(\mathfrak{g}))$.

(2) *If* (π, V) *is irreducible, then* $\mathcal{A}(r)$ *contains the image of* $\Delta^{(r-1)}(C)$ *under* $\pi^{\otimes r}$.

Proof. The first part is immediate from the second statement of Theorem 2.1.

For (2), note that since (π, V) is irreducible, $1 \otimes C$ and $C \otimes 1$ act as scalars on $V \otimes V$, and hence are in $\mathcal{A}(2)$. This is the case $r = 2$ of the second assertion. But $\Delta^{(r-1)}(C) = (\mathrm{id}^{\otimes(r-2)} \otimes \Delta) \circ \Delta^{(r-2)}(C)$. Moreover $\mathrm{id}^{\otimes(r-2)} \otimes \Delta(C_{ij})$ is clearly a linear combination of elements $C_{i'j'} \in \mathrm{U}(\mathfrak{g})^{\otimes r}$. It follows that if $\Delta^{(r-2)}(C) \in \mathcal{A}(r-1)$, then $\Delta^{(r-1)}(C) \in \mathcal{A}(r)$. This proves (2) by induction on r. □

It follows from Theorem 2.3 that if we have equality in Proposition 2.4(1), then $\mathcal{A}(r)$ is the full centraliser algebra of $\mathrm{U}(\mathfrak{g})$ on $V^{\otimes r}$. Thus the decomposition of $V^{\otimes r}$ as $\mathrm{U}(\mathfrak{g})$-module is reduced to investigation of the structure of $\mathcal{A}(r)$. One of the main concerns of this work is therefore the identification of $C(\mathcal{A}(r))$, and in particular we give sufficient conditions for equality in Proposition 2.4(1).

3. Strongly multiplicity free modules

Denote by $\mathcal{P}$ the lattice of weights of $\mathfrak{g}$ with respect to $\mathfrak{h}$, by $\mathcal{P}^+$ the dominant weights, and fix $\Lambda_0 \in \mathcal{P}^+$. Let (π, V) be a finite dimensional irreducible $\mathrm{U}(\mathfrak{g})$-module with highest weight Λ_0 and denote the set of weights occurring with non-zero multiplicity in V by Π. Assume that V is multiplicity free, i.e., that every weight space has dimension 1. To avoid the trivial case $V = \mathbb{C}$, we assume that $\dim V > 1$.

For any dominant weight $\Lambda \in \mathcal{P}^+$, write L_Λ for the simple $\mathfrak{g}$-module with highest weight Λ, and define the set $\widehat{\mathcal{P}}^+_\Lambda := \{\Lambda + \mu \,|\, \mu \in \Pi\} \cap \mathcal{P}^+$. Let $\mathcal{P}^+_\Lambda$ be the subset of $\widehat{\mathcal{P}}^+_\Lambda$ such that the simple module L_λ appears in $L_\Lambda \otimes V$. We shall require the following simple result.

Lemma 3.1. *(i). Given that* V *is (irreducible and) multiplicity free, for any finite dimensional irreducible* $\mathfrak{g}$*-module* (π_Λ, L_Λ)*,* $L_\Lambda \otimes V = \bigoplus_{\lambda \in \mathcal{P}^+_\Lambda} L_\lambda$*, where each irreducible submodule appears with multiplicity* 1.

(ii). If V *is further assumed to be minuscule in the sense that* Π *consists of a single Weyl group orbit, then for every* $\mu \in \Pi$ *such that* $\Lambda + \mu$ *is dominant, the irreducible module* $L_{\Lambda+\mu}$ *appears in* $L_\Lambda \otimes V$ *exactly once. Thus when* V *is minuscule, we have* $\widehat{\mathcal{P}}^+_\Lambda = \mathcal{P}^+_\Lambda$.

Proof. To see (i), recall the usual partial order on Π: for elements $\mu, \nu \in \Pi$, $\mu > \nu$ if $\mu = \nu + \beta$ for some non-zero $\beta \in \mathbb{Z}_+ R_+$. Since V is multiplicity free, it has an ordered basis $\{v_1, v_2, \ldots, v_d\}$ of weight vectors ($d = \dim V$), compatible with the partial order of Π in the sense that if $wt(v_i)$ denotes the weight corresponding to v_i, then $wt(v_i) > wt(v_j)$ implies $i < j$. That is, the weights satisfy: $wt(v_i) \geq wt(v_{i+1})$ for each i.

Any vector u in $L_\Lambda \otimes V$ can be uniquely expressed as $u = w_1 \otimes v_1 + \cdots + w_d \otimes v_d$. If u is a weight vector with weight λ, then for each i, $w_i = 0$ or w_i is a weight vector with weight $\lambda - wt(v_i)$. Suppose u is a highest weight vector, and let k be the largest index such that $w_k \neq 0$. The fact that u is annihilated by all positive root vectors of $\mathfrak{g}$ implies that w_k is also annihilated by all positive root vectors of $\mathfrak{g}$, and hence is a highest weight vector of L_Λ. Hence $wt(u) = \Lambda + wt(v_k)$.

If there is another highest weight vector $u' \in L_\Lambda \otimes V$ with the same weight $wt(u)$, then after adjustment by a non-zero scalar, we may assume that u' is of the form

$u' = w'_1 \otimes v_1 + \cdots + w'_{k-1} \otimes v_{k-1} + w_k \otimes v_k$. If $u - u' \neq 0$, let $j < k$ be the largest index such that $w_j - w'_j \neq 0$. Then $u - u'$ is another highest weight vector with weight $wt(u)$, and by the argument of the last paragraph, we see that $w_j - w'_j$ is a highest weight vector of L_Λ. But this is impossible as the weight of $w_j - w'_j$ is not equal to Λ. Hence any two highest weight vectors of the same weight in $L_\Lambda \otimes V$ are proportional. This completes the proof of (i).

In view of part (i), to prove (ii) we need only show that $L_{\Lambda+\mu}$ does appear in the tensor product if $\Lambda+\mu$ is dominant. But the Parthasarathy-Ranga Rao-Varadarajan conjecture proved by Kumar [Ku] states that for every w in the Weyl group of $\mathfrak{g}$, the irreducible module with highest weight equal to the unique dominant weight in the Weyl group orbit of $\Lambda + w(\Lambda_0)$ appears in $L_\Lambda \otimes V$. Since V is assumed to be minuscule, every $\mu \in \Pi$ is of the form $w(\Lambda_0)$, and we are done. □

We next define a special class of multiplicity free modules.

Definition 3.2. Let Π be the set of weights of the irreducible $\mathfrak{g}$-module (π, V). Then (π, V) is called *strongly multiplicity free* (smf) if each $\mu \in \Pi$ has multiplicity one, and for any pair of distinct elements μ and ν of Π, $\mu - \nu \in \mathbb{N}R_+ \cup (-\mathbb{N}R_+)$.

Remark 3.3. The condition that a multiplicity free module (π, V) be smf is equivalent to the requirement that its set Π of weights is linearly ordered under the partial ordering referred to in the proof of Lemma 3.1.

Before discussing the properties of smf modules which are relevant for our consideration of endomorphism algebras, we give a classification of the smf modules for the simple complex Lie algebras.

Theorem 3.4. *Let $\mathfrak{g}$ being a simple complex Lie algebra. The following is a complete list of the strongly multiplicity free irreducible modules:*

1. *the natural $\mathfrak{sl}_k$-module $\mathbb{C}^k$ and its dual, for $k > 2$;*
2. *the natural $\mathfrak{so}_{2k+1}$-module $\mathbb{C}^{2k+1}$ for $k \geq 2$;*
3. *the natural $\mathfrak{sp}_{2k}$-module $\mathbb{C}^{2k}$ for $k > 1$;*
4. *the 7 dimensional irreducible G_2-module;*
5. *all irreducible $\mathfrak{sl}_2$-modules of dimension greater than* 1.

Proof. If $\mathfrak{g} = \mathfrak{sl}_2$, there is one irreducible module of each dimension $k \geq 1$, and since the whole of $\mathcal{P}$ is totally ordered, and all irreducibles are multiplicity free, clearly those of dimension $k \geq 2$ are smf.

Assume now that the rank $\mathrm{rk}(\mathfrak{g})$ of $\mathfrak{g}$ is greater than 1. We claim that if V is strongly multiplicity free, then its highest weight Λ_0 must be a fundamental weight, i.e., there exists a unique simple root α_s such that for any simple root α_i we have

$$(\Lambda_0, \alpha_i^\vee) = \frac{2(\Lambda_0, \alpha_i)}{(\alpha_i, \alpha_i)} \quad = \quad \begin{cases} 1 \text{ if } & \alpha_i = \alpha_s, \\ 0 \text{ if } & \alpha_i \neq \alpha_s. \end{cases}$$

For any root $\alpha \in R$, we use the standard notation $\alpha^\vee$ for the corresponding coroot, viz. $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$. We shall need the following standard (and easily proved) facts concerning V. For $\alpha \in R_+$, let $e_\alpha, f_\alpha, h_\alpha$ be the usual elements of the Chevalley basis of $\mathfrak{g} \subset \mathrm{U}(\mathfrak{g})$. If $v \in V$ is a highest weight vector, then

$$\begin{aligned} e_\alpha f_\alpha v =&(\Lambda_0, \check{\alpha})v \\ e_\alpha f_\alpha^2 v =&2((\Lambda_0, \check{\alpha}) - 1) f_\alpha v. \end{aligned} \tag{3.1}$$

To prove the claim, first assume that there are two distinct simple roots α_s and α_j such that $(\Lambda_0, \check{\alpha_s}) \neq 0$ and $(\Lambda_0, \check{\alpha_j}) \neq 0$. Then by (3.1) both $f_{\alpha_s} v$ and $f_{\alpha_j} v$ are non-zero, and are weight vectors with respective weights $\Lambda_0 - \alpha_s$ and $\Lambda_0 - \alpha_j$, which are not comparable in the partial order on $\mathcal{P}$. Hence $(\Lambda_0, \alpha) \neq 0$ for just one simple root $\alpha = \alpha_s$.

To complete the proof of the claim, we show that $(\Lambda_0, \check{\alpha_s}) = 1$.

Let β be a simple root distinct from α_s such that $(\alpha_s, \beta) \neq 0$ (i.e., β is joined to α_s in the Dynkin diagram). Then $\alpha := \alpha_s + \beta$ is a root, and $(\Lambda_0, \alpha) = (\Lambda_0, \alpha_s) \neq 0$, whence by the argument just given, $f_\alpha v$ is a (non-zero) weight vector of weight $\Lambda_0 - \alpha = \Lambda_0 - \alpha_s - \beta$. However, if $\frac{2(\Lambda_0, \alpha_s)}{(\alpha_s, \alpha_s)} > 1$, then again by (3.1), $f_{\alpha_s}^2 v \neq 0$, and hence is a weight vector of weight $\Lambda_0 - 2\alpha_s$. Thus $\Lambda_0 - 2\alpha_s$ and $\Lambda_0 - \alpha_s - \beta$ are in Π, but are not comparable in the partial order on $\mathcal{P}$. This completes the proof of the claim.

We are therefore reduced to checking which fundamental weight modules of the simple complex Lie algebras are smf. All the weights of these modules have been tabulated in [BMP]. For the classical Lie algebras one sees easily that all the fundamental representations are multiplicity free. However, only those listed in the theorem satisfy the condition on Π.

For G_2, the two fundamental representations are respectively the 7 dimensional irreducible representation and the adjoint representation. Obviously the adjoint representation is not multiplicity free, but the 7 dimensional irreducible representation satisfies the condition on Π.

The weights and their multiplicities of all the fundamental representations of the other exceptional simple Lie algebras are explicitly given in [BMP]. Inspecting the results we see that these representations are either not multiplicity free or violate the condition on Π. □

Remark 3.5. The natural modules of $\mathfrak{sl}_k$ and $\mathfrak{sp}_{2k}$ in Theorem 3.4 are minuscule, but the natural module of $\mathfrak{so}_{2m+1}$ is not. Also note that the natural module of $\mathfrak{so}_{2m}$ is minuscule but not strongly multiplicity free.

We next construct some endomorphisms of $L_\Lambda \otimes V$ for strongly multiplicity free V and arbitrary dominant Λ, which will be used later to analyse the endomorphisms of tensor powers of V. Since the quadratic Casimir C of $\mathrm{U}(\mathfrak{g})$ is central, it acts via a scalar, which we denote by $\chi_\lambda(C)$, on the irreducible $\mathfrak{g}$-module L_λ. As is well known (see, e.g. [H, P.122]), $\chi_\lambda(C) = (\lambda + 2\rho, \lambda)$, where 2ρ denotes the sum of the positive roots of $\mathfrak{g}$.

Lemma 3.6. *Suppose V is strongly multiplicity free, and that Λ is any dominant weight. If $\lambda, \mu \in \widehat{\mathcal{P}}^+_\Lambda$, then $\chi_\lambda(C) = \chi_\mu(C)$ if and only if $\lambda = \mu$.*

Proof. For any two elements λ, μ of $\widehat{\mathcal{P}}^+_\Lambda$,

$$\chi_\lambda(C) - \chi_\mu(C) \quad = (\lambda + \mu + 2\rho, \lambda - \mu),$$

and this is zero if and only if $\lambda = \mu$, because $\lambda - \mu$ is either a sum of positive or negative roots of $\mathfrak{g}$, by the hypothesis concerning Π. □

We next introduce a weaker form of the smf condition.

Definition 3.7. The irreducible $\mathfrak{g}$-module $(\pi, V = L_{\Lambda_0})$ is called *weakly multiplicity free* (wmf) if each weight has multiplicity one, (so that by 3.1 $V \otimes V$ is multiplicity free), and for any pair of distinct weights μ and ν of $\mathcal{P}^+_{\Lambda_0}$, $\chi_\mu(C) \neq \chi_\nu(C)$, where $C \in \mathrm{U}(\mathfrak{g})$ is the Casimir element.

Note that a consequence of Lemma 3.6 is that any smf module is wmf.

The algebra $\mathrm{U}(\mathfrak{g})$, and hence the Casimir $C \in \mathrm{U}(\mathfrak{g})$, acts on $L_\Lambda \otimes V$ via $(\pi_\Lambda \otimes \pi) \circ \Delta$. Write $A_\Lambda := (\pi_\Lambda \otimes \pi)\Delta(C)$ for the operator induced by C. By Schur's Lemma, each irreducible submodule L_λ of $L_\Lambda \otimes V$ is an eigenspace of A_Λ with eigenvalue $\chi_\lambda(C)$. Thus by Lemma 3.1(i),

$$\prod_{\lambda \in \mathcal{P}^+_\Lambda} (A_\Lambda - \chi_\lambda(C)) = 0. \tag{3.2}$$

For $\lambda \in \mathcal{P}^+_\Lambda$, let $P[\lambda]$ be the $\mathrm{U}(\mathfrak{g})$-endomorphism of $L_\Lambda \otimes V$ defined by

$$P[\lambda] \;:=\; \prod_{\mu:\, \mu \neq \lambda} \frac{A_\Lambda - \chi_\mu(C)}{\chi_\lambda(C) - \chi_\mu(C)}. \tag{3.3}$$

By Lemma 3.6, $P[\lambda]$ is well defined. For any weight $\mu \in \mathcal{P}^+_\Lambda$, it is evident that $P[\lambda]$ acts on the summand L_μ of $L_\Lambda \otimes V$ as the identity if $\mu = \lambda$ and zero otherwise. We therefore refer to $P[\lambda]$ as a projection operator. The following statement is clear.

Lemma 3.8. *Assume that $L_\Lambda \otimes V$ is multiplicity free and that C has distinct eigenvalues on distinct summands of $L_\Lambda \otimes V$. Then the projection operators above have the following properties:*

(1) *The $P[\lambda]$ commute with the $\mathrm{U}(\mathfrak{g})$ action on $L_\Lambda \otimes V$. That is,*
$$[P[\lambda],\ (\pi_\Lambda \otimes \pi)\Delta(x)] \;=\; 0, \quad \forall x \in \mathrm{U}(\mathfrak{g}).$$

(2) *The $P[\lambda]$ form an orthogonal idempotent decomposition of $id_{L_\Lambda \otimes V}$, and*
$$P[\lambda](L_\Lambda \otimes V) = L_\lambda.$$

Proof. Part (1) follows from the fact that C is central in $\mathrm{U}(\mathfrak{g})$. Part (2) follows from Lemma 3.6, Equation (3.2) and the remarks above. □

We are now able to analyse $V^{\otimes 2}$ for V wmf completely.

Corollary 3.9. *Let V be a weakly multiplicity free $\mathfrak{g}$-module. Then*

(1) *The algebra $\mathcal{A}(2)$ is the full centraliser of $\mathrm{U}(\mathfrak{g})$, acting on $V^{\otimes 2}$ via $\Delta = \Delta^{(1)} : \mathrm{U}(\mathfrak{g}) \longrightarrow \mathrm{U}(\mathfrak{g})^{\otimes 2}$, i.e., in this case we have equality in Proposition 2.4(1).*

(2) *The endomorphism $s : V^{\otimes 2} \longrightarrow V^{\otimes 2}$ defined by $s(v \otimes v') = v' \otimes v$ lies in $\mathcal{A}(2)$.*

Proof. It is clear that $V^{\otimes 2}$ is multiplicity free as $\mathrm{U}(\mathfrak{g})$-module, i.e., it is a direct sum of pairwise non-isomorphic irreducible $\mathrm{U}(\mathfrak{g})$-modules L_λ, over $\lambda \in \mathcal{P}^+_{\Lambda_0}$. Thus $\mathrm{End}_{\mathrm{U}(\mathfrak{g})} V^{\otimes 2}$ is generated by the projections onto the summands L_λ, i.e. the projections $P[\lambda]$ of Lemma 3.8. Hence (1) will follow if we can show that $P[\lambda] \in \mathcal{A}(2)$ for

$\lambda \in \mathcal{P}^+_{\Lambda_0}$. But (3.3) shows that $P[\lambda]$ is a polynomial in $A_{\Lambda_0} = \pi^{\otimes 2}\Delta(C)$. Moreover Proposition 2.4(2) shows that $\pi^{\otimes 2}\Delta(C) \in \mathcal{A}(2)$, and the proof of (1) is complete.

For (2), we need merely observe that $s \in \mathrm{End}_{\mathrm{U}(\mathfrak{g})} V^{\otimes 2}$, and hence by (1), $s \in \mathcal{A}(2)$. □

This enables us to make the following general statement about $\mathcal{A}(r)$.

Theorem 3.10. *Let (π, V) be a wmf $\mathfrak{g}$-module, and let $\mathcal{A}(r) \subseteq \mathrm{End}_{\mathrm{U}(\mathfrak{g})} V^{\otimes r}$ be the algebra defined in §2. Then*

(1) *For any permutation $\sigma \in \mathrm{Sym}_r$, the endomorphism $\tau_\sigma : v_1 \otimes \cdots \otimes v_r \mapsto v_{\sigma(1)} \otimes \otimes \cdots \otimes v_{\sigma(r)}$ lies in $\mathcal{A}(r)$.*

(2) *For $r = 2, 3, \ldots$, $\mathcal{A}(r)$ is generated as associative algebra by $\mathcal{A}(r-1) \otimes 1$ together with the endomorphism $\pi^{\otimes r}(1 \otimes 1 \otimes \cdots \otimes \Delta(C))$ of $V^{\otimes r}$.*

Proof. The case where σ is the simple transposition (12) of (1) was proved in Corollary 3.9(2), and the same argument shows that for all simple transpositions $\sigma = (i, i+1)$ the corresponding τ_σ are in $\mathcal{A}(r)$. Since these involutions generate the symmetric group, $\tau_\sigma \in \mathcal{A}(r)$ for all $\sigma \in \mathrm{Sym}_r$.

We prove (2) by induction on r, the case $r = 2$ having been proved in Proposition 2.4(2). Note also that the argument of the proof of Proposition 2.4(2) shows that $\pi^{\otimes r}(1 \otimes 1 \otimes \cdots \otimes \Delta(C)) \in \mathcal{A}(r)$, whence the algebra $\mathcal{B}$ generated by $\mathcal{A}(r-1) \otimes 1$ and $\pi^{\otimes r}(1 \otimes 1 \otimes \cdots \otimes \Delta(C))$ is contained in $\mathcal{A}(r)$. We need to show that $\mathcal{B}$ contains all the elements $\pi^{\otimes r}(C_{ij})$ $(1 \le i < j \le r)$. If $i < j < r$, this is clear, so that we are reduced to the case when $j = r$. But again, the argument in the proof of Proposition 2.4(2) shows that $\pi^{\otimes r}(C_{r-1,r})$ is a linear combination of $\pi^{\otimes r}(1 \otimes 1 \otimes \cdots \otimes \Delta(C))$ and scalars, and hence lies in $\mathcal{B}$. Further, by (1), we also have all the endomorphisms $\tau_\sigma \in \mathcal{B}$ for $\sigma \in \mathrm{Sym}_{r-1}$. Since $\pi^{\otimes r}(C_{ir})$ may be obtained from $\pi^{\otimes r}(C_{r-1,r})$ by conjugation by a transposition in Sym_{r-1}, we see that all $\pi^{\otimes r}(C_{ij})$ lie in $\mathcal{B}$ and the proof of (2) is complete. □

The following general result is also useful.

Proposition 3.11. *Assume that $(\pi_1, V_1), \ldots, (\pi_r, V_r)$ are strongly multiplicity free irreducible $\mathfrak{g}$-modules.*

(1) *There is a decomposition $V_1 \otimes \cdots \otimes V_r = \oplus_i L_i^{(r)}$ into (not necessarily distinct) irreducibles, which, up to ordering the summands, is determined by the sequence $(V_1, \ldots, V_r)$.*

(2) *The projections $P_i^{(r)} : V_1 \otimes \cdots \otimes V_r \longrightarrow L_i^{(r)}$, regarded as endomorphisms of $V_1 \otimes \cdots \otimes V_r$ (i.e. $Res^{V_1 \otimes \cdots \otimes V_r}_{L_j^{(r)}} P_i^{(r)} = \delta_{ij} id_{L_j^{(r)}}$) lie in $\mathcal{A}(r)$, here interpreted as $(\pi_1 \otimes \cdots \otimes \pi_r) \circ \psi(T_r)$.*

Proof. We prove both statements by induction on r; the case $r = 1$ is trivial. If $V_1 \otimes \cdots \otimes V_r = \oplus_i L_i^{(r)}$ is an irreducible decomposition, then $V_1 \otimes \cdots \otimes V_r \otimes V_{r+1} = \oplus_i L_i^{(r)} \otimes V_{r+1}$, and by Lemma 3.1, $L_i^{(r)} \otimes V_{r+1}$ is multiplicity free for each i, and hence has a unique decomposition into irreducible submodules. This proves (1).

For (2), assume we have projections $P_i^{(r)} : V_1 \otimes \cdots \otimes V_r \longrightarrow L_i^{(r)}$ in $\mathcal{A}(r)$. If $L_i^{(r)} \otimes V_{r+1}$ has irreducible decomposition $L_i^{(r)} \otimes V_{r+1} = \oplus_{j=a}^b L_j^{(r+1)}$, write p_{ij} for the projection $L_i^{(r)} \otimes V_{r+1} \longrightarrow L_j^{(r+1)}$. Then by Lemmas 3.8 and 3.6, p_{ij} is a polynomial in $\pi_i^{(r)} \otimes \pi_{r+1}(C)$, where $\pi_i^{(r)}$ is the representation of $\mathrm{U}(\mathfrak{g})$ on $L_i^{(r)}$. But this is simply $\pi_1 \otimes \cdots \otimes \pi_{r+1}\Delta^{(r)}(C)$ which lies in $\mathcal{A}(r+1)$ by the appropriate generalisation of Proposition 2.4(2). Hence $P_j^{(r+1)} = p_{ij} \circ P_i^{(r)} \otimes 1 \in \mathcal{A}(r+1)$. □

Corollary 3.12. *Assume we are in the situation of Proposition 3.11. Let $V_1 \otimes \cdots \otimes V_r = \oplus_j M_j$ be the canonical decomposition of the left side into isotypic components. Then the projections $V_1 \otimes \cdots \otimes V_r \longrightarrow M_j$ all lie in $\mathcal{A}(r)$.*

Proof. Since each M_j is the sum of all the irreducible modules $L_i^{(r)}$ in a given isomorphism class, this is immediate from Proposition 3.11. □

The next statement is one of the main results of this paper.

Theorem 3.13. *If V is strongly multiplicity free, then for any integer $r \geq 2$,*

$$\mathrm{End}_{\mathfrak{g}}(V^{\otimes r}) = \mathcal{A}(r).$$

Equivalently, $C(\mathcal{A}(r)) = \mathrm{U}(\mathfrak{g})$.

The proof will be given in Section 5.

3.1. **Representations of Lie groups and Lie algebras.** Notice that Theorem 3.13 excludes the natural module for the orthogonal Lie algebra $\mathfrak{so}_{2m}$ of even dimension, as the corresponding statement is false in this case, although the analogous statement for the corresponding group is true. The reason underlying this is that in order to describe classical invariant theory in a unified manner using the algebra of infinitesimal braids, one needs to pass to the corresponding Lie group $O_{2m}(\mathbb{C})$.

Given any finite dimensional (holomorphic) representation (Υ, V) of G, that is, a finite dimensional vector space V with a homomorphism $\Upsilon : G \longrightarrow GL(V)$ of complex Lie groups, its differential $d\Upsilon := \pi$ gives rise to a representation of the Lie algebra $\mathfrak{g}$ (see [V, P. 105]). We denote the resulting $\mathfrak{g}$-module by (π, V).

Proposition 3.14. *Let (Υ, V) be a representation of the connected complex Lie group G with corresponding representation $(\pi = d\Upsilon, V)$ of $\mathfrak{g}$ =Lie(G). Then* $\mathrm{End}_G(V) = \mathrm{End}_{\mathfrak{g}}(V)$.

We shall use Proposition 3.14 to prove Theorem 3.13 case by case. Let G denote one of the complex Lie groups $SL_k(\mathbb{C})$, $O_k(\mathbb{C})$, $Sp_{2k}(\mathbb{C})$, or G_2. Writing $\mathfrak{g} = \mathrm{Lie}(G)$ for the corresponding Lie algebra, let V be one of the following $\mathfrak{g}$-modules: the l-th symmetric power (for some l) of the natural module when $\mathfrak{g}$ is $\mathfrak{sl}_2$, the natural module for the Lie algebra $\mathfrak{g}$ of G when $\mathfrak{g}$ is of classical type other than $\mathfrak{sl}_2$, and the seven dimensional irreducible module when $\mathfrak{g}$ is the Lie algebra of G_2. Then clearly these representations all lift to representations of the corresponding Lie groups, as do their tensor powers.

Theorem 3.15. *For the Lie groups and their representations listed above, the corresponding Lie algebras act, and if $\mathcal{A}(r)$ is the algebra of endomorphisms of $V^{\otimes r}$ defined in §1, we have*

$$\mathrm{Hom}_G(V^{\otimes r}, V^{\otimes r}) = \mathcal{A}(r), \quad \forall r \geq 2.$$

Remark 3.16. We shall prove Theorem 3.15 in Section 5. When G is connected, this will imply Theorem 3.13 for $\mathfrak{g}$ in view of Proposition 3.14. In the orthogonal case, we need some *ad hoc* arguments to complete the proof.

4. The symmetric group and Brauer algebra in $\mathcal{A}(r)$

We analyse the structure of the algebra $\mathcal{A}(r)$ in some detail in this section. It is well known that the symmetric group and Brauer algebra play a central role in classical invariant theory [W]. We shall see that particular representations of this algebra arise as subalgebras of $\mathcal{A}(r)$. The material in this section will be used in proving Theorem 3.13 and Theorem 3.15.

We shall use the same notation as in the last section. Throughout this section, the $\mathfrak{g}$ is either $\mathfrak{gl}_k$ or a simple Lie algebra. Let $V = L_{\Lambda_0}$ be an irreducible, weakly multiplicity free $\mathfrak{g}$-module. The projections to the irreducible submodules of $V\otimes V$ span $\mathrm{End}_{\mathfrak{g}}(V\otimes V)$.

4.1. **Permutations in the endomorphism algebra.** Consider the permutation

$$s : V\otimes V \longrightarrow V\otimes V, \quad w\otimes w' \longmapsto w'\otimes w,$$

which is $\mathrm{U}(\mathfrak{g})$-invariant. Define the maps $s_i : V^{\otimes r} \to V^{\otimes r}$, $i = 1, 2, \cdots, r-1$, by

$$s_i := \mathrm{id}^{\otimes(i-1)} \otimes s \otimes \mathrm{id}^{\otimes(r-i-1)}, \tag{4.1}$$

where id denote the identity map on V. Then it is well known that they satisfy the defining relations of the symmetric group Sym_r

$$s_i^2 = \mathrm{id}^{\otimes r}, \quad s_i s_j = s_j s_i, \quad |i-j| > 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},$$

thus generate a representation ϕ_r of Sym_r on $V^{\otimes r}$. Recall that we denote $\psi_r(T_r)$ by $\mathcal{A}(r)$.

Theorem 4.1. *Let V be an irreducible weakly multiplicity free $\mathfrak{g}$-module. Then $\mathbb{C}\phi_r(\mathrm{Sym}_r) \subseteq \mathcal{A}(r)$. If $|\mathcal{P}^+_{\Lambda_0}| = 2$, then $\mathbb{C}\phi_r(\mathrm{Sym}_r) = \mathcal{A}(r)$.*

Proof. The first statement is simply a rephrasing of Theorem 3.10.

Now by Lemma 3.1, we have

$$s = \sum_{\lambda\in\mathcal{P}^+_{\Lambda_0}} \epsilon(\lambda) P[\lambda], , \tag{4.2}$$

where $\epsilon(\lambda) \in \mathbb{C}$, and since $s^2 = 1$, $\epsilon(\lambda) = \pm 1$.

Since s is not a scalar multiple of $\mathrm{id}_{V\otimes V}$, when $|\mathcal{P}^+_{\Lambda_0}| = 2$, one of the $\epsilon(\lambda)$ in (4.2) is $+1$ and the other is -1. It follows that the two projections may be expressed as $(\mathrm{id}^{\otimes 2} + s)/2$ and $(\mathrm{id}^{\otimes 2} - s)/2$ respectively. Let ξ^+ and ξ^- denote the two (necessarily distinct) eigenvalues of t on $V\otimes V$. Then for every $i \le r$,

$$\psi_r(t_{i,i+1}) \quad = \quad \xi^+ \frac{\mathrm{id}^{\otimes r} + s_i}{2} + \xi^- \frac{\mathrm{id}^{\otimes r} - s_i}{2}.$$

Using this equation we can express $\psi_r(t_{ij})$ as

$$\psi_r(t_{ij}) \quad = \quad s_{j-1}s_{j-2}\cdots s_{i+1}\psi_r(t_{i,i+1})s_{i+1}\cdots s_{j-2}s_{j-1}.$$

This shows that $\phi_r(\mathbb{C}\mathrm{Sym}_r)$ contains $\mathcal{A}(r)$ and hence that $\phi_r(\mathbb{C}\mathrm{Sym}_r) = \mathcal{A}(r)$. □

Example 4.2. The natural representation of $\mathfrak{gl}_n$.

Choose the standard Borel subalgebra of upper triangular matrices, which contains the Cartan subalgebra consisting of the diagonal matrices. The dual $\mathfrak{h}^*$ of the Cartan subalgebra is spanned by $\{e_i \mid 1 \leq i \leq n\}$, and is equipped with a positive definite bilinear form (,) such that $(e_i, e_j) = \delta_{ij}$. The positive roots are $e_i - e_j$, $i < j$, which sum to

$$2\rho = \sum_{i=1}^{n} (n - 2i + 1) e_i.$$

We take V to be the natural $\mathfrak{gl}_n$-module, which has highest weight e_1.

$$V \otimes V \;\; = \;\; L_{2e_1} \oplus L_{e_1+e_2}.$$

The eigenvalues of the quadratic Casimir operator C in the irreducible modules V, L_{2e_1} and $L_{e_1+e_2}$ are respectively given by

$$\begin{aligned} \chi_{e_1}(C) &= (e_1 + 2\rho, e_1) = n, \\ \chi_{2e_1}(C) &= (2e_1 + 2\rho, 2e_1) = 2(n+1), \\ \chi_{e_1+e_2}(C) &= (e_1 + e_2 + 2\rho, e_1 + e_2) = 2(n-1). \end{aligned}$$

The eigenvalues of t in the irreducible submodules L_{2e_1} and $L_{e_1+e_2}$ of $V \otimes V$ are respectively given by

$$\zeta_{2e_1}(t) = 1, \quad \zeta_{e_1+e_2}(t) = -1.$$

Thus

$$P[2e_1] = (\psi_2(t) + 1)/2, \quad P[e_1 + e_2] = -(\psi_2(t) - 1)/2,$$

are the projections mapping $V \otimes V$ onto the respective irreducible submodules. The permutation operator is given by

$$s = P[2e_1] - P[e_1 + e_2] = \psi_2(t). \tag{4.3}$$

The natural module for $\mathfrak{sl}_n$ may be treated in exactly the same way.

4.2. **The Brauer algebra in $\mathcal{A}(r)$.** Suppose $V \otimes V$ contains the trivial 1-dimensional submodule $L_0 = \mathbb{C}$; write $V \otimes V = L_0 \oplus L_0^\perp$, and write $P[0]$ for the corresponding projection to L_0. Recall that the signs $\epsilon(\lambda)$ are defined by the decomposition (4.2).

Lemma 4.3. *For any* $\mathrm{U}(\mathfrak{g})$*-module endomorphism* $f : V \otimes V \longrightarrow V \otimes V$*, we have the following equations in* $\mathrm{End}_{\mathrm{U}(\mathfrak{g})}(V^{\otimes 3})$*.*

$$\begin{aligned} &\text{(i)} \quad (\mathrm{P}[0] \otimes id_{\mathrm{V}})(id_{\mathrm{V}} \otimes \mathrm{f})(\mathrm{P}[0] \otimes id_{\mathrm{V}}) &= \xi_f P[0] \otimes id_V; \\ &\text{(ii)} \quad (id_{\mathrm{V}} \otimes \mathrm{P}[0])(\mathrm{f} \otimes id_{\mathrm{V}})(id_{\mathrm{V}} \otimes \mathrm{P}[0]) &= \xi_f id_V \otimes P[0], \end{aligned}$$

where $\xi_f = \mathrm{tr}_{V \otimes V}(f)/(\dim_{\mathbb{C}} V)^2$*.*

Proof. Since both equations are proved in the same way, we prove (i). The left hand side is a $\mathrm{U}(\mathfrak{g})$-invariant endomorphism of $V^{\otimes 3}$ whose kernel contains $L_0^\perp \otimes V$, and whose image lies in $L_0 \otimes V$. But as $L_0 \otimes V \cong V$ is irreducible, it follows from Schur's Lemma that the left hand side of (i) is proportional to $P[0] \otimes \mathrm{id}_V$. To determine the proportionality constant ξ_f, we proceed as follows. For any two (finite dimensional) vector spaces W, W', since $\mathrm{End}_{\mathbb{C}}(W \otimes W') \cong \mathrm{End}_{\mathbb{C}}(W) \otimes \mathrm{End}_{\mathbb{C}}(W')$, we have a linear map $T_{W,W'} : \mathrm{End}_{\mathbb{C}}(W) \otimes \mathrm{End}_{\mathbb{C}}(W') \longrightarrow \mathrm{End}_{\mathbb{C}}(W')$, given by

$$T_{W,W'}(\sum h \otimes h') := \sum \mathrm{tr}_W(h)h'.$$

If W, W' are $\mathrm{U}(\mathfrak{g})$-modules, then $T_{W,W'}$ is a homomorphism of $\mathrm{U}(\mathfrak{g})$-modules. It is easy to see that for $g \in \mathrm{End}_{\mathbb{C}}(W \otimes W')$ and $g' \in \mathrm{End}_{\mathbb{C}}(W')$,

$$\mathrm{tr}_{W'}(T_{W,W'}(g)) = \mathrm{tr}_{W \otimes W'}(g) \text{ and } T_{W,W'}(g \circ (\mathrm{id}_W \otimes g')) = T_{W,W'}(g) \circ g'. \tag{4.4}$$

If W' is an irreducible $\mathrm{U}(\mathfrak{g})$-module, then by Schur's Lemma $T_{W,W'}(g) = c_g \mathrm{id}_{W'}$, and by (4.4), $c_g = tr_{W \otimes W'}(g)/\dim_{\mathbb{C}} W'$. In particular, $T_{V,V}(P[0]) = \frac{1}{\dim_{\mathbb{C}} V}\mathrm{id}_V$, and so

$$T_{V,V\otimes V}(P[0] \otimes \mathrm{id}_V) = \frac{1}{\dim_{\mathbb{C}} V}\mathrm{id}_{V\otimes V}. \tag{4.5}$$

Using this, let us compute the trace of both sides of (i). Since $P[0] \otimes \mathrm{id}_V$ is a projection to the subspace $L_0 \otimes V$, which has dimension $\dim V$, the trace of the right side is $\xi_f \dim V$. Moreover

$$\begin{aligned}\mathrm{tr}_{V^{\otimes 3}}(P[0] \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes f)(P[0] \otimes \mathrm{id}_V) &= \mathrm{tr}_{V^{\otimes 3}}(P[0] \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes f)\\ &= \mathrm{tr}_{V^{\otimes 2}} T_{V,V^{\otimes 2}}\left((P[0] \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes f)\right) \text{ by (4.4)}\\ &= \mathrm{tr}_{V^{\otimes 2}}\left(T_{V,V^{\otimes 2}}(P[0] \otimes \mathrm{id}_V) \circ f\right)\\ &= \mathrm{tr}_{V^{\otimes 2}}(\frac{1}{\dim_{\mathbb{C}} V}\mathrm{id}_{V\otimes V} \circ f) \text{ by (4.5)}\\ &= \frac{1}{\dim_{\mathbb{C}} V}\mathrm{tr}_{V^{\otimes 2}}(f).\end{aligned}$$

Comparing this with the trace of the right side, we obtain $\xi_f = \frac{1}{(\dim V)^2}\mathrm{tr}_{V^{\otimes 2}}(f)$, as stated. □

Now define the following elements of $\mathcal{A}(r)$. Write $\tau = P[0]$, and define

$$\tau_i = \mathrm{id}_V^{\otimes(i-1)} \otimes \epsilon(0)\tau \otimes \mathrm{id}_V^{\otimes(r-i-1)}, \quad i = 1, 2, \cdots, r-1.$$

Recall that the Brauer algebra $B_r(\delta)$ may be defined as the $\mathbb{C}[\delta^{\pm 1}]$-algebra generated by elements $S_i, E_i,\ i = 1, \ldots, r-1$ subject to the usual braid relations for the S_i, as well as $S_i^2 = 1$, the Temperley-Lieb relations $E_i^2 = \delta E_i$, $E_i E_{i\pm 1} E_i = E_i$ and $E_i E_j = E_j E_i$ if $|i-j| \geq 2$, and the relations $S_i E_i = E_i S_i = E_i$, $S_{i+1} E_i S_{i+1} = S_i E_{i+1} S_i$, and $S_i E_j = E_j S_i$ when $|i-j| \geq 2$.

Theorem 4.4. *Suppose V is weakly multiplicity free and self dual, so that the trivial module L_0 is a summand of $V \otimes V$. Then the assignment $S_i \mapsto s_i$, $E_i \mapsto \dim V \tau_i$, $i = 1, 2, \cdots, r-1$ defines a homomorphism from the Brauer algebra (with $\delta = \epsilon(0)\dim V$) to $\mathcal{A}(r)$. When $|\mathcal{P}^+_{\Lambda_0}| = 3$, this homomorphism is surjective.*

Proof. Note first that since the projections $P[\lambda]$ are in $\mathcal{A}(2)$ by Corollary 3.9, $\tau \in \mathcal{A}(2)$, and since the s_i are in $\mathcal{A}(r)$ (by Theorem 3.10), it follows that $\tau_i \in \mathcal{A}(r)$ for all i. Now Lemma 4.3, with $f = P[0]$, implies that

$$\tau_i \tau_{i\pm 1} \tau_i = (\dim V)^{-2}\tau_i, \quad \tau_i \tau_j = \tau_j \tau_i, \quad |i-j| > 1.$$

Hence the maps $\tau_i' := \dim V \tau_i$ satisfy the Temperley-Lieb relations with $\delta = \epsilon(0)\dim V$. Also, $s_i\tau_i = \tau_i s_i = \tau_i$, and the other Brauer relations are clearly satisfied by the τ_i' and s_j. The first statement is therefore clear.

Now we prove the second part. The two signs $\epsilon(\lambda)$ with $\lambda \neq 0$ are unequal. Thus the three projections can be respectively expressed as

$$\epsilon(0)\tau, \qquad (\mathrm{id}^{\otimes 2} + s - \tau)/2, \qquad (\mathrm{id}^{\otimes 2} - s + \tau)/2.$$

Also t acting on $V \otimes V$ has three pairwise distinct eigenvalues, which we denote by ξ^+, ξ^- and ξ^0. We have

$$\psi_r(t_{i,i+1}) \;=\; \xi^+(\mathrm{id}^{\otimes r} + s_i - \tau_i)/2 + \xi^-(\mathrm{id}^{\otimes r} - s_i + \tau_i)/2 + \xi^0\epsilon(0)\tau_i.$$

Using this equation we can express $\psi_r(t_{ij})$ as

$$\psi_r(t_{ij}) \;=\; s_{j-1}s_{j-2}\cdots s_{i+1}\psi_r(t_{i,i+1})s_{i+1}\cdots s_{j-2}s_{j-1}.$$

Therefore, the homomorphic image of the Brauer algebra generated by the s_i, τ_i is the whole of $\mathcal{A}(r)$. □

Example 4.5. The natural representation of $\mathfrak{so}_n$

Let V be the natural module of $\mathfrak{so}_n$. We have seen that V is strongly multiplicity free if $n = 2m+1$, but is not smf when n is even. We need to show that it is wmf, i.e. that all irreducible submodules of $V \otimes V$ have distinct Casimir eigenvalues. That is the content of the next lemma.

Lemma 4.6. *Let V be the natural module of the Lie algebra $\mathfrak{so}_{2m}$. Then $V \otimes V$ is multiplicity free, and $\chi_\lambda(C) \neq \chi_\mu(C)$ if $\lambda \neq \mu$ for any $\lambda, \mu \in \mathcal{P}^+_{\Lambda_0}$.*

Proof. We may assume that $m > 1$ as $\mathfrak{so}_2$ is isomorphic to $\mathfrak{gl}_1$. Also $\mathfrak{so}_4 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, so that this case is easy. Note also that it essentially reduces to Lemma 3.6 for case (5) of Theorem 3.4.

For $m > 2$, let e_i, $i = 1, 2, \ldots, m$, be an orthonormal basis of an m-dimensional Euclidean space. The positive roots of $\mathfrak{so}_{2m}$ are given by $e_i \pm e_j$, $i < j$, which sum to $2\rho = \sum_{i=1}^m (2m - 2i)e_i$. The highest weight of the natural module is $\Lambda_0 = e_1$, and $\Pi = \{\pm e_i | 1 \le i \le m\}$. By dimension, these are all the weights.

Now $\widehat{\mathcal{P}}^+_{\Lambda_0} = \mathcal{P}^+_{\Lambda_0} = \{2e_1, e_1 + e_2, 0\}$. Clearly $\chi_{2e_1}(C)$ and $\chi_{e_1+e_2}(C)$ are both different from $\chi_0(C) = 0$, and $\chi_{2e_1}(C) - \chi_{e_1+e_2}(C) = (3e_1 + e_2 + 4\rho, e_1 - e_2) > 0$. □

We shall also need explicit knowledge of the root system of $\mathfrak{so}_{2m+1}$. The positive roots are $e_i \pm e_j$, $i < j$, and e_i. The sum of the positive roots is given by $2\rho = \sum_{i=1}^m (2m + 1 - 2i)e_i$.

For both $n = 2m$ and $2m + 1$, V has highest weight e_1, and

$$V \otimes V \;=\; L_{2e_1} \oplus L_0 \oplus L_{e_1+e_2}. \tag{4.6}$$

The eigenvalues of the Casimir operator C on these irreducible modules are

$$\begin{aligned}
\chi_{e_1}(C) &= (e_1 + 2\rho, e_1) = n - 1,\\
\chi_{2e_1}(C) &= (2e_1 + 2\rho, 2e_1) = 2n,\\
\chi_{e_1+e_2}(C) &= (e_1 + e_2 + 2\rho, e_1 + e_2) = 2(n-2),\\
\chi_0(C) &= 0,
\end{aligned}$$

and the eigenvalues of the endomorphism t on $V \otimes V$ are

$$\zeta_{2e_1}(t) = 1, \quad \zeta_{e_1+e_2}(t) = -1, \quad \zeta_0(t) = 1 - n.$$

The projection operators mapping $V \otimes V$ onto the irreducible submodules are respectively given by

$$\begin{aligned} P[2e_1] &= \frac{(\psi_2(t)+1)(\psi_2(t)-1+n)}{2n}, \\ P[e_1+e_2] &= -\frac{(\psi_2(t)-1)(\psi_2(t)-1+n)}{2(n-2)}, \\ P[0] &= \frac{(\psi_2(t)-1)(\psi_2(t)+1)}{n(n-2)}. \end{aligned}$$

The permutation operator in the present case can be expressed as

$$s = P[2e_1] + P[0] - P[e_1+e_2] = \psi_2(t) + \frac{(\psi_2(t)-1)(\psi_2(t)+1)}{n-2}. \tag{4.7}$$

Example 4.7. The natural representation of $\mathfrak{sp}_{2m}$

The positive roots of $\mathfrak{sp}_{2m}$ are $e_i \pm e_j$, $i < j$, $2e_k$, and $2\rho = 2\sum_{i=1}^{m}(m-i+1)e_i$. The natural module V has highest weight e_1, and

$$V \otimes V = L_{2e_1} \oplus L_{e_1+e_2} \oplus L_0.$$

The relevant eigenvalues of C and t are

$$\begin{aligned} \chi_{e_1}(C) &= (e_1+2\rho, e_1) = 1+2m, \\ \chi_{2e_1}(C) &= (2e_1+2\rho, 2e_1) = 4m+4, \\ \chi_{e_1+e_2}(C) &= (e_1+e_2+2\rho, e_1+e_2) = 4m, \\ \chi_0(C) &= 0, \end{aligned}$$

$$\zeta_{2e_1}(t) = 1, \qquad \zeta_{e_1+e_2}(t) = -1, \quad \zeta_0(t) = -1-2m.$$

The projection operators are given by

$$\begin{aligned} P[2e_1] &= \frac{(\psi_2(t)+1)(\psi_2(t)+1+2m)}{2(2m+2)}, \\ P[e_1+e_2] &= -\frac{(\psi_2(t)-1)(\psi_2(t)+1+2m)}{4m}, \\ P[0] &= \frac{(\psi_2(t)-1)(\psi_2(t)+1)}{2m(2m+2)}. \end{aligned}$$

The permutation operator is given by

$$\begin{aligned} s &= P[2e_1] - P[e_1+e_2] - P[0] \\ &= \psi_2(t) + \frac{(\psi_2(t)-1)(\psi_2(t)+1)}{2m+2}. \end{aligned}$$

Note that in accord with Theorem 4.4, the projection operators have the same form as those in the $\mathfrak{so}_n$ case with t replaced by $-t$ and $2m$ by $-n$, since the sign $\epsilon(0)$ is different in the two cases.

5. Proof of Theorem 3.13 and Theorem 3.15

In this section we give the proofs of Theorems 3.13 and 3.15. The proof will be case by case, using our classification of smf modules.

5.1. **The Classical Lie groups.** We first consider the natural modules of the classical Lie groups GL_k, SL_k, O_k, and Sp_{2k}.

5.1.1. *The linear groups GL_k and SL_k.* It is a celebrated result of classical invariant theory known as Schur-Weyl duality (see [W] or [Hos]) that $\mathrm{End}_{GL_k(\mathbb{C})}(V^{\otimes r})$ is the group algebra of the symmetric group Sym_r in the representation ϕ_r. If $SL_k(\mathbb{C})$ is regarded as a subgroup of $GL_k(\mathbb{C})$, then $GL_k(\mathbb{C})$ is generated by $SL_k(\mathbb{C})$ and scalars. Since the scalars in $GL_k(\mathbb{C})$ act on $V^{\otimes r}$ as multiplication by constants, $\mathrm{Hom}_{SL_k(\mathbb{C})}(V^{\otimes r}, V^{\otimes r})$ and $\mathrm{Hom}_{GL_k(\mathbb{C})}(V^{\otimes r}, V^{\otimes r})$ coincide. By Remark 3.16, it follows that $\mathrm{Hom}_{\mathfrak{sl}_k(\mathbb{C})}(V^{\otimes r}, V^{\otimes r})$ and $\mathrm{Hom}_{\mathfrak{gl}_k(\mathbb{C})}(V^{\otimes r}, V^{\otimes r})$ are also both equal to $\phi_r(\mathbb{C}\mathrm{Sym}_s)$. In view of Theorem 4.1 this completes the proof of Theorems 3.15 and 3.13 for these cases.

5.1.2. *The symplectic group $Sp_{2k}(\mathbb{C})$.* It is a classical result of Brauer ([Br]) that $\mathrm{Hom}_{Sp_{2k}(\mathbb{C})}(V^{\otimes r}, V^{\otimes r})$ is the image of the Brauer algebra described in Theorem 4.4. Since $Sp_{2k}(\mathbb{C})$ is connected, it follows from Proposition 3.14 that the corresponding centraliser algebra in the Lie algebra case $\mathfrak{sp}_{2k}(\mathbb{C})$ is the same. Thus Theorem 4.4 implies both Theorems 3.15 and 3.13 in this case.

5.1.3. *The orthogonal groups $O_k(\mathbb{C})$.* Brauer proved in *op. cit.* that $\mathrm{End}_{O_k(\mathbb{C})}(V^{\otimes r})$ is the homomorphic image of the Brauer algebra described in Theorem 4.4. Hence, bearing in mind Theorem 4.4, we have, writing V for the natural $O_k(\mathbb{C})$-module,

$$\mathcal{A}(r) = \mathrm{End}_{O_k(\mathbb{C})}(V^{\otimes r}) \subseteq \mathrm{End}_{SO_k(\mathbb{C})}(V^{\otimes r}) = \mathrm{End}_{\mathfrak{so}_k(\mathbb{C})}(V^{\otimes r}), \tag{5.1}$$

where the last equality follows from Proposition 3.14. If $k = 2m+1$, then O_k is generated by $-\mathrm{id}_V$ and SO_k, and since $-\mathrm{id}_V$ acts on $V^{\otimes r}$ as a scalar, the inequality of (5.1) is an equality. Thus in the odd orthogonal case, both Theorems 3.15 and 3.13 follow.

Theorem 3.15 for the case of $O_{2m}(\mathbb{C})$ also follows from the above discussion.

5.2. **The case of G_2.** We now consider Theorem 3.13 and Theorem 3.15 in the case of G_2 and its Lie algebra. Since the group $G_2(\mathbb{C})$ is simply connected, in view of Proposition 3.14 and Remark 3.16, it suffices to consider the Lie group case.

5.2.1. *Invariant theory for G_2.* The G_2 endomorphism algebras of tensor powers of the seven dimensional irreducible module V are nothing other than G_2 invariants in tensor powers of V since V is self dual. We therefore recall the invariant theory for G_2 developed by Schwarz [S]. Let us first describe the 7-dimensional irreducible representation of G_2. We note that another approach to G_2 may be found in the interesting paper of Kuperberg [Kup].

Denote by α_1 and α_2 the simple roots of G_2, with α_1 being the shorter one. We take the bilinear form (,) to be normalised so that $(\alpha_1, \alpha_1) = 2$. The set of positive roots is given by

$$R_+ = \{\alpha_1, \alpha_2; \alpha_2 + \alpha_1;\ \alpha_2 + 2\alpha_1;\ \alpha_2 + 3\alpha_1;\ 2\alpha_2 + 3\alpha_1\};$$

and the fundamental weights are

$$\lambda_1 = 2\alpha_1 + \alpha_2, \qquad \lambda_2 = 3\alpha_1 + 2\alpha_2.$$

Note that both λ_1 and λ_2 are positive roots, with λ_1 being short and λ_2 long. In fact λ_2 is the maximal root, hence the module L_{λ_2} is isomorphic to the Lie algebra itself, which is 14 dimensional. Let V denote the 7 dimensional irreducible module L_{λ_1}, which is strongly multiplicity free by Theorem 3.4. The set of weights of V is $\Pi = \{0, \pm\alpha_1, \pm(\alpha_2 + \alpha_1), \pm(\alpha_2 + 2\alpha_1)\}$. We have

$$V \otimes V = L_{2\lambda_1} \oplus L_{\lambda_2} \oplus L_{\lambda_1} \oplus L_0,$$

where the dimension of $L_{2\lambda_1}$ is 27. The appearance of the trivial 1-dimensional submodule L_0 in $V \otimes V$ implies that V is self-dual, that is, V and its dual V^* are isomorphic. Moreover since $L_{\lambda_1} = V$, it follows that each tensor power of V contains L_0.

The eigenvalues of the quadratic Casimir in these irreducible modules are respectively given by

$$\chi_{\lambda_1}(C) = 12, \quad \chi_{\lambda_2}(C) = 24, \quad \chi_{2\lambda_1}(C) = 28, \quad \chi_0(C) = 0.$$

Let $P[\mu], \mu = 2\lambda_1, \lambda_2, \lambda_1, 0$, be the respective projections mapping $V \otimes V$ onto the irreducibles. Then the permutation map $s : V \otimes V \to V \otimes V$ can be expressed as

$$s = P[2\lambda_1] + P[0] - P[\lambda_2] - P[\lambda_1].$$

This shows that the symmetric and exterior squares of V are given by $S^2(V) = L_{2\lambda_1} \oplus L_0$ and $\wedge^2(V) = L_{\lambda_2} \oplus L_{\lambda_1}$.

As is well known (see, e.g., [S]), the Cayley algebra over $\mathbb{C}$ is the unique (up to isomorphism) 8-dimensional non-commutative, non-associative alternative algebra, which can be constructed in the following way. Let $A_{\mathbb{R}}$ denote the set of ordered pairs of quaternions with component wise addition and the following multiplication:

$$(a; b)(c; d) = (ac - \bar{d}b; da + b\bar{c}),$$

where $\bar{a}$ is the usual conjugation of quaternions. Then $A_{\mathbb{R}}$ is a central simple non-associative, non-commutative alternative algebra of dimension 8 over $\mathbb{R}$. If $x = (a; b) \in A_{\mathbb{R}}$, we define the trace of x by $\mathrm{tr}(x) = \mathrm{Re}(a)$, the real part of a. The complexification $A_{\mathbb{C}} := A$ of $A_{\mathbb{R}}$ is the Cayley algebra; it retains the complex valued linear form tr. The connected component of the automorphism group of A is the complex exceptional group G_2. The 7-dimensional G_2-module V is now realised as the subspace of A consisting of elements of trace zero.

For any non-negative integer m, let V^m be the direct sum of m copies of V. Let $(x_1, \cdots, x_m)$ be an arbitrary element in V^m. Following [S], we define three series of G_2-invariant polynomial functions in the coordinate ring $\mathbb{C}[V^m]$. First define A, B and C in $\mathbb{C}[V^2], \mathbb{C}[V^3]$ and $\mathbb{C}[V^4]$ respectively by

(5.2)

$$A(x, y) = -\mathrm{tr}(xy); \quad B(x, y, u) = -\mathrm{tr}(x(y(u)); \quad C(x, y, u, v) = \mathrm{skew}(\mathrm{tr}(x(y(u(v))),$$

for $x, y, u, v \in V$, where skew denotes skew symmetrisation over Sym_4. For any sequence $1 \le i_1 < \cdots < i_p \le n$ write $P_{i_1 i_2 \ldots i_p} : V^m \to V^p$ for the projection $V^m \to V_{i_1} \oplus \cdots \oplus V_{i_p}$, define $\alpha_{ij} \in \mathbb{C}[V^m]$ by $\alpha_{ij} = A \circ P_{ij}$. Similarly, define $\beta_{ijk} = B \circ P_{ijk}$ and $\gamma_{ijkl} = C \circ P_{ijkl}$.

Theorem 5.1. [S] *The algebra* $\mathbb{C}[V^m]^{G_2}$ *of* G_2*-invariant polynomials on* V^m *is generated by the polynomials* α_{ij}*,* β_{ijk} *and* γ_{ijkl}*.*

Now $\mathbb{C}[V^m]$ may be identified with the symmetric algebra $S((V^m)^*)$ on the dual space $(V^m)^*$ of V^m. Since $S((V^m)^*) \cong S((V^*)^m) \cong S(V^*)^{\otimes m}$, this algebra is $\mathbb{Z}^m$-graded (since $S(V^*)$ is $\mathbb{Z}$-graded, with the non-zero components all having non-negative degree), and for any m-tuple $\mathbf{d} = (d_1, \ldots, d_m)$ of integers, we denote by $S^{\mathbf{d}}((V^m)^*)$ the subspace of polynomials of multi-degree $(d_1, \ldots, d_m)$. Then $S((V^m)^*) = \oplus_{\mathbf{d}} S^{\mathbf{d}}((V^m)^*)$ is a G_2-invariant direct decomposition of $S((V^m)^*)$.

The homogeneous component $S^{(1,\ldots,1)}((V^m)^*) \cong (V^*)^{\otimes m}$, and since G_2 preserves the grading, Theorem 5.1 inter alia provides an explicit description of the invariant subspace $((V^*)^{\otimes m})^{G_2}$.

Remark 5.2. The map A of (5.2) is a G_2-invariant, non-degenerate pairing : $V \times V \to \mathbb{C}$. Hence A provides a canonical isomorphism of G_2-modules $V \xrightarrow{\sim} V^*$. We shall therefore consider the G_2-modules V and V^* as canonically identified in this way. In particular, with this identification, Theorem 5.1 is interpreted as an explicit description of the space $(V^{\otimes m})^{G_2}$.

Using the identification described in Remark 5.2, the G_2-invariant maps A, B and C of (5.2) give rise to G_2 equivariant maps among the tensor powers of V as follows. First, the maps A, B and C themselves provide

$$A : V^{\otimes 2} \to \mathbb{C}, \qquad B : V^{\otimes 3} \to \mathbb{C}, \qquad C : V^{\otimes 4} \to \mathbb{C}. \tag{5.3}$$

It is useful to note that while A is symmetric, both B and C are skew symmetric.

Now $A \in (V^{\otimes 2})^* \cong (V^*)^{\otimes 2}$, which by Remark 5.2 is canonically identified with $V^{\otimes 2}$. Thus we have the G_2-equivariant map

$$\Phi : \mathbb{C} \to V \otimes V, \quad c \mapsto cA.$$

Similarly, we obtain from B the maps

$$B' : V^{\otimes 2} \to V, \quad \Psi' : V \to V^{\otimes 2}, \quad \Psi : \mathbb{C} \to V^{\otimes 3};$$

and from C the maps

$$C' : V^{\otimes 3} \to V, \quad C'' : V^{\otimes 2} \to V^{\otimes 2}, \quad \Theta' : V \to V^{\otimes 3}, \quad \Theta : \mathbb{C} \to V^{\otimes 4}.$$

The maps A, B, C and their relatives described above are the contraction and expansion operators of [HZ]. As shown in [S, HZ] (proof of Proposition 3.22 in [S] and Lemma 5.1. (c) in [HZ]), C'' restricts to an isomorphism on $\wedge^2 V$, thus $C'' = aP[\lambda_1] + bP[\lambda_2]$ with $a, b \neq 0$. We further note that $a \neq b$.

Now by Remark 5.2, we have, for any integer n, an isomorphism of G_2 modules $(V^*)^{\otimes 2n} \cong (V^*)^{\otimes n} \otimes V^{\otimes n} \cong \mathrm{Hom}_{\mathbb{C}}(V^{\otimes n}, V^{\otimes n})$. Taking G_2-invariants, one obtains

$$\left((V^*)^{\otimes 2n}\right)^{G_2} \cong \mathrm{Hom}_{G_2}(V^{\otimes n}, V^{\otimes n}), \tag{5.4}$$

which leads to the following easy application of Theorem 5.1.

Lemma 5.3. [HZ] *The algebra* $\mathrm{Hom}_{G_2}(V^{\otimes n}, V^{\otimes n})$ *is generated by the place permutations in* $\phi_n(\mathrm{Sym}_n)$ *together with the contraction and expansion maps defined above.*

Some clarification is required as to how $\mathrm{Hom}_{G_2}(V^{\otimes n},\ V^{\otimes n})$ is generated by the given elements. Let $\Omega : V^{\otimes k} \to \mathbb{C}$ be any map constructed by tensoring the maps A, B, C (e.g., $\Omega = A^{\otimes 3} \otimes B^{\otimes 5} \otimes C \otimes A \otimes C^{\otimes 2}$), and let $U : \mathbb{C} \to V^{\otimes l}$ be constructed by tensoring Φ, Ψ, Θ (e.g., $U = \Phi^{\otimes 7} \otimes \Theta \otimes \Phi \otimes \Psi^{\otimes 6} \otimes \Theta^{\otimes 3}$); by composition one obtains $U \circ \Omega : V^{\otimes k} \to V^{\otimes l}$. Next, let $X : V^{\otimes p} \to V^{\otimes q}$ be constructed by tensoring B', C', Ψ', Θ', C'', and id (e.g., $X = (B')^{\otimes 2} \otimes \mathrm{id} \otimes C'' \otimes (\Psi')^{\otimes 3} : V^{\otimes 10} \to V^{\otimes 11}$). Finally, define the map $(U \circ \Omega) \otimes X : V^{\otimes (k+p)} \to V^{\otimes (l+q)}$. When $k + p = l + q = n$, we can compose $(U \circ \Omega) \otimes X$ with permutations σ_1, σ_2 in $\phi_n(\mathrm{Sym}_n)$ to obtain

$$F = \sigma_2 \circ ((U \circ \Omega) \otimes X) \circ \sigma_1.$$

Then $\mathrm{Hom}_{G_2}(V^{\otimes n},\ V^{\otimes n})$ is spanned by maps of the form F.

5.2.2. *Proof of Theorem 3.15 for G_2.* By Theorem 4.1, the permutations are in $\mathcal{A}(n)$, whence we only need show that all maps of the form $(U \circ \Omega) \otimes X$ are in $\mathcal{A}(n)$. This will be done in a series of reductions.

(1). Assume that there is a factor C in Ω. Note that $C = (A \otimes A) \circ (\mathrm{id} \otimes C'' \otimes \mathrm{id})$. Let $\hat{\Omega}$ be the map obtained by replacing this C in Ω by $A \otimes A$. Then $(U \circ \Omega) \otimes X = \Big((U \circ \hat{\Omega}) \otimes X\Big) \circ \big(\mathrm{id}^{\otimes r} \otimes C'' \otimes \mathrm{id}^{\otimes s}\big)$ for some integers s and r. As C'' belongs to $\mathcal{A}(n)$ by the case $n = 2$, which is known to be true, our task reduces to showing that $(U \circ \Omega) \otimes X \in \mathcal{A}(r)$ if Ω contains no factor C. A similar argument using $\Theta = (\mathrm{id} \otimes C'' \otimes \mathrm{id}) \circ (\Phi \otimes \Phi)$ shows that we may also assume that U has no factor Θ.

(2). Consider $(U \circ \Omega) \otimes X$ with Ω containing B factors, and U containing Ψ factors. Applying permutations if necessary, we may assume that all the B's are at the end of Ω, and the Ψ's at the end of U. Now

$$\Psi \circ B : V^{\otimes 3} \to V^{\otimes 3}$$

is the projection which maps $V^{\otimes 3}$ onto its unique 1-dimensional submodule. It can be constructed from the triple co-product of the quadratic Casimir element of G_2, and hence is contained in $\mathcal{A}(3)$. Therefore, we are further reduced to the case where $(U \circ \Omega) \otimes X$ does not simultaneously contain factors B and Ψ.

Assume that we are in the situation where Ω contains k factors B, but U contains no Ψ. The other case is similar. We may replace each factor B in Ω by $A \circ (\mathrm{id} \otimes B')$. In this way, $(U \circ \Omega) \otimes X$ can be expressed as a composition of maps of the form $\sigma_2 \circ (Z \otimes X') \circ \sigma_1$ where Z is constructed from A and Φ only, while X' is constructed from B', Ψ', C', C'', Θ' and id.

(3). Consider a map X of the form of X' above. If X has a factor C'' acting on the i-th and $i + 1$-th factors in $V^{\otimes n}$, then $(U \circ \Omega) \otimes X = \Big((U \circ \Omega) \otimes \hat{X}\Big) \circ (\mathrm{id}^{\otimes (i-1)} \otimes C'' \otimes \mathrm{id}^{\otimes (n-i-1)})$, where $\hat{X}$ is obtained from X by replacing C'' by the identity map on the tensor product $V \otimes V$. As C'' belongs to $\mathcal{A}(2)$, we only need to show $(U \circ \Omega) \otimes X \in \mathcal{A}(n)$ when X contains no C''.

The relation $C' = (\mathrm{id} \otimes A) \circ (C'' \otimes \mathrm{id})$ and the previous paragraph show that any factor C' may be eliminated from X, and so is also θ' in a similar way. Hence we are reduced to showing that $Z \otimes X \in \mathcal{A}(n)$, where Z is constructed from A and Φ only, while X is constructed from B', Ψ' and id only.

(4). It is important to observe that the total number of factors B' and Ψ' in X must be even. Thus modulo permutations, X may be assumed to be either of the form $(B' \otimes \Psi')^{\otimes k} \otimes (B')^{\otimes 2l} \otimes \mathrm{id}^{\otimes p}$ or $(B' \otimes \Psi')^{\otimes k} \otimes (\Psi')^{\otimes 2l} \otimes \mathrm{id}^{\otimes p}$.

Note that $B' \otimes \Psi' = (B' \otimes \Psi') \circ (P[\lambda_1] \otimes \mathrm{id})$. As $V \otimes V$ is multiplicity free, we can always express $B' \otimes \Psi'$ as $T \circ (P[\lambda_1] \otimes \mathrm{id})$, where T is a linear combination of the projection operators $P_2[\mu]$, $\mu = 2\lambda_1, 0, \lambda_1, \lambda_2$, described in Proposition 3.11. Thus $(B' \otimes \Psi')^{\otimes k}$ belongs to $\mathcal{A}(3k)$. Therefore, we now only need to show that all $Z \otimes X \in \mathcal{A}(n)$ when $X = (B')^{\otimes 2l} \otimes \mathrm{id}^{\otimes p}$ or $(\Psi')^{\otimes 2l} \otimes \mathrm{id}^{\otimes p}$.

Since these two cases are almost identical, we shall examine the first one only. In order for $Z \otimes X$ to be in $\mathrm{Hom}_{G_2}(V^{\otimes n}, V^{\otimes n})$, Z should be the tensor product of $\Phi^{\otimes l}$ with an appropriate tensor power of $P[0]$. Note that $B' \otimes B' = (\mathrm{id} \otimes A) \circ (B' \otimes \Psi' \otimes \mathrm{id})$. Thus $Z \otimes X$ can be expressed as a composition $F \circ H$, where F is the image in $\mathcal{A}(n)$ of an element of the Brauer algebra in the representation described in Theorem 4.4, and H is an element of $\mathcal{A}(n)$ constructed from $B' \otimes \Psi'$ and id. Obviously $F \circ H$ belongs to $\mathcal{A}(n)$. This completes the proof of Theorem 3.13 for the case G_2.

5.3. **The higher dimensional representations of $\mathfrak{gl}_2$ and $\mathfrak{sl}_2$.** Let V be the $(l+1)$-dimensional irreducible $\mathfrak{gl}_2$-module. We realise $V^{\otimes r}$ as follows. Denote by $\mathbb{C}[\mathbf{X}]$ the algebra of polynomials in the indeterminates x_i^p, $p = 1, 2$ and $i = 1, 2, \dots, r$. If M_i is the subalgebra $\mathbb{C}[x_i^1, x_i^2]$ of $\mathbb{C}[\mathbf{X}]$, then $\mathbb{C}[\mathbf{X}] \cong M_1 \otimes \cdots \otimes M_r$. Let $e_i^{pq} := x_i^p \frac{\partial}{\partial x_i^q}$ $(p, q = 1, 2)$. If g^{pq} $(p, q = 1, 2)$ denotes the usual basis of $\mathfrak{gl}_2$, the map $g^{pq} \mapsto e_i^{pq}$ defines an action of $\mathfrak{gl}_2$ on M_i for each index i. If $\mathcal{U}_2$ denotes the universal enveloping algebra of $\mathfrak{gl}_2$, then the operators $\{e_i^{pq} \mid p, q = 1, 2;\ \ i = 1, \dots, r\}$ generate an action of $\mathcal{U}_2^{\otimes r}$ on $\mathbb{C}[\mathbf{X}]$ in the obvious way. With these identifications, observe that $\Delta^{(r)}(g^{pq})$ acts as $e^{pq} := \sum_{i=1}^r e_i^{pq} = \sum_{i=1}^r x_i^p \frac{\partial}{\partial x_i^q}$ on $\mathbb{C}[\mathbf{X}]$ (for each p, q).

Now each subalgebra M_i is $\mathbb{N}$-graded by degree, whence $\mathbb{C}[\mathbf{X}]$ is $\mathbb{N}^r$-graded. Let $M_{i,d}$ be the homogeneous subspace of M_i of degree d. If $\mathbf{d} = (d_1, \dots, d_r) \in \mathbb{N}^r$, then $\mathbb{C}[\mathbf{X}]_{\mathbf{d}} = M_{1,d_1} \otimes \cdots \otimes M_{r,d_r}$ denotes the degree $\mathbf{d}$ subspace of $\mathbb{C}[\mathbf{X}]$. The above action of $\mathfrak{gl}_2$ leaves invariant the subspaces $M_{i,d}$ for each i, d, and hence we have the $\mathcal{U}_2^{\otimes r}$-module decomposition

$$\mathbb{C}[\mathbf{X}] = \oplus_{\mathbf{d} \in \mathbb{N}^r} \mathbb{C}[\mathbf{X}]_{\mathbf{d}}. \tag{5.5}$$

Clearly $V \cong M_{i,l}$ for each i, and so $V^{\otimes r}$ may (and will) be identified with $\mathbb{C}[\mathbf{X}]_{(l,l,\dots,l)}$. A generating set for the algebra $\mathcal{A}(r)$ of endomorphisms of the $\mathcal{U}_2$-module $\mathbb{C}[\mathbf{X}]$ (or any of the submodules $\mathbb{C}[\mathbf{X}]_{\mathbf{d}}$) is provided by the images of the elements C_{ij} of (2.2). Since $(g^{11}, g^{12}, g^{21}, g^{22})$ and $(g^{11}, g^{21}, g^{12}, g^{22})$ are dual bases of $\mathfrak{gl}_2$, we have

Remark 5.4. The operators $\{A_{ij} := \sum_{p,q=1,2} e_i^{pq} e_j^{qp} \mid 1 \leq i < j \leq r\}$ generate the associative algebra $\mathcal{A}(r)$ of $\mathcal{U}_2$-endomorphisms of $\mathbb{C}[\mathbf{X}]_{\mathbf{d}}$ for any degree $\mathbf{d} = (d_1, \dots, d_r)$.

We wish to show that for $\mathbf{d} = (l, \dots, l)$ this is the whole endomorphism algebra $\mathcal{E}$. In fact we shall prove the stronger result

Theorem 5.5. *For any degree $\mathbf{d} = (d_1, \dots, d_r)$, the endomorphism algebra $\mathcal{E}$ of $\mathbb{C}[\mathbf{X}]_{\mathbf{d}}$ regarded as a $\mathcal{U}_2$-module (or $\mathfrak{gl}_2$-module, or $\mathfrak{sl}_2$-module), coincides with the algebra $\mathcal{A}(r)$ of Remark 5.4.*

Our strategy for proving Theorem 5.5 will be to use Howe duality to obtain a characterisation of $\mathcal{E}$, and from that to identify a generating set for $\mathcal{E}$ using a result of Millson and Toledano Laredo, whose elements will be shown to lie in $\mathcal{A}(r)$.

Proof of Theorem 5.5. Define linear operators E_{ij} on $\mathbb{C}[\mathbf{X}]$ by

$$E_{ij} = \sum_{p=1}^{2} x_i^p \frac{\partial}{\partial x_j^p}, \quad i, j = 1, 2, \ldots, r.$$

Let $\{U_{ij} \mid 1 \leq i, j \leq r\}$ be the usual basis of $\mathfrak{gl}_r$. The map $U_{ij} \mapsto E_{ij}$ defines an action of $\mathfrak{gl}_r$, and hence its universal enveloping algebra $\mathcal{U}_r$, on $\mathbb{C}[\mathbf{X}]$, and the actions of $\mathcal{U}_2$ and $\mathcal{U}_r$ commute. In fact $(\mathfrak{gl}_2, \mathfrak{gl}_r)$ forms a dual reductive pair on $\mathbb{C}[\mathbf{X}]$ in the sense of [Ho] or [Hos]. This implies that as a module for $\mathcal{U}_2 \otimes \mathcal{U}_r$, $\mathbb{C}[\mathbf{X}]$ is multiplicity free, and from [Hos, Theorem 2.1.2] it follows that as $\mathcal{U}_2 \otimes \mathcal{U}_r$-module,

$$\mathbb{C}[\mathbf{X}] \cong \oplus_{\lambda \in \Lambda_2} L_\lambda^{\mathcal{U}_2} \otimes L_\lambda^{\mathcal{U}_r}, \tag{5.6}$$

where Λ_2 is the set of all partitions with two parts, and if $\lambda = (\lambda_1 \geq \lambda_2)$, then $L_\lambda^{\mathcal{U}_2}$ (resp. $L_\lambda^{\mathcal{U}_r}$) is the irreducible $\mathcal{U}_2$-module (resp. $\mathcal{U}_r$-module) with highest weight (λ_1, λ_2) (resp. $(\lambda_1, \lambda_2, 0, \ldots, 0)$, where there are $r-2$ zeros).

The basis elements U_{ii}, $i = 1, \ldots, r$ form a basis of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{gl}_r$. They act on $\mathbb{C}[\mathbf{X}]$ via the E_{ii}, and if $f \in \mathbb{C}[\mathbf{X}]$ has degree $\mathbf{d} = (d_1, \ldots, d_r)$, then $E_{ii} f = d_i f$. Hence if H is the (commutative) subalgebra of $\mathcal{U}_r$ generated by the U_{ii} $(i = 1, \ldots, r)$ then $\mathbb{C}[\mathbf{X}] = \oplus_{\mathbf{d}} \mathbb{C}[\mathbf{X}]_{\mathbf{d}}$ is the decomposition of $\mathbb{C}[\mathbf{X}]$ into H-invariant subspaces; that is, the decomposition (5.5) of $\mathbb{C}[\mathbf{X}]$ by degree is precisely its decomposition into weight spaces with respect to H as a representation of $\mathcal{U}_r$. Since the action of $\mathcal{U}_2$ preserves degree, this implies that the decompositions (5.5) and (5.6) are related as follows (using the notation above).

For any degree $\mathbf{d} = (d_1, \ldots, d_r)$, we have

$$\mathbb{C}[\mathbf{X}]_{\mathbf{d}} = \oplus_{\lambda \in \Lambda_2} L_\lambda^{\mathcal{U}_2} \otimes (L_\lambda^{\mathcal{U}_r})_{\mathbf{d}}, \tag{5.7}$$

where $(L_\lambda^{\mathcal{U}_r})_{\mathbf{d}}$ is the $\mathbf{d}$-weight space of the $\mathcal{U}_r$ module $L_\lambda^{\mathcal{U}_r}$.

The decomposition (5.7) is precisely the decomposition of $\mathbb{C}[\mathbf{X}]_{\mathbf{d}}$, regarded as a $\mathcal{U}_2$-module into isotypic components. It is therefore clear that

$$\mathrm{End}_{\mathcal{U}_2}(\mathbb{C}[\mathbf{X}]_{\mathbf{d}}) \cong \oplus_{\lambda \in \Lambda_2} \mathrm{End}_{\mathbb{C}}((L_\lambda^{\mathcal{U}_r})_{\mathbf{d}}). \tag{5.8}$$

By the argument of Corollary 3.12, the projections to all isotypic components of $\mathbb{C}[\mathbf{X}]_{\mathbf{d}}$ lie in $\mathcal{A}(r)$, so for the proof of the Theorem, it suffices to show that all endomorphisms of the multiplicity spaces $(L_\lambda^{\mathcal{U}_r})_{\mathbf{d}}$ are induced by $\mathcal{A}(r)$.

To prove the latter assertion, we introduce, following [MT, §1] the Casimir subalgebra $\mathcal{C}_r$ of $\mathcal{U}_r$. This is defined as the subalgebra of $\mathcal{U}_r$ generated by the elements $\{U_{ij}U_{ji} + U_{ji}U_{ij}, U_{ii} \mid i, j = 1, \ldots, r\}$. Note that the generators of $\mathcal{C}_r$ preserve degree, whence $\mathcal{C}_r$ acts on $\mathbb{C}[\mathbf{X}]_{\mathbf{d}}$, and hence on the summands of the decomposition (5.7). Moreover it follows from [MT, Remark 6.2, after Theorem 6.1] that $\mathcal{C}_r$ acts irreducibly on $(L_\lambda^{\mathcal{U}_r})_{\mathbf{d}}$, and hence that its generators induce the whole of $\mathrm{End}_{\mathbb{C}}((L_\lambda^{\mathcal{U}_r})_{\mathbf{d}})$. It therefore suffices to show that the generators of $\mathcal{C}_r$ induce endomorphisms of $\mathbb{C}[\mathbf{X}]_{\mathbf{d}}$ which lie in $\mathcal{A}(r)$.

To see this, note first that U_{ii} acts on $\mathbb{C}[\mathbf{X}]_{\mathbf{d}}$ as E_{ii}, i.e. as the scalar d_i. Thus certainly the endomorphisms induced by the U_{ii} are in $\mathcal{A}(r)$. Next, $U_{ij}U_{ji}$ acts on

$\mathbb{C}[\mathbf{X}]_{\mathbf{d}}$ as $E_{ij}E_{ji}$, and it will clearly suffice to show that the latter lie in $\mathcal{A}(r)$. But, for any $i \neq j$,

$$\begin{aligned}
E_{ij}E_{ji} =& \sum_{p,q=1,2} x_i^p \frac{\partial}{\partial x_j^p} x_j^q \frac{\partial}{\partial x_i^q} \\
=& \sum_{p,q=1,2} x_i^p \left(\delta_{pq} \frac{\partial}{\partial x_i^q} + x_j^q \frac{\partial^2}{\partial x_j^p \partial x_i^q} \right) \\
=& \sum_{p} x_i^p \frac{\partial}{\partial x_i^p} + \sum_{p,q=1,2} x_i^p x_j^q \frac{\partial^2}{\partial x_j^p \partial x_i^q} \\
=& E_{ii} + \sum_{p,q=1,2} x_i^p \frac{\partial}{\partial x_i^q} x_j^q \frac{\partial}{\partial x_j^p} \\
=& E_{ii} + \sum_{p,q=1,2} e_i^{pq} e_j^{qp} \\
=& E_{ii} + A_{ij}
\end{aligned}$$

Since E_{ii} acts as a scalar, it follows that $E_{ij}E_{ji} \in \mathcal{A}(r)$, which completes the proof of Theorem 5.5. □

This completes the proof of Theorem 3.15 for $\mathfrak{g} = \mathfrak{gl}_2$ and V as above. The case of $\mathfrak{sl}_2$ is implied by the result on $\mathfrak{gl}_2$. By Remark 3.16, this also proves the relevant statement of Theorem 3.15.

6. Quantised universal enveloping algebras and representations of the braid group

6.1. **Quantised universal enveloping algebras.** Let $\mathfrak{g}$ be a simple Lie algebra of rank $rk(\mathfrak{g})$, and let $\mathfrak{b}$ and $\mathfrak{h}$ respectively denote a Borel subalgebra and a Cartan subalgebra of $\mathfrak{g}$. Assume the bilinear form $(\ ,\ )$ on $\mathfrak{h}^*$ is normalised so that any short root α satisfies $(\alpha, \alpha) = 2$ for all algebras except that in the case of $\mathfrak{so}_{2m+1}$, $(\alpha, \alpha) = 1$. If $\{\alpha_i\}$ is the set of simple roots of $\mathfrak{g}$, the Cartan matrix $A = (a_{ij})$ of $\mathfrak{g}$ is defined by $a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$. Let l be the exponent of the weight lattice of $\mathfrak{g}$ modulo the root lattice. Let z be an indeterminate over $\mathbb{C}$ and set $q = z^{\ell^2(\alpha,\alpha)/2}$ where α is any short root. Note that the exponent of z is an integer.

In this work, the Jimbo version [D, J] of the quantised universal enveloping algebra $\mathrm{U}_q(\mathfrak{g})$ is defined to be the unital associative algebra over the field $\mathbb{C}(z)$ of rational functions of the indeterminate z, generated by elements $k_i^{\pm 1}$, E_i, F_i, $i = 1, 2, \ldots, rk(\mathfrak{g})$, subject to the following relations:

$$\begin{aligned}
& k_i k_j = k_j k_i, \quad k_i k_i^{-1} = 1, \\
& k_i E_j k_i^{-1} = q^{(\alpha_i, \alpha_j)} E_j, \quad k_i F_j k_i^{-1} = q^{-(\alpha_i, \alpha_j)} F_j, \\
& [E_i,\ F_j] = \delta_{ij} \tfrac{k_i - k_i}{q_i - q_i^{-1}}, \\
& \textstyle\sum_{t=0}^{1-a_{ij}} (-1)^t \begin{bmatrix} 1-a_{ij} \\ t \end{bmatrix}_{q_i} (E_i)^t E_j (E_i)^{1-a_{ij}-t} = 0, \quad i \neq j, \\
& \textstyle\sum_{t=0}^{1-a_{ij}} (-1)^t \begin{bmatrix} 1-a_{ij} \\ t \end{bmatrix}_{q_i} (F_i)^t F_j (F_i)^{1-a_{ij}-t} = 0, \quad i \neq j,
\end{aligned} \tag{6.1}$$

where $\begin{bmatrix} s \\ t \end{bmatrix}_x = \frac{[s]_x!}{[s-t]_x![t]_x!}$ denotes the Gauss polynomial in x, with $[m]_x! = \prod_{k=1}^m [k]_x$, and $[m]_x = \frac{x^m - x^{-m}}{x - x^{-1}}$ and where for any simple root α_i, $q_i = q^{(\alpha_i,\ \alpha_i)/2}$.

Recall that $\mathrm{U}_q(\mathfrak{g})$ has the structure of a Hopf algebra. Denote its comultiplication by Δ, and its antipode by S; then explicitly,

$$\begin{aligned} \Delta(k_i) &= k_i \otimes k_i, \quad \Delta(E_i) = E_i \otimes k_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + k_i^{-1} \otimes F_i, \\ S(E_i) &= -E_i k_i^{-1}, \qquad S(F_i) = -k_i F_i, \qquad S(k_i) = k_i^{-1}. \end{aligned}$$

We shall also consider the quantised universal enveloping algebra $\mathrm{U}_q(\mathfrak{gl}_k)$ of $\mathfrak{gl}_k$. In place of the k_i, we have generators K_a, $1 \le a \le k$, and the first three sets of relations in (6.1) are replaced by

$$\begin{aligned} &K_a K_a = K_b K_a, \quad K_a K_a^{-1} = 1, \\ &K_a E_j K_a^{-1} = q^{(e_a, \alpha_j)} E_j, \quad K_a F_j K_a^{-1} = q^{-(e_a, \alpha_j)} F_j, \\ &[E_i,\ F_j] = \delta_{ij} \frac{K_i K_{i+1}^{-1} - K_i^{-1} K_{i+1}}{q - q}, \end{aligned}$$

while the Serre relations remain the same. The comultiplication and antipode for $\mathrm{U}_q(\mathfrak{gl}_k)$ are similar to the case when $\mathfrak{g}$ is simple. In the remainder of the paper, $\mathfrak{g}$ will either be a simple Lie algebra or $\mathfrak{gl}_k$ for some k.

We shall consider only finite dimensional $\mathrm{U}_q(\mathfrak{g})$-modules which are direct sums of weight spaces, i.e., simultaneous eigenspaces of the elements k_i; further we confine ourselves to "type-$(1, \ldots, 1)$ modules"; this means that corresponding to each weight vector w, there exists a $\mu \in \mathfrak{h}^*$ such that $k_i w = q_i^{\frac{2(\mu,\alpha_i)}{(\alpha_i,\alpha_i)}} w$, for all i. For $\mathrm{U}_q(\mathfrak{gl}_k)$ this condition is replaced by $K_a w = q^{(e_a,\mu)} w$, for all a. We shall refer to μ as the weight of w. In particular, the weight of a highest weight vector of an irreducible module will be called the highest weight of the module. The advantage of our convention is that this notion of weights coincides with that of the case of Lie algebras. We shall denote by $\mathcal{C}_q(\mathfrak{g})$ the category of such $\mathrm{U}_q(\mathfrak{g})$-modules, which is known to be semisimple [L2]. Furthermore, $\mathcal{C}_q(\mathfrak{g})$ forms a braided tensor category [Ka].

As in the classical case, define $\Delta^{(r-1)} : \mathrm{U}_q(\mathfrak{g}) \longrightarrow \mathrm{U}_q(\mathfrak{g})^{\otimes r}$ recursively by $\Delta^{(r-1)} = (\mathrm{id}^{\otimes(r-2)} \otimes \Delta) \circ \Delta^{(r-2)}$. This makes any tensor product of $\mathrm{U}_q(\mathfrak{g})$-modules into a $\mathrm{U}_q(\mathfrak{g})$-module, and we shall be interested in the decomposition of such modules into irreducibles.

The category $\mathcal{C}_q(\mathfrak{g})$ has a braiding, conveniently described by adapting to the present situation Drinfel'd's theory [D] of the universal R-matrix in the context where $\mathrm{U}_q(\mathfrak{g})$ is defined over a power series ring $\mathbb{C}[[h]]$. Let Θ be the quasi-R matrix [L2], which belongs to an appropriate completion of $\mathrm{U}_q(\mathfrak{g}) \otimes \mathrm{U}_q(\mathfrak{g})$. Let $U_\pm$ be the subalgebra of $\mathrm{U}_q(\mathfrak{g})$ generated by the elements E_i (resp. F_i). Then by [KR] there exist elements $\{\mathbf{E}_s\}$ of U_+ and $\{\mathbf{F}_s\}$ of U_- respectively, which may be used to express the quasi R-matrix as

$$\Theta = 1 \otimes 1 + (q - q^{-1}) \sum \mathbf{E}_s \otimes \mathbf{F}_s. \tag{6.2}$$

Given any two modules V_1 and V_2 in $\mathcal{C}_q(\mathfrak{g})$, define the map

$$\Upsilon_{V_1,V_2} : V_1 \otimes V_2 \longrightarrow V_1 \otimes V_2, \qquad w_1 \otimes w_2 \mapsto q^{(\mu_1,\mu_2)} w_1 \otimes w_2, \tag{6.3}$$

where w_1 and w_2 have weights μ_1 and μ_2 respectively. This defines a U_q-endomorphism of $V_1 \otimes V_2$ for any pair of $\mathrm{U}_q(\mathfrak{g})$ modules V_1, V_2, and we speak of Υ as a "functorial endomorphism", whose evaluation at (V_1, V_2) is Υ_{V_1,V_2}. Now let

$$R = \Theta\Upsilon. \tag{6.4}$$

We shall also consider $R^T = \Theta^T\Upsilon$, where $\Theta^T = 1 \otimes 1 + (q - q^{-1}) \sum \mathbf{F}_s \otimes \mathbf{E}_s$.

The following results of Drinfel'd are well-known.

Theorem 6.1. *Let W_i, $i = 1, 2, 3$ be finite dimensional $\mathrm{U}_q(\mathfrak{g})$-modules. Then*

(a) *For all $x \in \mathrm{U}_q(\mathfrak{g})$, $R\Delta(x) = \Delta'(x)R$, where Δ' is the opposite comultiplication.*

(b) *R acts on $(W_1 \otimes W_2) \otimes W_3$ by $R_{13}R_{23}$, and on $W_1 \otimes (W_2 \otimes W_3)$ by $R_{13}R_{12}$.*

(c) *Part (b) implies the Yang-Baxter equation*

$$R_{12}R_{13}R_{23} \quad = \quad R_{23}R_{13}R_{12}. \tag{6.5}$$

Next let ϖ be the functorial endomorphism, which attaches to any $\mathrm{U}_q(\mathfrak{g})$-module W the endomorphism $\varpi_W : W \longrightarrow W$, defined by $\varpi_W : w \mapsto q^{-(\mu,\mu)}w$, where μ is the weight of w. Let K be a product of the $k_i^{\pm 1}$ in $\mathrm{U}_q(\mathfrak{g})$ such that $Kw = q^{2(\mu,\rho)}w$ for any weight vector w of weight μ, where ρ is the half sum of the positive roots of $\mathfrak{g}$. We have the following well-known result (see, e.g. [KS]).

Theorem 6.2. *Define the functorial endomorphism*

$$v := K^{-1}\left(1 + (q - q^{-1})\sum_t S(\mathbf{F}_t)\mathbf{E}_t\right)\varpi. \tag{6.6}$$

Then v has the following properties:

(a) *The evaluation of v at any $\mathrm{U}_q(\mathfrak{g})$-module is invertible, and its inverse v^{-1} acts on an irreducible $\mathrm{U}_q(\mathfrak{g})$-module L_λ with highest weight λ as multiplication by the scalar $q^{(\lambda+2\rho,\lambda)}$.*

(b) *For any pair W_1 and W_2 of $\mathrm{U}_q(\mathfrak{g})$-modules in $\mathcal{C}_q(\mathfrak{g})$, v^{-1} acts on $W_1 \otimes W_2$ as $(v^{-1} \otimes v^{-1})R^T R$.*

Note that the evaluations of v^{-1} are the quantum analogues of the classical quadratic Casimir.

6.2. **Representations of the braid group.** Let B_r denote the braid group on r strings. It is generated by elements b_i, $i = 1, 2, \ldots, r-1$, with relations

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}, \qquad b_i b_j = b_j b_i, \quad |i - j| > 1. \tag{6.7}$$

Given a representation of $\mathrm{U}_q(\mathfrak{g})$, for any integer r, the universal R-matrix of $\mathrm{U}_q(\mathfrak{g})$ provides a representation of the r-string braid group B_r on a tensor space. To see this, fix a $\mathrm{U}_q(\mathfrak{g})$-module (π, V_q), and define the linear maps

$$s : V_q \otimes V_q \to V_q \otimes V_q, \quad w \otimes w' \mapsto w' \otimes w,$$
$$\check{R} = s \circ R : V_q \otimes V_q \to V_q \otimes V_q.$$

Then it respectively follows from parts (a) and (b) of Theorem 6.1 that $\check{R}$ is a $\mathrm{U}_q(\mathfrak{g})$-module automorphism,

$$\check{R}(\pi \otimes \pi)\Delta(x) = (\pi \otimes \pi)\Delta(x)\check{R}, \quad \forall x \in \mathrm{U}_q(\mathfrak{g}), \tag{6.8}$$

and satisfies the following form of the Yang-Baxter equation

$$(6.9)\quad (\check{R}\otimes \mathrm{id}_{V_q})(\mathrm{id}_{V_q}\otimes \check{R})(\check{R}\otimes \mathrm{id}_{V_q}) = (\mathrm{id}_{V_q}\otimes \check{R})(\check{R}\otimes \mathrm{id}_{V_q})(\mathrm{id}_{V_q}\otimes \check{R}).$$

The following result is well-known (see, e.g., [R]).

Theorem 6.3. *For any* $\mathrm{U}_q(\mathfrak{g})$*-module* V_q*, there is a representation*

$$\psi_r : B_r \longrightarrow GL(V_q^{\otimes r})$$

of the r*-string braid group* B_r *with generators* $b_1, \ldots, b_{r-1}$ *on* $V_q^{\otimes r}$*, defined by*

$$\psi_r(b_i) = \mathrm{id}_{V_q}^{\otimes(i-1)} \otimes \check{R} \otimes \mathrm{id}_{V_q}^{\otimes(r-i)}, \qquad i = 1, 2, \cdots r-1.$$

This is clear from the Yang-Baxter equation (6.9).

Denote the image $\mathbb{C}(z)\psi_r(\mathrm{B}_r)$ by $\mathcal{B}(r)$.

Lemma 6.4. $\mathcal{B}(r)$ *is a subalgebra of* $\mathrm{End}_{\mathrm{U}_q(\mathfrak{g})}({V_q}^{\otimes r})$.

Proof. We have $\Delta^{(r-1)} = (\mathrm{id}^{\otimes(i-1)} \otimes \Delta \otimes \mathrm{id}^{\otimes(r-i-1)})\Delta^{(r-2)}$ for any $i < r$ because of the co-associativity of Δ. By (6.8),

$$[\check{R}_{i\,i+1},\, \Delta^{(r-1)}(x)] = 0, \quad \forall x \in \mathrm{U}_q(\mathfrak{g}).$$

Therefore, $\psi_i(b_i) \in \mathrm{End}_{\mathrm{U}_q(\mathfrak{g})}({V_q}^{\otimes r})$ for all i. □

We wish to identify $\mathcal{B}(r)$, in particular to determine when it is the whole commuting algebra $\mathrm{End}_{\mathrm{U}_q(\mathfrak{g})}({V_q}^{\otimes r})$. Consider the action of v^{-1} on ${V_q}^{\otimes r}$ for $r > 2$. By repeatedly applying part (b) of Theorem 6.2, we obtain

$$\begin{aligned}\Delta^{(r-1)}(v^{-1}) \;=\; & \underbrace{(v^{-1}\otimes\cdots\otimes v^{-1})}_{r}\,\check{R}_{12}\check{R}_{12}\cdot\check{R}_{23}\check{R}_{12}\check{R}_{12}\check{R}_{23}\\ & \cdot\check{R}_{34}\check{R}_{23}\check{R}_{12}\check{R}_{12}\check{R}_{23}\check{R}_{34}\,\cdot\,\cdots\\ & \cdot\check{R}_{r-1\,r}\check{R}_{r-2\,r-1}\cdots\check{R}_{2\,3}\check{R}_{12}\check{R}_{12}\check{R}_{23}\cdots\check{R}_{r-2\,r-1}\check{R}_{r-1\,r}.\end{aligned}$$

In terms of $\psi_r(b_i)$, this equation can be rewritten as

$$\begin{aligned}\pi^{\otimes r}\circ\Delta^{(r-1)}(v^{-1}) \;=\; & \underbrace{(v^{-1}\otimes\cdots\otimes v^{-1})}_{r}\,\psi_r(b_1)\psi_r(b_1)\cdot\psi_r(b_2)\psi_r(b_1)\psi_r(b_1)\psi_r(b_2)\\ & \cdot\psi_r(b_3)\psi_r(b_2)\psi_r(b_1)\psi_r(b_1)\psi_r(b_2)\psi_r(b_3)\cdots\\ & \cdot\psi_r(b_{r-1})\psi_r(b_{r-2})\cdots\psi_r(b_1)\psi_r(b_1)\cdots\psi_r(b_{r-2})\psi_r(b_{r-1}).\end{aligned}$$

Assume that V_q is the irreducible $\mathrm{U}_q(\mathfrak{g})$-module L_{Λ_0} with highest weight Λ_0. Then v^{-1} acts on V_q as the scalar $q^{(\Lambda_0+2\rho,\Lambda_0)}$ by part (a) of Theorem 6.2. Thus in this case $\pi^{\otimes r}\left(\Delta^{(r-1)}(v^{-1})\right)$ belongs to $\mathcal{B}(r)$. Also note that $\pi^{\otimes k}\left(\Delta^{(k-1)}(v^{-1})\right)\otimes id_{V_q}^{\otimes(r-k)} \in \mathcal{B}(r)$ for all $k \le r$, and these operators commute with one another.

Observe that $\check{R}^2$ and $R^T R$ are equal, regarded as maps on $V_q\otimes V_q$. Assume that $V_q\otimes V_q$ is multiplicity free, and write $P[\mu]$ for the projection to the irreducible component with highest weight μ. Then by part (b) of Theorem 6.2, we have the following formulae (cf. [R])

$$(6.10)\quad \check{R}^2 = \sum_{\mu\in\mathcal{P}^+_{\Lambda_0}} q^{\chi_\mu(C)-2\chi_{\Lambda_0}(C)}P[\mu], \qquad \check{R} = \sum_{\mu\in\mathcal{P}^+_{\Lambda_0}} \epsilon(\mu)q^{\frac{1}{2}(\chi_\mu(C)-2\chi_{\Lambda_0}(C))}P[\mu],$$

where the $\epsilon(\mu) = \pm 1$ are the eigenvalues of the permutation s in the classical limit $q \to 1$.

Example 6.5. *The natural* $\mathrm{U}_q(\mathfrak{gl}_n)$*-module and a representation of the Hecke algebra.*

Let V_q be the natural $\mathrm{U}_q(\mathfrak{gl}_n)$-module. Then $V_q \otimes V_q$ decomposes as the direct sum of L_{2e_1} and $L_{e_1+e_2}$. Denote by $P[2e_1]$ and $P[e_1+e_2]$ respectively the projections onto these irreducible components. The calculations of eigenvalues of the Casimir in Section 3 immediately give $\check{R} = qP[2e_1] - q^{-1}P[e_1+e_2]$, which leads to

$$(\check{R} - q)(\check{R} + q^{-1}) = 0.$$

Thus as maps on $V_q^{\otimes r}$, $(\psi_r(b_i) - q)(\psi_r(b_i) + q^{-1}) = 0$ for all i. This fact together with (6.7) imply (cf. [J]) that $\mathcal{B}(r)$ is a homomorphic image of the Hecke algebra $H(r)$. The statement that $\mathcal{B}(r)$ is the whole endomorphism algebra in this case is usually referred to as Schur-Jimbo duality:

$$\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{gl}_n)}(V^{\otimes r}, V^{\otimes r}) = \mathcal{B}(r), \quad \forall r \geq 2. \tag{6.11}$$

Example 6.6. *The natural modules for* $\mathrm{U}_q(\mathfrak{so}_n)$ *and* $\mathrm{U}_q(\mathfrak{sp}_{2m})$ *and representations of the Birman-Wenzl algebra.*

The natural $\mathrm{U}_q(\mathfrak{g})$-module V_q has highest weight e_1. As in the classical case,

$$V_q \otimes V_q = L_{2\epsilon_1} \oplus L_{\epsilon_1+\epsilon_2} \oplus L_0,$$

where L_0 is the trivial module. Let $P[\lambda]$, $\lambda = 2e_1, e_1+e_2$, 0 be the projections onto L_λ. Then using the eigenvalues of the quadratic Casimir operator in the relevant irreducible representations computed in Section 3, we have

$$\begin{aligned} \check{R} &= qP[2e_1] - q^{-1}P[e_1+e_2] + q^{1-n}P[0], && \text{for } \mathfrak{so}_\mathrm{n}, \\ \check{R} &= qP[2e_1] - q^{-1}P[e_1+e_2] - q^{-2m-1}P[0], && \text{for } \mathfrak{sp}_{2\mathrm{m}}. \end{aligned} \tag{6.12}$$

Lemma 6.7. *Maintaining the above notation, we have*

$$\begin{aligned}
(P[0] \otimes id_{V_q})(id_{V_q} \otimes P[0])(P[0] \otimes id_{V_q}) &= \tfrac{1}{(\dim_q V_q)^2} P[0] \otimes id_{V_q}, \\
(id_{V_q} \otimes P[0])(P[0] \otimes id_{V_q})(id_{V_q} \otimes P[0]) &= \tfrac{1}{(\dim_q V_q)^2} id_{V_q} \otimes P[0], \\
(P[0] \otimes id_{V_q})(id_{V_q} \otimes \check{R})(P[0] \otimes id_{V_q}) &= \tfrac{q^{\chi_{e_1}(C)}}{\dim_q V_q} P[0] \otimes id_{V_q}, \\
(id_{V_q} \otimes P[0])(\check{R} \otimes id_{V_q})(id_{V_q} \otimes P[0]) &= \tfrac{q^{\chi_{e_1}(C)}}{\dim_q V_q} id_{V_q} \otimes P[0],
\end{aligned}$$

where $\dim_q V_q = \mathrm{tr}_{V_q}(K)$ *is the quantum dimension of* V_q.

Proof. The proof is analogous to that of Lemma 4.3; we sketch a proof of the first relation. Recall [Ja, 5.3, p.71] that for any finite dimensional $\mathrm{U}_q(\mathfrak{g})$-module W and $\phi \in \mathrm{End}_{\mathbb{C}}(W)$, the quantum trace $\mathrm{tr}_{q,W}(\phi) = \mathrm{tr}_W(\phi K)$ is a $\mathrm{U}_q(\mathfrak{g})$-module homomorphism to the trivial $\mathrm{U}_q(\mathfrak{g})$-module.

The left side of the first relation is a $\mathrm{U}_q(\mathfrak{g})$-module map from $L_0 \otimes V_q$ to itself. But $L_0 \otimes V_q \simeq V_q$ is irreducible, hence the map must be of the form $c\ \mathrm{id}_{P[0]\otimes \mathrm{id}_{V_q}}$, where c is a scalar.

To determine the scalar c, we take the quantum trace of both sides, following the method of [ZGB] (cf. Lemma 4.3 above). For any pair W_1 and W_2 of finite dimensional modules over $\mathrm{U}_q(\mathfrak{g})$ define $T^q_{W_2,W_1} : \mathrm{End}_{\mathbb{C}}(W_1 \otimes W_2) \longrightarrow \mathrm{End}_{\mathbb{C}} W_1$ by

$T^q_{W_2,W_1}(\sum h \otimes h') := \sum \mathrm{tr}_{q,W_2}(h')h$. Then $T^q_{W_2,W_1}$ is a homomorphism of $\mathrm{U}_q(\mathfrak{g})$-modules, and since $\Delta(K) = K \otimes K$, in analogy with the classical case, we have for $f \in \mathrm{End}_{\mathbb{C}}(W_1 \otimes W_2)$,

$$\mathrm{tr}_{q,W_1}(T^q_{W_2,W_1}(f)) = \mathrm{tr}_{q,W_1\otimes W_2}(f).$$

If W_1 is irreducible and $f \in \mathrm{End}_{\mathrm{U}_q}(W_1 \otimes W_2)$, then by Schur's Lemma, $T^q_{W_2,W_1} f = b\ \mathrm{id}_{W_1}$ for some constant b, and using the trace relation above, $b = \frac{\mathrm{tr}_{q,W_1\otimes W_2}(f)}{\dim_q(W_1)}$. Using this, it is clear that $T^q_{V_q,V_q}(P[0]) = \frac{1}{\dim_q(V_q)}\mathrm{id}_{V_q}$.

The argument now proceeds exactly as that in Lemma 4.3.

To prove the last two relations, one also requires the relation

$$(\mathrm{id}_{V_q} \otimes tr_{V_q})\{(\mathrm{id}_{V_q} \otimes K) \circ \check{R}\} = q^{\chi_{e_1}(C)}\mathrm{id}_{V_q},$$

which may be found in [ZGB]. □

Now define the following endomorphisms of $V_q^{\otimes r}$:

$$\tau_i = \mathrm{id}^{\otimes(i-1)} \otimes \epsilon(0)P[0] \otimes \mathrm{id}^{\otimes(r-i-1)}, \qquad i = 1, 2, \cdots r-1,$$

where $\epsilon(0)$ is equal to 1 for $\mathfrak{so}_n$ and -1 for $\mathfrak{sp}_{2m}$. Then in analogy with Theorem 4.4, (6.12) and Lemma 6.7 imply the well-known fact that

Proposition 6.8. *The $\psi_r(b_i)$ and τ_i together generate a representation of the Birman-Wenzl-Murakami algebra (see, e.g.,* [BW]*) on $V_q^{\otimes r}$.*

7. Strongly multiplicity free modules for quantum groups

The notion of strongly multiplicity free irreducible modules extends to quantised enveloping algebras in the obvious way. Let (π, V_q) be an irreducible $\mathrm{U}_q(\mathfrak{g})$-module with highest weight Λ_0. To remove the trivial case, we assume $\dim L_{\Lambda_0} > 1$. Recall the notion of weights from Section 6.1, and we denote by Π the set of the weights of V_q.

Definition 7.1. The irreducible $\mathrm{U}_q(\mathfrak{g})$-module V_q is called strongly multiplicity free if for any distinct elements μ and ν in Π, $\mu - \nu \in \mathbb{N}R_+ \cup (-\mathbb{N}R_+)$.

The strongly multiplicity free $\mathrm{U}_q(\mathfrak{g})$-modules are closely related to the strongly multiplicity free modules of the corresponding Lie algebra $\mathfrak{g}$. This connection follows from the result of [L1, Theorem 4.12] which states that when z (and therefore q) is specialised to 1, a suitable lattice in the irreducible $\mathrm{U}_q(\mathfrak{g})$-module L_λ with highest weight λ specialises to the irreducible $\mathrm{U}(\mathfrak{g})$-module with the same highest weight. We describe this specialisation here. Now L_λ is generated by a highest weight vector w^+. Take any basis E consisting of elements of the form $F_{i_1}F_{i_2}\cdots F_{i_k}w^+$. Let $\mathbb{C}(z)_1$ be the subring of elements of $\mathbb{C}(z)$ with no pole at $z = 1$. Then $(L_\lambda)^{reg} := \mathbb{C}(z)_1 E$ is stable under the action of all the generators E_i, F_i, $k_i^{\pm 1}$ and $\frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}$ of $\mathrm{U}_q(\mathfrak{g})$. Define an action of $\mathbb{C}(z)_1$ on $\mathbb{C}$ with z acting as 1. Then $(L_\lambda)^{reg} \otimes_{\mathbb{C}(z)_1} \mathbb{C}$ is isomorphic to the irreducible $\mathrm{U}(\mathfrak{g})$-module with highest weight λ ([L1], *loc. cit.*).

In view of Theorem 3.4 and the above remarks, the following lemma provides a classification of the strongly multiplicity free modules.

Lemma 7.2. *An irreducible* $\mathrm{U}_q(\mathfrak{g})$*-module is strongly multiplicity free if and only if the irreducible* $\mathrm{U}(\mathfrak{g})$*-module with the same highest weight is strongly multiplicity free.*

The quantum analogue of part (i) of Lemma 3.1 is evidently true. Thus the tensor product $L_\Lambda \otimes V_q$ of a multiplicity free module V_q with any irreducible module (π_Λ, L_Λ) is multiplicity free. Write $L_\Lambda \otimes V_q = \bigoplus_{\lambda\in\mathcal{P}^+_\Lambda} L_\lambda$, where the definition of $\mathcal{P}^+_\Lambda$ is the same as in the classical case of Lie algebras.

Lemma 7.3. *Assume that* V_q *is strongly multiplicity free. Then* v^{-1} *acts on the irreducible submodules of* $L_\Lambda \otimes V_q$ *as multiplication by scalars, which are distinct for distinct submodules.*

Proof. The eigenvalues of v^{-1} on the irreducible submodule L_λ is $q^{\chi_\lambda(C)}$. The assertion now follows from Lemma 3.6. □

If $L_\Lambda \otimes V_q$ is multiplicity free, v^{-1} acts on $L_\Lambda \otimes V_q$ by

$$(\pi_\Lambda \otimes \pi)(v^{-1}) = \sum_{\mu\in\mathcal{P}^+_\Lambda} q^{\chi_\mu(C)} P[\mu].$$

As in the classical case, when v^{-1} has distinct eigenvalues on the irreducible submodules of $L_\Lambda \otimes V_q$, the projections can be reconstructed from $(\pi_\Lambda \otimes \pi)(v^{-1})$ through the formula

$$P[\lambda] \;:=\; \prod_{\mu(\neq\lambda)} \frac{(\pi_\Lambda \otimes \pi)(v^{-1}) - q^{\chi_\mu(C)}}{q^{\chi_\lambda(C)} - q^{\chi_\mu(C)}}, \quad \lambda \in \mathcal{P}^+_\Lambda. \tag{7.1}$$

The following result may be proved in the same way as Proposition 3.11, so we omit the details.

Lemma 7.4. *Assume that* V_q *is a strongly multiplicity free* $\mathrm{U}_q(\mathfrak{g})$*-module.*

1. *For each integer* $r \geq 0$ *there is a decomposition* $V_q^{\otimes r} = \oplus_i L_i^r$ *of* $V_q^{\otimes r}$ *into irreducibles which is canonical in the sense that, up to ordering the summands, it depends only on* V_q
2. *The projections* $P_i^r : V_q^{\otimes r} \longrightarrow L_i^r$*, regarded as endomorphisms of* $V_q^{\otimes r}$ *(i.e.* $Res^{V_q^{\otimes r}}_{L_j^r} P_i^r = \delta_{ij} id_{L_j^r}$*) lie in* $\mathcal{B}(r)$*.*

The second formula in (6.10) and (7.1) together provide a more transparent explanation of Lemma 6.4 in the case when V_q is strongly multiplicity free. The converse of Lemma 6.4 is also true in this case:

Theorem 7.5. *If* V_q *is strongly multiplicity free, then for any integer* $r \geq 2$,

$$\mathrm{End}_{\mathrm{U}_q(\mathfrak{g})}(V_q^{\otimes r}) = \mathcal{B}(r).$$

In the next section we shall in fact prove a stronger version (Theorem 8.5) of the theorem, which includes the quantised universal enveloping algebra of the even dimensional orthogonal Lie algebras. The proof proceeds essentially by reducing to the classical case.

8. Proof of Theorem 7.5

When V_q is the natural module for $\mathrm{U}_q(\mathfrak{gl}_k)$, Theorem 7.5 is the celebrated Schur-Jimbo duality stated in (6.11). When $\mathrm{U}_q(\mathfrak{sl}_k)$ is regarded as a Hopf subalgebra of $\mathrm{U}_q(\mathfrak{gl}_k)$, we have

$$\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{gl}_k)}(V_q^{\otimes r}, V_q^{\otimes r}) = \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{sl}_k)}(V_q^{\otimes r}, V_q^{\otimes r}).$$

This proves Theorem 7.5 in these cases.

Now in all the remaining cases of Theorem 7.5, the modules involved are self dual. Therefore, as finite dimensional vector spaces, $\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{g})}(V_q^{\otimes r}, V_q^{\otimes r}) \cong \left(V_q^{\otimes 2r}\right)^{\mathrm{U}_q(\mathfrak{g})}$. We shall complete the proof of the theorem by showing that the dimension of $\mathcal{B}(r)$ is equal to the dimension of $\left(V_q^{\otimes 2r}\right)^{\mathrm{U}_q(\mathfrak{g})}$ by reducing to the classical case following the approach of [L1]. Note that [LR, Corollary 5.22] shows that Theorem 7.5 is true for the natural modules of $\mathrm{U}_q(\mathfrak{so}_{2m+1})$ and $\mathrm{U}_q(\mathfrak{sp}_{2m})$.

8.1. **Preliminaries.** We recall some facts required for the proof of the theorem.

8.1.1. *The 7-dimensional irreducible* $\mathrm{U}_q(G_2)$*-module.* Denote by V_q the 7- dimensional irreducible module over $\mathrm{U}_q(G_2)$. Exactly as in the classical case, we have

$$V_q \otimes V_q = L_{2\lambda_1} \oplus L_0 \oplus L_{\lambda_1} \oplus L_{\lambda_2}.$$

In terms of the projections onto these irreducible submodules, the $\check{R}$-matrix can be expressed as

$$\check{R} = q^2 P[2\lambda_1] + q^{-12} P[0] - q^{-6} P[\lambda_1] - P[\lambda_2].$$

The existence of the the trivial 1-dimensional submodule L_0 in $V_q \otimes V_q$ shows that V_q is self dual.

8.1.2. *The algebras* $\mathrm{U}_q(\mathfrak{o}_n)$*.* We introduce extra generators $\sigma^{\pm 1}$ to enlarge $\mathrm{U}_q(\mathfrak{so}_n)$ obtaining a new algebra, which we denote by $\mathrm{U}_q(\mathfrak{o}_n)$. Here $\sigma^{\pm 1}$ are mutual inverses, and have the following properties: If n is odd, σ commutes with all the generators of $\mathrm{U}_q(\mathfrak{so}_n)$. When $n = 2m$, we label the last two simple roots of $\mathfrak{g}$ as $\alpha_{m-1} = e_{m-1} - e_m$ and $\alpha_m = e_{m-1} + e_m$. Then

$$\begin{aligned} \sigma E_{m-1}\sigma^{-1} &= E_m, & \sigma E_m \sigma^{-1} &= E_{m-1}, \\ \sigma F_{m-1}\sigma^{-1} &= F_m, & \sigma F_m \sigma^{-1} &= F_{m-1}, \\ \sigma k_{m-1}\sigma^{-1} &= k_m, & \sigma k_m \sigma^{-1} &= k_{m-1}, \end{aligned}$$

while all the other generators commute with σ. Then $\mathrm{U}_q(\mathfrak{o}_n)$ is the associative algebra generated by $\mathrm{U}_q(\mathfrak{so}_n)$ and $\sigma^{\pm 1}$. We extend the comultiplication and antipode of $\mathrm{U}_q(\mathfrak{so}_n)$ to $\mathrm{U}_q(\mathfrak{o}_n)$ by letting $\Delta(\sigma) = \sigma \otimes \sigma$, and $S(\sigma) = \sigma^{-1}$. Then $\mathrm{U}_q(\mathfrak{o}_n)$ acquires the structure of a Hopf algebra. Now σ acts as an automorphism on $\mathrm{U}_q(\mathfrak{so}_n)$ by conjugation. The R-matrix of $\mathrm{U}_q(\mathfrak{so}_n)$ is invariant under the automorphism, i.e., $\Delta(\sigma)$ commutes with the R-matrix.

We extend the natural $\mathrm{U}_q(\mathfrak{so}_n)$-module V_q to a $\mathrm{U}_q(\mathfrak{o}_n)$-module by requiring that

(i) For odd n, σ acts on the highest weight vector of V_q by -1;
(ii) For even n, σ acts on the highest weight vector of V_q by 1.

Then $\mathrm{U}_q(\mathfrak{o}_n)$ also acts on tensor powers of V_q through the comultiplication. A $\mathrm{U}_q(\mathfrak{o}_n)$-module will be a sum of submodules of $V_q^{\otimes r}$, $r \geq 0$.

Every irreducible $\mathrm{U}_q(\mathfrak{so}_{2m+1})$-submodule of $V_q^{\otimes r}$ lifts to an irreducible $\mathrm{U}_q(\mathfrak{o}_{2m+1})$ - module, but isomorphic irreducible $\mathrm{U}_q(\mathfrak{so}_{2m+1})$-modules may become non-isomorphic $\mathrm{U}_q(\mathfrak{o}_{2m+1})$-modules. Consider the joint kernel of the maps $\mathrm{id}^{\otimes(i-1)} \otimes (\check{R}-q)(\check{R}-q^{-2m}) \otimes \mathrm{id}^{\otimes(2m+1-1-i)}$ in $V_q^{\otimes(2m+1)}$, $i = 1, 2, \dots, 2m+1$, which is a 1-dimensional $\mathrm{U}_q(\mathfrak{o}_{2m+1})$-submodule of $V_q^{\otimes(2m+1)}$. Let $\det_q$ be a generator of this module, which is an $\mathrm{U}_q(\mathfrak{so}_{2m+1})$-invariant, but σ acts as -1. Therefore, isomorphic $\mathrm{U}_q(\mathfrak{so}_{2m+1})$-modules L_λ and $\det_q \otimes L_\lambda$ are non-isomorphic as $\mathrm{U}_q(\mathfrak{o}_{2m+1})$-modules.

Let $\lambda \in \sum_{i=1}^m \mathbb{Z}e_i$ be a dominant $\mathfrak{so}_{2m}$ weight. If $l_m := (\lambda, e_m) = 0$, the irreducible $\mathrm{U}_q(\mathfrak{so}_{2m})$-module L_λ lifts to an is irreducible $\mathrm{U}_q(\mathfrak{o}_{2m})$-module. If $l_m \neq 0$, let $\lambda' = \lambda - 2l_m e_m$. Then $L_\lambda \oplus L_{\lambda'}$ lifts to an irreducible module over $\mathrm{U}_q(\mathfrak{o}_{2m})$. There also exists a 1-dimensional $\mathrm{U}_q(\mathfrak{o}_{2m})$-module $\mathbb{C}(z)\det_q$ which is defined similarly as in the case of $\mathrm{U}_q(\mathfrak{o}_{2m+1})$. On this module, σ acts as -1.

8.2. **Completion of the proof of Theorem 7.5.** Henceforth V_q will denote the natural $\mathrm{U}_q(\mathfrak{g})$-module for $\mathfrak{g} = \mathfrak{o}_k$ or $\mathfrak{sp}_{2k}$, the 7-dimensional irreducible module for $\mathrm{U}_q(G_2)$, and the irreducible $l+1$-dimensional module for $\mathrm{U}_q(\mathfrak{sl}_2)$. Denote the highest weight of V_q by Λ_0 as before. Recall that V_q is multiplicity free and self dual, and that the irreducible submodules of $V_q \otimes V_q$ have distinct v^{-1}-eigenvalues.

We have defined a $\mathbb{C}(z)_1$-lattice $(L_\lambda)^{reg}$ for any irreducible $\mathrm{U}_q(\mathfrak{g})$-module L_λ in Section 7. Let $(V_q^{\otimes r})^{reg} := (V_q)^{reg} \otimes_{\mathbb{C}(z)_1} \cdots \otimes_{\mathbb{C}(z)_1} (V_q)^{reg}$. This $\mathbb{C}(z)_1$ module is clearly stable under the action of the generators E_i, F_i, $k_i^{\pm 1}$, $\frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}$, and also the generator σ if $\mathfrak{g}$ is $\mathfrak{o}_k$. Specialising z to 1 in $(V_q^{\otimes r})^{reg}$, we obtain the $\mathrm{U}(\mathfrak{g})$-module $(V_q^{\otimes r})^{reg} \otimes_{\mathbb{C}(z)_1} \mathbb{C}$. Here $\mathrm{U}(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$ if $\mathfrak{g} \neq \mathfrak{o}_k$, and $\mathrm{U}(\mathfrak{so}_k)$ augmented by an automorphism corresponding to σ if $\mathfrak{g} = \mathfrak{o}_k$. If $V = (V_q)^{reg} \otimes_{\mathbb{C}(z)_1} \mathbb{C}$, then $V^{\otimes r} = (V_q^{\otimes r})^{reg} \otimes_{\mathbb{C}(z)_1} \mathbb{C}$.

Proposition 8.1. *Let V_q be any irreducible $\mathrm{U}_q(\mathfrak{g})$-module. For any positive integer r,*

$$\dim_{\mathbb{C}(z)} \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{g})}(\mathbb{C}(z), V_q^{\otimes r}) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{U}(\mathfrak{g})}(\mathbb{C}, V^{\otimes r}), \tag{8.1}$$

where V is the specialisation of V_q at $z = 1$. (Note that for this result only, V_q may be any irreducible $\mathrm{U}_q(\mathfrak{g})$-module).

Proof. The projection which maps $V_q^{\otimes r}$ onto the isotypic component of the trivial $\mathrm{U}_q(\mathfrak{g})$-module is given by

$$Q = \begin{cases} \frac{1}{2}(\Delta^{(r-1)}(\sigma) + 1) \prod_{\mu \in \Theta_r \setminus 0} \frac{\Delta^{(r-1)}(v^{-1}) - q^{\chi_\mu(C)}}{1 - q^{\chi_\mu(C)}} & \text{if } \mathfrak{g} = \mathfrak{o}_k, \\ \prod_{\mu \in \Theta_r \setminus 0} \frac{\Delta^{(r-1)}(v^{-1}) - q^{\chi_\mu(C)}}{1 - q^{\chi_\mu(C)}} & \text{otherwise,} \end{cases}$$

where Θ_r denotes the set of the highest weights of the isotypic components of $V_q^{\otimes r}$. Now the numerator of each factor of Q vanishes at the specialisation $z = 1$ (cf. (6.6)). Since $z - 1$ has multiplicity 1 in the denominator of each factor, it follows that the $\mathbb{C}(z)_1$-module $(V_q^{\otimes r})^{reg}$ is stable under the action of Q. Let

$$\left((V_q^{\otimes r})^{reg}\right)^0 = Q(V_q^{\otimes r})^{reg}, \quad \left((V_q^{\otimes r})^{reg}\right)^\perp = (1 - Q)(V_q^{\otimes r})^{reg}.$$

Then $\mathbb{C}(z)\left((V_q^{\otimes r})^{reg}\right)^0 \cong \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{g})}(\mathbb{C}(z), V_q^{\otimes r})$.

Clearly $V^{\otimes r}$ is spanned by $\left((V_q^{\otimes r})^{reg}\right)^0 \otimes_{\mathbb{C}(z)_1} \mathbb{C}$ and $\left((V_q^{\otimes r})^{reg}\right)^\perp \otimes_{\mathbb{C}(z)_1} \mathbb{C}$. Since the quadratic Casimir operator of $\mathrm{U}(\mathfrak{g})$ annihilates $\left((V_q^{\otimes r})^{reg}\right)^0 \otimes_{\mathbb{C}(z)_1} \mathbb{C}$ and acts on $\left((V_q^{\otimes r})^{reg}\right)^\perp \otimes_{\mathbb{C}(z)_1} \mathbb{C}$ as an automorphism, we have the decomposition

$$V^{\otimes r} = \left((V_q^{\otimes r})^{reg}\right)^0 \otimes_{\mathbb{C}(z)_1} \mathbb{C} \bigoplus \left((V_q^{\otimes r})^{reg}\right)^\perp \otimes_{\mathbb{C}(z)_1} \mathbb{C}$$

of $V^{\otimes r}$ as $\mathrm{U}(\mathfrak{g})$-module. The proposition follows. □

Lemma 8.2. *Let $P[\lambda]$, $\lambda \in \mathcal{P}^+_{\Lambda_0}$, be the projections which map $V_q \otimes V_q$ onto its irreducible components. Then $P[\lambda](V_q \otimes V_q)^{reg} \subset (V_q \otimes V_q)^{reg}$ for all λ. Thus every $P[\lambda]$ specialises to the projection $P[\lambda]^0$ which maps $(V_q \otimes V_q)^{reg} \otimes_{\mathbb{C}(z)_1} \mathbb{C}$ onto the irreducible $\mathrm{U}(\mathfrak{g})$-submodule with highest weight λ. More precisely, for all $w \in (V_q \otimes V_q)^{reg}$,*

$$(P[\lambda]w) \otimes_{\mathbb{C}(z)_1} 1 = P[\lambda]^0(w \otimes_{\mathbb{C}(z)_1} 1).$$

Remark 8.3. The projections $P[\lambda]^0$ were denoted by $P[\lambda]$ in Sections 3-5. Here we add the superscript 0 to differentiate from the quantum case.

Proof of Lemma 8.2. It is clear from (6.6) and (7.1) that the projection $P[\lambda]$ has neither a zero nor pole at $z = 1$. The first statement is immediate.

Define $(L_\lambda)_{reg} = P[\lambda](V_q \otimes V_q)^{reg}$. A vector $w \in (L_\lambda)_{reg}$ specialises to 0 only if it belongs to $(z-1)(L_\lambda)_{reg}$. In that case, there exists a vector w' in $(L_\lambda)_{reg}$ but not in $(z-1)(L_\lambda)_{reg}$ such that $w = (z-1)^a w'$ for some positive integer a. Hence there exist highest weight vectors of $(L_\lambda)_{reg}$ which do not vanish when z is specialised to 1. Thus $(L_\lambda)_{reg} \otimes_{\mathbb{C}(z)_1} \mathbb{C}$ is a non-trivial indecomposable $\mathrm{U}(\mathfrak{g})$-module, which must be irreducible [L1] by complete reducibility of finite dimensional $\mathrm{U}(\mathfrak{g})$-modules. It follows that $V^{\otimes 2} = (V_q \otimes V_q)^{reg} \otimes_{\mathbb{C}(z)_1} \mathbb{C}$ decomposes as

$$V^{\otimes 2} = \bigoplus_{\lambda \in \mathcal{P}^+_{\Lambda_0}} (L_\lambda)_{reg} \otimes_{\mathbb{C}(z)_1}$$

into irreducible $\mathrm{U}(\mathfrak{g})$-submodules. If $P[\lambda]^0$ are the projections which map $V^{\otimes 2}$ onto these irreducible components, we have

$$P[\lambda]^0(w \otimes 1) = (P[\lambda]w) \otimes 1, \quad \forall w \in (V_q \otimes V_q)^{reg},$$

as stated. □

Remark 8.4. Lemma 8.2 can also be proved by explicitly constructing the lattice $(L_\lambda)_{reg}$.

Proof of Theorem 7.5. Define endomorphisms of $V_q^{\otimes r}$ as follows.

$$P_i[\lambda] = \mathrm{id}_{V_q}^{\otimes(i-1)} \otimes P[\lambda] \otimes \mathrm{id}_{V_q}^{\otimes(r-i-1)}, \quad \lambda \in \mathcal{P}^+_{\Lambda_0}, \ i = 1, 2, \dots, r-1,$$

and denote by $\mathbf{P}$ the set of monomials in the projections $P[\lambda]_i$. Then $\mathbf{P}$ spans $\mathcal{B}(r)$ over $\mathbb{C}(z)$. Similarly, define their classical analogues on $V^{\otimes r}$ as follows.

$$P_i[\lambda]^0 = \mathrm{id}_V^{\otimes(i-1)} \otimes P[\lambda]^0 \otimes \mathrm{id}_V^{\otimes(r-i-1)}, \quad \lambda \in \mathcal{P}^+_{\Lambda_0}, \ i = 1, 2, \dots, r-1.$$

Clearly the $P_i[\lambda]^0$ generate the algebra $\mathcal{A}(r)$.

For any element $Q = P_{i_1}[\lambda_1]\cdots P_{i_k}[\lambda_k] \in \mathbf{P}$, let $Q^0 = P_{i_1}[\lambda_1]^0\cdots P_{i_k}[\lambda_k]^0$ be its classical specialisation. It follows from Lemma 8.2 by induction on k that

$$(Qw)\otimes_{\mathbb{C}(z)_1} 1 = Q^0(w\otimes_{\mathbb{C}(z)_1} 1), \quad \forall w \in (V_q^{\otimes r})^{reg}.$$

If a set of elements $Q_1, \dots, Q_k$ of $\mathbf{P}$ is $\mathbb{C}(z)$-linearly dependent, then there exist $c_1(z), \dots, c_n(z) \in \mathbb{C}(z)_1$, not all in $(z-1)\mathbb{C}(z)_1$, such that $\sum_i c_i(z)Q_i = 0$ as an element of $\mathcal{B}(r)$. Hence $\sum_i c_i(1)Q_i^0 = 0$ is a non-trivial linear relation in $\mathcal{A}(r)$. Since $\mathcal{A}(r)$ is spanned by monomials in the elements $P_i[\lambda]^0$, $\dim_{\mathbb{C}(z)}\mathcal{B}(r) \geq \dim_{\mathbb{C}}\mathcal{A}(r)$. Further, $\dim_{\mathbb{C}}\mathcal{A}(r) = \dim_{\mathbb{C}}\mathrm{Hom}_{\mathrm{U}(\mathfrak{g})}(\mathbb{C}, V^{\otimes 2r})$ by Theorem 3.15. It follows that

$$\dim_{\mathbb{C}(z)}\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{g})}(\mathbb{C}(z), V_q^{\otimes 2r}) \geq \dim_{\mathbb{C}(z)}\mathcal{B}(r) \geq \dim_{\mathbb{C}}\mathrm{Hom}_{\mathrm{U}(\mathfrak{g})}(\mathbb{C}, V^{\otimes 2r}). \tag{8.2}$$

But by Proposition 8.1, the two inequalities in (8.2) are equalities. This completes the proof of Theorem 7.5. □

It is clear that $\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{o}_{2m+1})}(V_q^{\otimes r}, V_q^{\otimes r}) = \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{so}_{2m+1})}(V_q^{\otimes r}, V_q^{\otimes r})$ as σ acts on $V_q^{\otimes r}$ by $(-1)^r$. Also $\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{gl}_2)}(V_q^{\otimes r}, V_q^{\otimes r}) = \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{sl}_2)}(V_q^{\otimes r}, V_q^{\otimes r})$. Thus we have proved the following theorem, which strengthens Theorem 7.5.

Theorem 8.5. *Let* $\mathrm{U}_q(\mathfrak{g})$ *be* $\mathrm{U}_q(\mathfrak{gl}_k)$, $\mathrm{U}_q(\mathfrak{sl}_k)$, $\mathrm{U}_q(\mathfrak{o}_k)$, $\mathrm{U}_q(\mathfrak{so}_{2m+1})$ $\mathrm{U}_q(\mathfrak{sp}_{2k})$, *or* $U_q(G_2)$. *Let* V_q *denote the natural module over* $\mathrm{U}_q(\mathfrak{g})$ *if* $\mathfrak{g}$ *is* $\mathfrak{gl}_k$, $\mathfrak{sl}_k$ $(k>2)$, $\mathfrak{o}_k$, $\mathfrak{sp}_{2k}$, *the* 7*-dimensional irreducible module if* $\mathfrak{g}$ *is the Lie algebra of* G_2, *and the* l*-th symmetric power of the natural module if* $\mathfrak{g}$ *is* $\mathfrak{gl}_2$ *or* $\mathfrak{sl}_2$. *Then*

$$\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{g})}(V_q^{\otimes r}, V_q^{\otimes r}) = \mathcal{B}(r).$$

Remark 8.6. The arguments above may be adapted to prove the following quantum analogue of Theorem 5.5. Let $V_{1,q}, \cdots, V_{r,q}$ be arbitrary irreducible modules for $\mathrm{U}_q(\mathfrak{sl}_2)$. Then the endomorphism algebra of $V_{1,q}\otimes\cdots\otimes V_{r,q}$ is generated by operators of the form $1\otimes\cdots\otimes 1\otimes\Delta^{(i)}\left(\frac{v^{-1}-1}{q-1}\right)\otimes 1\otimes\cdots\otimes 1$.

Appendix A. Proof of Theorem 2.2

We shall prove Theorem 2.2 by showing that the T_r-module $(\psi_r, V^{\otimes r})$ is unitarisable. To do this, we need to recall some notions on $*$-algebras and their unitarisable modules. A complex associative algebra $\mathcal{A}$ is called a $*$-algebra if it has an anti-linear anti-involution $\omega : \mathcal{A}\longrightarrow\mathcal{A}$ (called a $*$-structure). An $\mathcal{A}$-module (ϕ, W) is called unitarisable if there exists a positive definite Hermitian form $\langle\ |\ \rangle : W\times W\longrightarrow\mathbb{C}$ such that

$$\langle v|\phi(a)u\rangle = \langle\phi(\omega(a))v|u\rangle, \quad \forall a\in\mathcal{A}, v, u\in W.$$

Let $\mathcal{A}'$ be another $*$-algebra with a $*$-structure ω', and let (ϕ', W') be a unitarisable $\mathcal{A}'$ module with the positive definite Hermitian form $\langle\ |\ \rangle' : W'\times W'\longrightarrow\mathbb{C}$. Now $\mathcal{A}\otimes\mathcal{A}'$ is a $*$-algebra with the $*$-structure $\omega\otimes\omega'$, and $(\phi\otimes\phi', W\otimes W')$ forms a unitarisable $\mathcal{A}\otimes\mathcal{A}'$-module with a positive definite hermitian form $\langle\langle\ |\ \rangle\rangle : (W\otimes W')\times(W\otimes W')\longrightarrow\mathbb{C}$ defined by $\langle\langle u\otimes u'|v\otimes v'\rangle\rangle = \langle u|v\rangle\langle u'|v'\rangle'$.

Now the algebra T_r admits a $*$-structure defined by

$$\omega : T_r\to T_r, \quad t_{ij}\mapsto t_{ij}, \quad \forall i<j. \tag{A.1}$$

Let $\mathfrak{g}$ denote either $\mathfrak{gl}_k$ or a finite dimensional semisimple Lie algebra. Then the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ is a $*$-algebra with the $*$-structure defined by

$$\theta : \mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g}), \qquad X_\alpha \mapsto X^\alpha, \quad \forall \alpha, \tag{A.2}$$

where $\{X_\alpha\}$ and $\{X^\alpha\}$ are bases of $\mathfrak{g}$ dual with respect to the non-degenerate bilinear form on $\mathfrak{g}$. Furthermore, all finite dimensional $\mathrm{U}(\mathfrak{g})$-modules are unitarisable.

Proof of Theorem 2.2. Clearly every unitarisable module over a $*$-algebra is semisimple, and so the theorem will follow if we show that $(\psi_r, V^{\otimes(r+1)})$ is a unitarisable T_r-module.

From the above discussion of $*$-algebras it is clear that $\mathrm{U}(\mathfrak{g})^{\otimes r}$ forms a $*$-algebra with the $*$-structure $\theta^{\otimes r}$, where θ is defined by (A.2). But clearly $\theta^{\otimes r}(C_{ij}) = C_{ij}$, $\forall i < j$. Thus the map $\psi : T_r \to \mathrm{U}(\mathfrak{g})^{\otimes r}$ preserves the $*$-structures of the algebras in the sense that $\theta \circ \psi = \psi \circ \omega$. Therefore, every unitarisable $\mathrm{U}(\mathfrak{g})^{\otimes r}$-module has the structure of a unitarisable T_r-module, and the theorem follows. □

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